A NUMERICAL SOLUTION TO THE MATRIX $\mathcal{H}_2/\mathcal{H}_\infty$ OPTIMAL CONTROL PROBLEM

G. D. HALIKIAS$^1$, I. M. JAIMOUKHA$^2$ AND D. A. WILSON$^1$

$^1$Department of Electronic and Electrical Engineering, University of Leeds, Leeds, LS2 9JT, U.K.
$^2$Interdisciplinary Research Centre for Process Systems Engineering and Department of Electrical and Electronic Engineering, Imperial College of Science, Technology and Medicine, London, SW7-2BY, U.K.

SUMMARY

In this paper a numerical solution is obtained to the problem of minimizing an $\mathcal{H}_2$-type cost subject to an $\mathcal{H}_\infty$-norm constraint. The method employed is based on the convex alternating projection algorithm and generalizes a recent technique to the multivariable case. The solution is derived in terms of the Markov parameters of an FIR filter of arbitrary length; this is finally approximated by a low-order IIR filter using Hankel-norm model-reduction techniques. The results are illustrated with a numerical example. © 1997 by John Wiley & Sons, Ltd.

1. INTRODUCTION

In this paper the following mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem is considered: given rational matrix functions $F(z), G(z) \in \mathcal{L}_\infty^{p \times l}$, find $H(z) \in \mathcal{H}_\infty^{p \times l}$ which minimizes the cost,

$$\gamma_2 = \inf \|F(z) - H(z)\|_2$$

subject to the constraint,

$$\|G(z) - H(z)\|_\infty \leq \gamma_\infty$$

where $\gamma_\infty$ is a specified positive number.

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems have recently received considerable attention for their application to robust control: a natural application of problems of this type involves the optimization of an $\mathcal{H}_2$-type performance measure subject to an $\mathcal{H}_\infty$-norm bound on an appropriate closed-loop.

This paper was recommended for publication by editor C. de Souza

*Correspondence to: G. D. Halikias, Department of Electronic and Electrical Engineering, University of Leeds, Leeds, LS2 9JT, U.K.

Contract grant sponsor: Engineering and Physical Science Research Council
Contract grant number: GR/J42533

CCC 1049-8923/97/070711-16$17.50
© 1997 by John Wiley & Sons, Ltd.

Received 4 November 1994
Revised 2 April 1996
system which guarantees robust stability in the presence of model uncertainty. At present, mixed
problems of the type considered in this paper are typically solved via convex programming
methods introduced in Reference 2 or via non-differential optimization techniques. Both
methods are computationally intensive and typically result in high-order optimal controllers. The
resulting optimization problems are in principle infinite-dimensional and thus (in the first case)
they are typically approximated by a sequence of finite-dimensional convex programmes (see e.g.,
the work in Reference 4 which relies on $n$-grid approximation techniques). As a result, alternative
methods have been developed which seek to minimize an ‘entropy-like’ upper bound of the
$H_2$ cost, subject to an $H_\infty$-norm constraint. Despite certain attractive features of this approach (the
solution is obtained via a truly finite-dimensional convex programme and the degree of the
optimal controller is shown not to exceed the degree of the generalized plant), they suffer from the
fact that they are essentially sub-optimal in character and do not seem to generalize naturally to
more general settings (e.g., to the case of two independent vectors of exogenous inputs).

This paper extends the method given in Reference 1 for solving the mixed $H_2/H_\infty$ optimization
problem to the matrix case. First, the problem is formulated as a finite-dimensional Hilbert-space
minimum-distance problem to a set which is characterized as a finite intersection of closed convex
sets. Next, the individual projections to these sets are obtained, which allows the so-called convex
alternating projection method\textsuperscript{6} to be applied to the problem.

The layout of the paper is as follows: Section 2 gives a few definitions and outlines some basic
background theory. In Section 3 the optimization problem is posed in a geometric setting and
a solution is obtained via the convex alternating projection algorithm. In Section 4, a number of
issues associated with the realization of the optimal $H(z)$ from its Markov parameters are
addressed: to circumvent the state dimension inflation and the numerical ill-conditioning of the
Kalman–Ho algorithm (which, in principle, could be employed at this stage to obtain a minimal
realization of $H(z)$ from its Markov parameters), a Hankel-norm approximation technique
applied to FIR filters is developed. A computer example included in Section 5 illustrates the
results in the paper and compares the numerical efficiency of the algorithm with other convex
programming optimization methods. Finally, Section 6 contains the paper’s conclusions.

2. NOTATION AND PRELIMINARIES

Let $D$ denote the open unit disk $D = \{ \zeta \in \mathbb{C} : |\zeta| < 1 \}$ with $\overline{D}$ and $\partial D$ its closure and boundary, respectively. $L_2(\partial D)$ denotes the Hilbert space of $p \times l$ matrix-valued functions defined on the unit circle as,
\[ L_2 = \left\{ F(\zeta) : \frac{1}{2\pi} \int_0^{2\pi} \text{trace}[F^*(e^{i\theta})F(e^{i\theta})] \, d\theta < \infty \right\} \]
with inner product,
\[ \langle F, G \rangle = \frac{1}{2\pi} \int_0^{2\pi} \text{trace}[F^*(e^{i\theta})G(e^{i\theta})] \, d\theta < \infty \]
$H_2(\partial D)$ ($\mathcal{H}_2(\partial D)$) denotes the (closed) subspace of $L_2(\partial D)$ of all matrix-valued functions analytic in $\mathbb{C} \setminus \overline{D}$ ($D$). $L_\infty(\partial D)$ is the space of all uniformly bounded matrix-valued functions in $\partial D$, i.e., all functions defined on the unit circle whose norm,
\[ \| F \|_\infty = \sup_{\theta \in [0, 2\pi]} \tilde{\sigma}[F(e^{i\theta})] \]
is finite. Here $\tilde{\sigma}(\cdot)$ denotes the largest singular value. $H_\infty(\partial D)$ ($\mathcal{H}_\infty(\partial D)$) is the closed subspace of $L_\infty(\partial D)$ with functions analytic in $\mathbb{C} \setminus \overline{D}$ ($D$), while $\mathcal{H}_\infty(-k)(\partial D)$ denotes the set of all (matrix)
functions in \( \mathcal{L}_\infty(\partial D) \) with no more than \( k \) poles in \( D \). Spaces of real-rational functions will be denoted with the prefix \( \mathbb{R} \).

Next, let \( P_+ \) and \( P_- \) denote the orthogonal projections
\[
P_+ : \mathcal{L}_2(\partial D) \rightarrow \mathcal{H}_2(\partial D), \quad P_- : \mathcal{L}_2(\partial D) \rightarrow \mathcal{H}_2(\partial D)
\]
and consider the optimization problems,
\[
d_2 = \inf \{ \| F - H \|_2 : H \in \mathcal{H}_\infty^{p \times l} \}
\]
and
\[
d_\infty = \inf \{ \| G - H \|_\infty : H \in \mathcal{H}_\infty^{p \times l} \}
\]
in which \( F \) and \( G \) are \( p \times l \) (rational) matrix functions in \( \mathcal{L}_\infty \). Since \( \mathcal{L}_\infty(\partial D) \subseteq \mathcal{L}_2(\partial D) \) and \( \mathcal{H}_\infty(\partial D) \subseteq \mathcal{H}_2(\partial D) \), \( d_2 \) is the distance from \( F \) to \( \mathcal{H}_\infty(\partial D) \) in the \( \mathcal{L}_2 \)-norm, while \( d_\infty \) is the distance from \( G \) to \( \mathcal{H}_\infty(\partial D) \) in the \( \mathcal{L}_\infty \)-norm. Since \( \mathcal{L}_2 \) is a Hilbert space, the solution to the first problem is unique and can be obtained by projection, i.e., \( d_2 = \| P_- F \|_2 \). The solution to the second problem may be obtained from Nehari’s theorem as \( d_\infty = \| \Gamma_G \| \) in which \( \Gamma_G \) is the Hankel operator associated with \( G \), defined as
\[
\Gamma_G : \mathcal{H}_2(\partial D) \rightarrow \mathcal{H}_2(\partial D), \quad \Gamma_G = P_+ M_G|_{\mathcal{H}^2(\partial D)}
\]
in which \( M_G \) denotes the multiplication operator \( M_G : \mathcal{L}_2(\partial D) \rightarrow \mathcal{L}_2(\partial D), M_G f = G f \). In addition, it is well known, that the infimum in (5) is actually attained.\(^7\) In the scalar case the minimizing \( H(z) \) is unique and may be obtained from the two Schmidt vectors of \( \Gamma_G \) corresponding to \( \| \Gamma_G \| \). In the matrix case, the optimal \( H(z) \) is hardly ever unique; the optimal solution set may be parametrized via a linear fractional transformation of the set of contractions which are analytic in \( D \).\(^7\)

3. MIXED \( \mathcal{H}_2/\mathcal{H}_\infty \) OPTIMIZATION

The mixed \( \mathcal{H}_2/\mathcal{H}_\infty \) optimization (1)–(2) is considerably harder than the individual optimization problems (3) and (4). In this section, we present a number of preliminary results for the solution of this problem and generalize a numerical technique proposed in Reference 1 based on the alternating projection algorithm to the matrix case.

First note that the optimization problem (1)–(2) has a solution if and only if \( \gamma_\infty \geq d_\infty \). Consider the case \( \gamma_\infty = d_\infty \): if either \( p = 1 \) or \( l = 1 \), the solution to (4) is unique and thus problem (1)–(2) is trivial. In the matrix case, on the other hand, (i.e., if \( p > 1 \) and \( l > 1 \), a continuum of solutions typically exists and thus problem (1)–(2) is non-trivial. Although it is possible, in this case, using results from super-optimal interpolation theory, to translate the problem to an optimization similar in form to (1)–(2) in terms of a function of reduced dimension,\(^8\) it will be assumed in the sequel that \( \gamma_\infty > d_\infty \).

Problem (1)–(2) may be formulated as
\[
\gamma_2(\gamma_\infty) = \inf_{H(z) \in C_{\gamma_\infty}} \| F(z) - H(z) \|_2
\]
where \( \gamma_\infty \in \mathbb{R}_+ \) and
\[
C_{\gamma_\infty} = \{ H(z) \in \mathcal{H}_\infty^{p \times l} : \| G(z) - H(z) \|_\infty \leq \gamma_\infty \}
\]
Since \( F(z) \in \mathbb{R} \mathcal{L}_\infty^{p \times l} \), there exists a unique decomposition \( F(z) = F_+(z) + F_-(z) \) where \( F_+(z) \in \mathbb{R} \mathcal{H}_2 \) and \( F_-(z) \in \mathbb{R} \mathcal{H}_2 \). Let \( \gamma_\infty' = \| \Gamma_G \| \), then \( \gamma_2(\gamma_\infty) \) is well-defined for all \( \gamma_\infty \geq \gamma_\infty' \). Now, for all
Lemma 3.1

Let $\gamma_\infty \geq \gamma'_\infty$. We have

$$\gamma_2(\gamma_\infty) = \inf_{H(z) \in C_\gamma} \|F(z) - H(z)\|_2 \geq \inf_{H(z) \in \mathcal{H}_\infty^p \times l} \|F(z) - H(z)\|_2$$

$$\geq \inf_{H(z) \in \mathcal{H}_\infty^p \times l} \|F(z) - H(z)\|_2 = \|F_-(z)\|_2$$

where the second inequality follows since $\mathcal{H}_\infty^p \times l \subseteq \mathcal{H}_2^p \times l$. On the other hand, let $\gamma'^\infty = \|G(z) - F_+(z)\|_\infty$ (recall that $G(z) \in \mathbb{R} \mathcal{L}_\infty^p \times l$ and $F_+(z) \in \mathbb{R} \mathcal{H}_\infty^p \times l$, so that $\gamma'^\infty$ is finite). Then $F_+(z) \in C_{\gamma'^\infty}$ and so $\gamma_2(\gamma_\infty) \leq \|F(z) - F_+(z)\|_2 = \|F_-(z)\|_2$. Hence, $\gamma_2(\gamma'^\infty) = \|F_-(z)\|_2 = \gamma'^\infty$. Also, $F_+(z) \in C_{\gamma'^\infty}$ for all $\gamma_\infty \geq \gamma'^\infty$. Hence, $\gamma_2(\gamma_\infty) = \gamma_2(\gamma'^\infty) = \gamma'^\infty$ for all $\gamma_\infty \geq \gamma'^\infty$. Further, $F_+(z) \notin C_{\gamma_\infty}$ for any $\gamma_\infty < \gamma'^\infty$, which, in view of the uniqueness of the orthogonal projection implies that $\gamma_2(\gamma_\infty) > \gamma'^\infty$ for all $\gamma_\infty < \gamma'^\infty$. Thus the domain of interest for the function $\gamma_2(\cdot)$ is the closed interval $[\gamma'^\infty, \gamma_\infty]$. (6) is an infinite-dimensional optimization problem for which no analytic solution seems, in general, possible (unless $\gamma_\infty \geq \gamma'^\infty$). One way of simplifying the problem is to restrict $H(z)$ to the class of fixed-order finite-impulse response filters,

$$S_N = \left\{ H(z) = \sum_{k=0}^{N} H_k z^{-k} : H_k \in \mathbb{R}^{p \times l} \right\}$$

In this case, a number of finite-dimensional convex-programming optimization techniques (e.g., see Reference 2) may be employed to obtain a numerical solution to the problem:

$$\hat{\gamma}_2^N(\gamma_\infty) = \inf_{H(z) \in C_{\gamma_\infty} \cap S_N} \|F(z) - H(z)\|_2$$

Clearly, $\gamma_2(\gamma_\infty) = \lim_{N \to \infty} \{\hat{\gamma}_2^N(\gamma_\infty)\}$. Unfortunately, convex programming algorithms tend to be slow and the optimization problem becomes intractable for large values of $N$. In the last part of this section, an alternative optimization technique will be presented, based on the convex alternating projection algorithm, which generalizes the results in Reference 1 to the matrix case. Before that, however, certain properties of $\gamma_2^N(\gamma_\infty)$ are established. The convexity of the cost function and the constraint set follows from the convexity of the two norms and suggests the geometric interpretation of the problem used later in the section.

**Lemma 3.1**

Let $\gamma'^1, \gamma'^2 \in (\gamma'^\infty, \infty)$. Then $\gamma_2^N(\gamma_\infty)$ is a decreasing convex function in the interval $(\gamma'^1, \gamma'^2, \infty)$.

**Proof.** Let $\gamma_\infty, \gamma'^1 \in (\gamma'^\infty, \infty)$ and suppose that

$$\gamma'^1 = \gamma_2^N(\gamma_\infty) = \|F(z) - H_1(z)\|_2, \quad H_1(z) \in C_{\gamma_\infty} \cap S_N, \quad i = 1, 2.$$ 

Let $\alpha \in [0, 1]$ and set $\gamma_\infty = \alpha \gamma'^1 + (1 - \alpha) \gamma'^2$, $H(z) = \alpha H_1(z) + (1 - \alpha) H_2(z) \in \mathcal{H}_\infty^p \times l \cap S_N$. Now,

$$\|G(z) - H(z)\|_\infty = \|\alpha [G(z) - H_1(z)] + (1 - \alpha) [G(z) - H_2(z)]\|_\infty \leq \alpha \gamma'^1 + (1 - \alpha) \gamma'^2$$

$$= \gamma'^1$$

and so, $H(z) \in C_{\gamma_\infty} \cap S_N$. Hence,

$$\gamma_2^N(\gamma_\infty) \leq \|F(z) - H(z)\|_2 = \|\alpha [F(z) - H_1(z)] + (1 - \alpha) [F(z) - H_2(z)]\|_2$$

$$\leq \alpha \gamma'^1 + (1 - \alpha) \gamma'^2$$
which proves that \( \gamma_2^N(\cdot) \) is convex in the internal \((\gamma_1^N, \infty)\). The decreasing property of \( \gamma_2^N(\cdot) \) follows immediately since \( C_{\gamma_2^N} \subseteq C_{\gamma_2^N}^1 \) if \( \gamma_1^N \leqslant \gamma_2^N \). □

Next, consider again problem (1)–(2) and note that it may be written as:

\[
\gamma_2 = \inf \left\{ \| F(z) - H(z) \|_2 : H(z) \in E_1 \cap E_2 \right\}
\]

(7)

where \( E_1 = \mathcal{H}_2 \) is a (closed) subspace of \( \mathcal{L}_2 \) and

\[
E_2 = \{ \Phi(z) \in \mathcal{L}_2 : \| G(z) - \Phi(z) \|_\infty \leqslant \gamma_\infty \}
\]

is a convex set in \( \mathcal{L}_2 \). Next we define a ‘discretized’ version of (7): let \( F(z) \) and \( G(z) \) have Laurent expansions,

\[
F(z) = \sum_{k = -\infty}^{\infty} F_k z^{-k}, \quad G(z) = \sum_{k = -\infty}^{\infty} G_k z^{-k}
\]

and define:

\[
H(z) = \sum_{k = 0}^{N - 1} H_k z^{-k} + \sum_{k = 1}^{N} H_{N-k}^T z^k
\]

where \( N = 2^{n_0} \) and \( n_0 \) is a positive integer. (Note that \( H(z) \in S_{N-1} \) if \( H_{N-k} = 0, k = 1, 2, \ldots, \frac{N}{2} \)). Then,

\[
F(z) - H(z) = \sum_{k = 0}^{N - 1} (F_k - H_k) z^{-k} + \sum_{k = 1}^{N} (F_{-k} - H_{N-k}) z^k + \sum_{k \geqslant N}^{N - 1} F_k z^{-k} + \sum_{k \geqslant \frac{N}{2} + 1}^{N} F_{-k} z^k
\]

and

\[
\| F(z) - H(z) \|_2^2 = \text{trace} \left\{ \sum_{k = 0}^{N - 1} (F_k - H_k)^T (F_k - H_k) + \sum_{k = 1}^{N} (F_{-k} - H_{N-k})^T (F_{-k} - H_{N-k}) + \sum_{k \geqslant \frac{N}{2}}^{N - 1} F_k^T F_k + \sum_{k \geqslant \frac{N}{2} + 1}^{N} F_{-k}^T F_{-k} \right\}
\]

(8)

Next, define the truncated matrix polynomials,

\[
F_T(z) = \sum_{k = 0}^{\frac{N}{2} - 1} F_k z^{-k} + \sum_{k = 1}^{\frac{N}{2}} F_{N-k} z^k, \quad G_T(z) = \sum_{k = 0}^{\frac{N}{2} - 1} G_k z^{-k} + \sum_{k = 1}^{\frac{N}{2}} G_{N-k} z^k
\]

(in which \( F_k = F_k, G_k = G_k \) \( k = 0, 1, \ldots, \frac{N}{2} - 1 \); \( F_{N-k} = F_{-k}, G_{N-k} = G_{-k} \) \( k = 1, 2, \ldots, \frac{N}{2} \)) and the corresponding DFT pair,

\[
\hat{F}_k = F_T \exp \left( j \frac{2\pi k}{N} \right) = \sum_{n = 0}^{N-1} F_n \exp \left( -j \frac{2\pi kn}{N} \right), \quad k = 0, 1, \ldots, N - 1
\]

\[
F_n = \frac{1}{N} \sum_{k = 0}^{N-1} \hat{F}_k \exp \left( j \frac{2\pi kn}{N} \right), \quad n = 0, 1, \ldots, N - 1
\]

(9)
(and similarly for \( \{H_k\} \) and \( \{G_k\} \)). Also let,
\[
\Phi \in \mathbb{C}^{pN \times pN}; \quad \Phi = \left[ \Phi(s, t) \right]_{s=1, \ldots, N}^{t=1, \ldots, N}; \quad \Phi(s, t) = I_p \exp \left( -j \frac{2\pi(s-1)(t-1)}{N} \right)
\]
and
\[
\hat{\Phi} \in \mathbb{C}^{pN \times pN}; \quad \hat{\Phi} = \left[ \hat{\Phi}(s, t) \right]_{s=1, \ldots, N}^{t=1, \ldots, N}; \quad \hat{\Phi}(s, t) = N^{-1} I_p \exp \left( j \frac{2\pi(s-1)(t-1)}{N} \right)
\]
which define in matrix form the relation of the Fourier coefficients in (9) in the time and frequency domains. Next, let \( \Phi, \hat{\Phi} \) be partitioned as
\[
\Phi = \left[ \Phi_1 \; \Phi_2 \right]; \quad \Phi_1, \Phi_2 \in \mathbb{C}^{pN \times pN}
\]
and
\[
\hat{\Phi} = \left[ \hat{\Phi}_1 \right]; \quad \hat{\Phi}_1, \hat{\Phi}_2 \in \mathbb{C}^{pN \times pN}
\]
and define the subspace of \( \mathbb{C}^{pN \times 1} \):
\[
S = \{ X \in \mathbb{C}^{pN \times 1}, \hat{\Phi}_2 X = 0 \} = \{ \Phi_1 Y : Y \in \mathbb{C}^{pN \times 1} \}
\]
(the fact that these two subspaces are identical follows immediately from \( \Phi \hat{\Phi} = I \)). From (7), \( H(z) \) is constrained to be causal (\( H(z) \in \mathcal{H}_2^{p \times 1} \)); since \( \mathcal{H}_2 \) can be identified with the set of matrix functions analytic in \( \mathbb{C} \setminus D \) whose positive Fourier coefficients vanish identically, i.e.,
\[
\mathcal{H}_2^{\mathbb{C} \setminus D} = \{ F(\zeta) \in L_2^\mathbb{C}(\zeta) : \frac{1}{2\pi} \int_0^{2\pi} F(e^{j\theta}) e^{j\nu \theta} \, d\theta = 0, \; \forall n < 0 \}
\]
we restrict \( \{ \hat{H}_k \} \) as:
\[
\hat{H} = \left[ \begin{array}{c} \hat{H}_0 \\ \vdots \\ \hat{H}_{N-1} \end{array} \right] \in S
\]
i.e., we constrain the Fourier coefficients of \( H(z) \) so that the anti-causal coefficients vanish. On noting that (8) may be written in terms of \( \{ \hat{H}_k, \hat{F}_k \} \) as,
\[
\|F(z) - H(z)\|^2 = \text{trace} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} (\hat{F}_k - \hat{H}_k)^* (\hat{F}_k - \hat{H}_k) + \sum_{k=0}^{N-1} \hat{F}_k^T \hat{F}_k + \sum_{k=0}^{N-1} \hat{F}_k^T \hat{F}_{-k} \right\}
\]
we define the optimization problem:
\[
\min \text{trace} \left\{ (\hat{F} - \hat{H})^* (\hat{F} - \hat{H}) \right\} \quad (10)
\]
in which
\[
\hat{F} = \left[ \begin{array}{c} \hat{F}_0 \\ \vdots \\ \hat{F}_{N-1} \end{array} \right]
\]
subject to

\[ \mathbf{H} \in \mathcal{C}_1 \cap \mathcal{C}_2; \quad \mathcal{C}_1 = \mathbb{S}; \quad \mathcal{C}_2 = \begin{bmatrix} C_0 \\ \vdots \\ C_{N-1} \end{bmatrix} \] (11)

where

\[ C_k = \{ \Phi \in \mathbb{C}^{p \times l} : \sigma(\hat{G}_k - \Phi) \leq \gamma_{\infty} \}, \quad k = 0, \ldots, N - 1 \]

The constraint on \( \mathbf{H} \) in (11) is the discretized version of the constraint on \( H(z) \) in (7). Considering the space,

\[ W = \mathbb{C}^{p \times l} \times \cdots \times \mathbb{C}^{p \times l} \times \mathbb{C}^{p \times l} \]

(the Cartesian product being taken \( N \) times) equipped with the inner product

\[ \langle A, B \rangle = \sum_{k=1}^{N} \text{trace}(A_k^* B_k) \] (12)

(where \( A \in W \) is represented as \( A = (A_1, A_2, \ldots, A_N) \) and the two sets defined in (11), we conclude that (10) may be interpreted as the projection of \( \hat{F} \) onto the intersection of a subspace and a convex subset of \( W \). With this geometric interpretation, the convex alternating projection algorithm may be employed to obtain a numerical solution to the problem. This is summarized in the following theorem.

**Theorem 3.2**

Let \( \{ M_1, M_2, \ldots, M_n \} \) be a family of closed convex sets in a Hilbert space \( \mathcal{H} \), and \( M \) the closed convex set defined by \( M = M_1 \cap M_2 \cap \ldots \cap M_n \), assumed to be non-empty. Let \( x_0 \) be any vector in \( \mathcal{H} \) and define the sequence \( \{ x_i \}_{i=0}^{\infty} \) by,

\[
\begin{align*}
x_1 &= P_1 x_0, \\
x_2 &= P_2 x_1, \\
&\vdots \\
x_n &= P_n x_{n-1}, \\
x_{n+1} &= P_1 (x_n - z_1), \\
x_{n+2} &= P_2 (x_n - z_2), \\
&\vdots \\
x_{2n} &= P_n (x_{2n-1} - z_n)
\end{align*}
\]

1st cycle

\[
\begin{align*}
x_{n+1} &= x_n - x_0, \\
z_{n+1} &= x_{n+1} - (x_n - z_1) \\
z_{n+2} &= x_{n+2} - (x_{n+1} - z_2) \\
&\vdots \\
z_{2n} &= x_{2n} - (x_{2n-1} - z_n)
\end{align*}
\]

2nd cycle

etc, where \( P_i \) denotes the projection onto \( M_i \). Then the sequence \( \{ x_i \} \) converges weakly to the orthogonal projection \( P_M x_0 \) of \( x_0 \) onto \( M \). The convex set \( M \) is empty if and only if the sequence does not converge.

**Proof.** See Reference 6. \( \Box \)
For the related algorithm which can offer accelerated convergence by employing directional information see References 1 and 6. The projection onto $\mathcal{C}_1 = S$ can be obtained by transforming to the time domain, setting the anti-causal coefficients to zero and then transforming back to the frequency domain via the FFT transform; this is computationally more efficient than projecting onto the range of $\Phi_1$ or the kernel of $\Phi_2$ via a direct linear-algebraic technique. The calculation of the projection onto $\mathcal{C}_2$ defined in (11) is given next.

**Lemma 3.3**

Let $K$ be the inner product space $(\mathbb{C}^{p \times l}, \langle \cdot, \cdot \rangle)$ with $\langle A, B \rangle = \text{trace} (A^* B)$ and define

$$C_m = \{ \Phi \in \mathbb{C}^{p \times l} : \tilde{\sigma}(\hat{G}_k - \Phi) \leq \gamma_\infty \} \quad (p \geq l)$$

Then $C_m$ is a closed convex set in $K$. For any $A \in \mathbb{C}^{p \times l}$ the projection of $A$ onto $C_m$ is given by:

$$P(A) = \tilde{G}_k + U \begin{bmatrix} \Sigma^0 & 0 \\ 0 & 0 \end{bmatrix} V^*, \quad \text{for } \tilde{\sigma}(A - \hat{G}_k) > \gamma_\infty$$

$$= A, \quad \text{for } \tilde{\sigma}(A - \hat{G}_k) \leq \gamma_\infty$$

in which

$$\Sigma^0 = \text{diag}\{ \sigma^0_1, \ldots, \sigma^0_n \}, \quad \sigma^0_i := \min(\gamma_\infty, \sigma_i)$$

where $A - \hat{G}_k$ has a singular value decomposition,

$$A - \hat{G}_k = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*$$

with $\Sigma = \text{diag}\{ \sigma_1, \ldots, \sigma_n \}$ $(\sigma_i \geq \sigma_{i+1})$.

**Proof.** Since $C_m$ is a closed convex set, the projection of $A$ onto $C_m$ is uniquely determined by the condition,

$$\text{Re} \langle A - \hat{A}, \hat{A} - \Phi \rangle \geq 0, \quad \forall \Phi \in C_m$$

or,

$$\text{Re} \text{trace}\{ (A - \hat{A})(\hat{A} - \Phi)^* \} \geq 0, \quad \forall \Phi \in C_m$$

This can be written as

$$\text{Re} \text{trace}\left\{ U \begin{bmatrix} \Sigma - \Sigma^0 \\ 0 \end{bmatrix} V^*(\hat{G}_k^* - \Phi^* + V[\Sigma^0 \ 0] U^*) \right\}$$

which can be rearranged as

$$\text{trace}\{ \Sigma^0 (\Sigma - \Sigma^0) \} - \text{Re} \text{trace}\left\{ U \begin{bmatrix} \Sigma - \Sigma^0 \\ 0 \end{bmatrix} V^*(\Phi^* - \hat{G}_k^*) \right\}$$

on using the properties of the trace function. Let the singular values of $\hat{G}_k - \Phi$ be $\{ \sigma_i^\hat{G}_k \}_{i=1}^n$ in non-increasing order of magnitude. Then,

$$0 \leq \sigma_i^\hat{G}_k \leq \gamma_\infty$$
and hence (Reference 10, p. 436),

$$\text{Re} \left\{ \text{trace} \left\{ U \left[ \Sigma - \Sigma^0 \right] V^* (\Phi^* - \hat{G}_k) \right\} \right\} \leq \sum_{i=1}^{n} \sigma_i^{\#} (\sigma_i - \sigma_i^{\#})$$

$$\leq \gamma_{\infty} \sum_{i=1}^{n} (\sigma_i - \sigma_i^{\#}) = \gamma_{\infty} \sum_{i \in I} (\sigma_i - \gamma_{\infty})$$

where

$$I = \{ i: \sigma_i > \gamma_{\infty} \}$$

Also,

$$\text{trace} \left\{ \Sigma^0 (\Sigma - \Sigma^0) \right\} = \sum_{i=1}^{n} \sigma_i^{\#} (\sigma_i - \sigma_i^{\#}) = \sum_{i \in I} \gamma_{\infty} (\sigma_i - \gamma_{\infty})$$

Hence,

$$\text{Re} \left\{ (A - \hat{A}) (\hat{A}^* - \Phi^*) \right\} \geq \gamma_{\infty} \sum_{i \in I} (\sigma_i - \gamma_{\infty}) - \gamma_{\infty} \sum_{i \in I} (\sigma_i - \gamma_{\infty}) = 0$$

and thus

$$\text{Re} \left\langle A - \hat{A}, \hat{A} - \Phi \right\rangle \geq 0, \quad \forall \Phi \in C_k$$

as required. A dual result can be obtained when $p \leq l$. 

It now follows from the definition of $W$, that the projection of

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix} \in W$$

onto $\tilde{C}_2$ is given by

$$\tilde{P}(A) = \begin{bmatrix} P(A_1) \\ \vdots \\ P(A_N) \end{bmatrix}$$

(with respect to the inner product defined in (12)).

4. HANKEL-NORM APPROXIMATION OF MATRIX FIR FILTERS

The application of the alternating projection algorithm to the $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem (1)–(2) gives the optimal $H(z)$ in terms of its Markov parameters. At this stage, a minimal state-space realization of $H(z)$ may be obtained via the Kalman–Ho algorithm.\(^{11}\) To avoid the numerical ill-conditioning of this technique and the potentially high-order resulting realization, an alternative approach is developed in this section based on Hankel-norm approximation techniques.

An important advantage of Hankel-norm approximation methods over other model-reduction techniques is that they offer tight bounds on the infinity norm of the approximation error; in particular, it is shown in Reference 7 how to construct approximations $H_k(z)$ of $H(z)$ with
\[ \text{deg}(H_k) \leq k \text{ such that,} \]

\[ \| H(z) - H_k(z) \|_{\infty} \leq \sum_{i > k + 1} \sigma_i[H(z)] \]

where \( \sigma_i[H(z)] \) denotes the \( i \)th Hankel singular value of \( H(z) \) (indexed in non-increasing order of magnitude). This (a priori) bound on the approximation error is of particular significance in our case, since it can be used to select the degree of the approximation \( H_k(z) \) without violating the \( L_\infty \)-norm constraint (2). Suppose for example that \( H(z) \in C_{\gamma_{\infty}} \) with,

\[ \| G(z) - H(z) \|_{\infty} = \gamma_{\infty}^* < \gamma_{\infty} \]

Provided that \( k \) is chosen so that,

\[ \sum_{i > k + 1} \sigma_i[H(z)] \leq \gamma_{\infty} - \gamma_{\infty}^* \]

the \( k \)th order approximation \( H_k(z) \) is still admissible since

\[ \| G(z) - H_k(z) \|_{\infty} \leq \| G(z) - H(z) \|_{\infty} + \| H(z) - H_k(z) \|_{\infty} \]

\[ \leq \gamma_{\infty}^* + \sum_{i > k + 1} \sigma_i[H(z)] \]

\[ \leq \gamma_{\infty} \]

The degree of approximation may be reduced further (at the expense of an increased deviation from optimality), by imposing a tighter \( L_\infty \)-norm bound to (2) than is actually required (say \( \gamma_{\infty} := \gamma_{\infty} - \delta, \delta > 0 \)) and choosing an approximation order \( k \) which satisfies,

\[ \sum_{i > k + 1} \sigma_i[H(z)] \leq \gamma_{\infty} - \| G(z) - H(z) \|_{\infty} ; \quad \| G(z) - H(z) \|_{\infty} \leq \gamma_{\infty} - \delta \]

As a result of this approximation, the corresponding 'cost' \( \| F(z) - H_k(z) \|_2 \) may also increase from its optimal level \( \| F(z) - H(z) \|_2 = \gamma_2 \), say. Now,

\[ \| G(z) - H_k(z) \|_2 \leq \| G(z) - H(z) \|_2 + \| H(z) - H_k(z) \|_2 \]

\[ = \gamma_2 + \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \text{trace} \left[ (H(e^{j\theta}) - H_k(e^{j\theta}))(H(e^{j\theta}) - H_k(e^{j\theta}))^* \right] d\theta \]

\[ = \gamma_2 + \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sum_{i=1}^{\min(p, l)} \sigma_i^2 \left[ (H(e^{j\theta}) - H_k(e^{j\theta})) \right] d\theta \]

\[ \leq \gamma_2 + \sqrt{\min(p, l)} \| H(z) - H_k(z) \|_{\infty} \]

and hence, the deviation of the 2-norm cost from its optimal level is guaranteed to be no worse than

\[ \delta = \sqrt{\min(p, l)} \sum_{i > k + 1} \sigma_i[H(z)] \]
where \( \min(p, l) \) is the minimum input/output dimension of \( H(z) \). Thus, the Hankel singular values of \( H(z) \) provide information about the worst-case deterioration in both the infinity- and 2-norm costs, and hence can be used to ‘trade-off’ deviation from optimality with the complexity of the approximation \( H_k(z) \). It should also be noted that even if \( \|G(z) - H(z)\|_\infty = \gamma \) (which is likely to happen when the two objectives really compete with one another), approximating \( H(z) \) by a lower-order system \( H_k(z) \) does not necessarily imply infeasibility in the infinity norm constraint: Consider for example the problem:

\[
\begin{align*}
\min_{H(z) \in H_+} & \quad \|F(z) - H(z)\|_2 \\
\text{subject to,} & \quad \|G(z) - H(z)\|_\infty \leq 0.5
\end{align*}
\]

with,

\[
F(z) = \begin{bmatrix} 0.5 & 0 \\ 0 & \varepsilon z^{-m} \end{bmatrix}, \quad G(z) = \begin{bmatrix} 0.5 + 1 \varepsilon & 0 \\ 0 & 0 \end{bmatrix}
\]

where \( 0 < \varepsilon \leq 0.5 \) and \( m > 0 \). Clearly, the optimal solution is given by the \( m \)th order filter,

\[
H(z) = \begin{bmatrix} 0.5 & 0 \\ 0 & \varepsilon z^{-m} \end{bmatrix}
\]

since it is both feasible and gives zero 2-norm cost. Now it is clear that the constraint is still feasible if \( H(z) \) is replaced by a lower-order approximation, for example the constant approximation

\[
H_0(z) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}
\]

For a full discussion of a similar problem concerning hierarchical \( \mathcal{H}_\infty \) control, see Reference 12.

One way of carrying out the Hankel-norm approximation is to transform \( H(z) \) to the \( s \)-domain via an appropriate bilinear transformation, perform the model reduction in this domain (e.g., using the formulae in Reference 7), and, finally, transform back to discrete-time using the inverse bilinear transformation. To avoid the accumulation of numerical errors arising from this procedure, it is preferable to carry out the approximation directly in the \( z \)-domain; this is achieved by applying certain results of Reference 3 to our case.

The following theorem parametrizes all solutions \( H_k(z) \) to the Hankel norm approximation problem \( \|H(z) - H_k(z)\|_\infty \leq \gamma \), in which \( H(z) \) is a (matrix) FIR filter and \( H_k(z) \) is a (matrix) IIR filter of degree \( \deg(H_k) \leq k \). Note that: (a) The parametrization is given in descriptor form, so that it applies both to the sub-optimal \( (\sigma_{k+1}[H(z)] < \gamma < \sigma_k[H(z)]) \) and the optimal \( (\gamma = \sigma_{k+1}[H(z)]) \) case, and (b) the generator of all solutions is defined directly in terms of the Markov parameters of \( H(z) \).

**Theorem 4.1**

Let

\[
\tilde{H}(z) = H(z) - H_0 = H_1 z^{-1} + H_2 z^{-2} + \cdots + H_n z^{-n}; \quad H_i \in \mathbb{C}^{p \times l}
\]
Then:

1. The Hankel singular values of \( \vec{H}(z) \), \( \sigma_i[\vec{H}(z)] \) (indexed in non-increasing order) are given by the singular values of the (Hankel) matrix,

\[
R_1 = \begin{bmatrix}
    H_1 & H_2 & H_3 & \cdots & H_n \\
    H_2 & H_3 & 0 & \cdots & 0 \\
    H_3 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    H_n & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

2. Suppose that \( \sigma_k[\vec{H}(z)] \leq \gamma < \sigma_k[\vec{H}(z)] \) and define:

\[
P = R_2 R_2^* := \begin{bmatrix}
    H_n & 0 & \cdots & 0 \\
    H_{n-1} & H_n & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    H_1 & H_2 & \cdots & H_n
\end{bmatrix} \begin{bmatrix}
    H_n & 0 & \cdots & 0 \\
    H_{n-1} & H_n & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    H_1 & H_2 & \cdots & H_n
\end{bmatrix}^*
\]

(13)

partitioned as,

\[
P = (P_{ij})_{i=1,2,\ldots,n}^{j=1,2,\ldots,n}, \quad P_{ij} = \min(i,j) \sum_{k=1}^{\min(i,j)} H_{n-i+k} H_{n-j+k}^*
\]

Then all \( X(z) \in \mathcal{H}_\infty^{-k}(\partial D) \) such that \( \| \vec{H}(z) + X(z) \|_\infty \leq \gamma \) are generated via,

\[
\{S_{11} + S_{12} U (I - S_{22} U)^{-1} S_{21} : U \in \mathcal{H}_\infty(\partial D), \| U \|_\infty \leq \gamma^{-1}\}
\]

where

\[
S_a = \begin{bmatrix}
    S_{11} & S_{12} \\
    S_{21} & S_{22}
\end{bmatrix} = \begin{bmatrix}
    D_{q11} & \gamma I_p \\
    \gamma I_l & 0
\end{bmatrix} + \begin{bmatrix}
    0 & C_{q12} \\
    0 & C_{q22}
\end{bmatrix} \begin{bmatrix}
    (z + 1) I_{np} - A_{q12} \\
    -A_{q21} - A_{q22}
\end{bmatrix}^{-1} \begin{bmatrix}
    0 & 0 \\
    B_{q21} & B_{q22}
\end{bmatrix}
\]

in which

\[
A_{q12}(i,j) = \gamma^2 \delta_{ij} I_p - P_{ij}, \quad A_{q21}(i,j) = (\delta_{ij} + \delta_{i,j+1}) I_p
\]

\[
A_{q22}(i,j) = P_{ij} + \gamma^2 (\delta_{ij}(1 + 1) + \delta_{i,j}) I_p, \quad D_{q11} = \sum_{i=1}^{n} (-1)^{i+1} H_i
\]

\[
B_{q21}(i) = H_{n-i+1}, \quad B_{q22} = (-1)^{n+1} \gamma \delta_{11} I_p
\]

\[
C_{q12}(j) = \sum_{i=1}^{n} (-1)^{n+1} P_{ij}, \quad C_{q22}(j) = \gamma \sum_{r=1}^{j} (-1)^{j-r} H_{n-r+1}^*
\]

where \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, n \) and \( \delta_{ij} \) denotes the Kronecker delta.
Proof

1. First note that an output balanced state-space realization of $\tilde{H}(z)$ is given by,

$$
A = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
I_p & 0 & 0 & \cdots & 0 & 0 \\
0 & I_p & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_p & 0 \\
\end{bmatrix} \in \mathbb{R}^{np \times np}, \quad B = \begin{bmatrix}
H_n \\
H_{n-1} \\
\vdots \\
H_2 \\
H_1 \\
\end{bmatrix} \in \mathbb{R}^{np \times l}, \quad C^T = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
I_p \\
\end{bmatrix} \in \mathbb{R}^{np \times p}
$$

By considering the discrete-time Lyapunov equation,

$$
P = AP A^T + BB^T
$$

it is also straightforward to verify (13), from which part (1) follows immediately, since $R_2$ is a permutation of $R_1$.

2. From Reference 13, Theorem 4.3 and Reference 14, the generator of all $k$th order Hankel-norm approximations of $\tilde{H}(z)$ may be written in descriptor form as,

$$
S_a = \begin{bmatrix}
CX^{-1}B & \gamma I_p \\
\gamma I_l & 0
\end{bmatrix} + \begin{bmatrix}
0 & CX^{-1}P \\
0 & C_0
\end{bmatrix} \begin{bmatrix}
I_{np} & 0 \\
0 & 0
\end{bmatrix} - \begin{bmatrix}
- I_{np} & \gamma^2 I - P \\
X & A_0
\end{bmatrix}^{-1} \begin{bmatrix}
0 & 0 \\
B & B_0
\end{bmatrix}
$$

where

$$
X = I + A, \quad A_0 = P - \gamma^2 XAX^{-1}, \quad B_0 = \gamma XX^{-1}C^T, \quad C_0 = \gamma B^T X^{-T}
$$

The result follows after a long sequence of straightforward algebraic manipulations. This completes the proof. \qed

5. EXAMPLE

In this section, the results of the paper are illustrated by means of a computer example. We consider the problem,

$$
\inf_{H(z) \in \mathcal{X}_\infty(\mathbb{D})} \| F(z) - H(z) \|_2
$$

subject to the constraint,

$$
\| G(z) - H(z) \|_\infty \leq \gamma_\infty = 2
$$

where,

$$
F(z) = \frac{1}{z^2 - 0.5 - 0.2} \begin{bmatrix}
z - 0.5 & 2z - 1 \\
- z + 0.4 & z + 0.4
\end{bmatrix}, \quad G(z) = \frac{1}{z^2 - 0.9 + 0.18} \begin{bmatrix}
z - 0.6 & 2z - 1.2 \\
- z - 0.3 & - 2z + 0.6
\end{bmatrix}
$$

Since $\| G - F \|_\infty = 5.4087 > 2$, the problem is non-trivial in the sense that the unconstrained optimal approximation is infeasible. The alternating projection algorithm of Section 3 was applied to the problem, using an FFT with $k = 32$ frequency points. The exit condition for the iteration was set as $\| H_k(z) - H_{k-1}(z) \|_2 \leq 10^{-7}$ where $H_k$ results from two consecutive
projections and $k$ is the iteration index. This convergence condition was satisfied after 55 iterations, (taking approximately 35-55 seconds of cpu time) resulting in a matrix FIR filter $H(z) = H_{55}(z)$ with $\|G(z) - H(z)\|_\infty = 2$ (to four significant digits) and $\|F(z) - H(z)\|_2 = 1.29165$.

Next, a matrix IIR approximation of $H(z)$ was obtained using the technique in Theorem 4.1: the first twelve Hankel singular values of $H(z)$ are shown in Table 1.

The approximation bound $\gamma$ (see Theorem 4.1) was chosen as $\gamma = 0.0022$ resulting in an approximation with eight stable poles. A plot of the largest singular value of the approximation error (with the anti-causal component included) is shown in Figure 1, along with $\gamma$; note that this is over-bounded by $\gamma$ at all frequencies (choosing $\gamma = s_9$ results in a singular value which is flat at $s_9$ at all frequencies).

The maximum singular value of the approximation error $H(z) - \hat{H}(z)$ (with the anti-causal component removed) is shown in Figure 2, along with its theoretical upper bound. In this case, the relatively high order of the approximation implies that $H(z)$ and its approximation have almost identical frequency responses.

Finally, Figure 3 shows the two singular values of the system $G(z) - \hat{H}(z)$. Note that the infinity norm bound $\|G(z) - \hat{H}(z)\|_\infty \leq 2$ remains in force. The transfer function $\hat{H}(z)$ was obtained as:

$$\hat{H}(z) = \frac{1}{d(z)} \begin{bmatrix} n_{11}(z) & n_{12}(z) \\ n_{21}(z) & n_{22}(z) \end{bmatrix}$$

where

$$d(z) = z^8 - 2.07527z^7 + 3.10140z^6 - 3.40090z^5 + 2.75725z^4 - 1.69717z^3$$

$$+ 0.70967z^2 - 0.17351z + 0.01875$$

Table 1. The first twelve Hankel singular values of $H(z)$

<table>
<thead>
<tr>
<th>$s_1$ = 3.37591</th>
<th>$s_2$ = 1.16702</th>
<th>$s_3$ = 0.40307</th>
<th>$s_4$ = 0.05626</th>
<th>$s_5$ = 0.01922</th>
<th>$s_6$ = 0.00705</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_7$ = 0.00361</td>
<td>$s_8$ = 0.00279</td>
<td>$s_9$ = 0.00215</td>
<td>$s_{10}$ = 0.00193</td>
<td>$s_{11}$ = 0.00173</td>
<td>$s_{12}$ = 0.00166</td>
</tr>
</tbody>
</table>

Figure 1. Plot of the largest singular value of the approximation error $H(z) - \hat{H}(z)$, including anti-causal component
To compare the efficiency of the algorithm with standard convex optimization methods, the ellipsoid algorithm was applied to the problem. $H(z)$ was restricted to a matrix FIR filter with 16

$$n_{11}(z) = -0.00704z^8 - 0.99205z^7 + 1.66180z^6 - 2.42050z^5 + 2.42052z^4 - 1.78203z^3$$
$$+ 0.98206z^2 - 0.31635z + 0.04691$$

$$n_{12}(z) = 0.00426z^8 - 2.00451z^7 + 3.35861z^6 - 4.86881z^5 + 4.86158z^4 - 3.57416z^3$$
$$+ 1.96624z^2 - 0.63331z + 0.09380$$

$$n_{21}(z) = -0.44224z^8 + 1.38261z^7 - 2.38353z^6 + 2.81894z^5 - 2.70285z^4 + 1.80614z^3$$
$$- 0.92510z^2 + 0.28178z - 0.03247$$

$$n_{22}(z) = 0.24385z^8 + 0.80323z^7 - 1.12653z^6 + 2.06110z^5 - 1.98230z^4 + 1.55883z^3$$
$$- 0.84643z^2 + 0.26907z - 0.03971$$

© 1997 by John Wiley & Sons, Ltd.
Markov parameters, resulting in a constrained optimization problem in 64 variables. Initially, the vector of variables \( \mathbf{x} \) was constrained to lie within the hypersphere \( \| \mathbf{x} \| \leq 10 \). The infinity norm calculations were performed via Hamiltonian decomposition techniques, while the subgradients were computed using standard methods.\(^2\) The algorithm exhibited very slow convergence especially near the optimum. It was finally terminated after \( 2 \times 10^3 \) iterations (corresponding approximately to 20 hours of processor time). At this stage, \( H(z) \) satisfied the constraint \( \| G - H \|_\infty = 2 - 10^{-5} \) approximately and the corresponding value of \( \| F - H \|_2 \) was obtained as \( 1.2940 \). When the ellipsoid algorithm was initialized with the Markov parameters obtained via the convex alternating projection method, no further reduction in cost was obtained.

6. CONCLUSIONS

In this note a numerical solution to the \( \mathcal{H}_2/\mathcal{H}_\infty \) problem has been derived. The method employed is based on the convex alternating projection algorithm and generalizes a technique of Reference 1 to the matrix case. The problem treated here corresponds to the so-called ‘1-block’ case; extensions of the method to general-distance problems is currently under investigation. A new method of approximating the solution by a low degree IIR filter using Hankel-norm approximation methods has finally been developed.

ACKNOWLEDGEMENTS

Financial support under the Engineering and Physical Science Research Council grant GR/J42533 is acknowledged.

REFERENCES