Modelling and $H^\infty$ control of a single-link flexible manipulator

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Abstract: The aim of this study is to investigate motion in the horizontal and the vertical planes of a single-link flexible manipulator. The manipulator is modelled including gravity terms, and it is verified experimentally that the horizontal and the vertical motions are decoupled for a cylindrically symmetrical link and payload. The mathematical model is used to design a mixed-sensitivity $H^\infty$ controller. The sensitivity weighting function is used to obtain the required disturbance rejection properties, including zero sensitivity to a force disturbance in the steady state (i.e. integral action control). The control sensitivity weighting function is chosen to guarantee stability despite variation in the payload mass. The controller is compared experimentally with a proportional-plus-integral controller with velocity feedback and shown to give improved vibration control performance in the vertical plane.

Keywords: flexible manipulator, robust control, $H^\infty$ control, vibration control

NOTATION

$A_y, A_z$ coefficient matrices of state space plant model (the horizontal and the vertical planes respectively)
$B_y, B_z$ driving matrices of state space plant model (the horizontal and the vertical planes respectively)
$C_y, C_z$ output matrices of state space plant model (the horizontal and the vertical planes respectively)
$C_{yi}$ damping coefficient for mode $i$ in the horizontal plane
$E$ Young’s modulus of elasticity for the link material
$F_y(x, t), F_z(x, t)$ forces acting on the link in the horizontal and the vertical planes respectively
$g$ acceleration due to gravity
$g_n$ component of $g$ normal to the longitudinal axis of the link
$G_i(s)$ plant model transfer function
$G_t(s)$ plant model transfer function (the vertical plane, payload $i$)
$H$ $H^\infty$ norm of the cost function
$h$ minimum $H^\infty$ norm of the cost function
$h_{\min}$
$I$ second moment of area for the link cross-section
$j_b$ polar moment of inertia of the link per unit length about the longitudinal axis
$J_{hy}, J_{hz}$ hub moments of inertia in the horizontal and the vertical planes respectively
$J_p$ payload moment of inertia
$J_{ty}$ total inertia of hub, link and payload in the horizontal plane, referred to the hub controller transfer function
$L$ length of the link
$L_g$ position of the hub centre of gravity, as a distance from the hub axis
$m$ mass per unit length of the link
$M_b$ link mass
$M_p$ payload mass
$M_{t}(x, t)$ bending moments in the horizontal and the vertical planes respectively
$Q_{yi}(t), Q_{zi}(t)$ temporal descriptions of motion for mode $i$ in the horizontal and the vertical planes respectively

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Industrial robot manipulators are currently designed with links which are sufficiently stiff for structural deflection to be negligible during normal operation. If this design constraint were not used, so that link flexing was allowed to occur, the following benefits may accrue:

1. **Increased payload capacity.** The ratio of allowable payload weight to robot weight is greater.
2. **Reduced energy consumption.** Lighter links require less powerful actuators.
3. **Cheaper construction.** Lighter robots will require fewer materials and smaller actuators.
4. **Faster movements.** Lighter links and the acceptance of appreciable link deformation will allow higher accelerations.
5. **Longer reach.** A more slender construction allows longer reach where there are access and space restrictions.
6. **Safer operation.** The compliance and low inertia of a flexible manipulator decrease the likelihood that damage will result from physical interaction between the arm and the working environment.

However, link flexibility causes significant technical problems. Most importantly, control algorithms are required which compensate for both the vibration and the static deflection which this flexibility will cause. This is the main research issue and is addressed in this paper.

Initial research on the modelling of flexible links was undertaken by Book [1]. The first significant experimental control results were obtained by Cannon and Schmitz [2]. Through the 1980s, there was a general increase in the scope and quantity of research conducted on both the modelling and the control of flexible manipulator systems. This was partly motivated by the space programme, with projects such as the development of the lightweight Canada arm for the Space Shuttle [3]. This was accompanied by the general desire to control a wider set of flexible structures, particularly lightweight space structures.

A variety of modelling approaches have been applied to flexible manipulators. In a survey of modelling techniques for single-link manipulators the following approaches were identified [4, 5]:

(a) the Lagrange equations and modal expansion (assumed modes method) [6],
(b) the Lagrange equation and finite element method [7],
(c) the Newton–Euler and modal expansion [8],
(d) the Hamilton principle and modal expansion [2] and
(e) the frequency domain and singular perturbation techniques [9].

There are additional hardware requirements for flexible manipulators, as measurement of joint angles alone is no longer sufficient to determine end-effector position.
and orientation. It is usual that measurements of the structural deformation of the link, in terms of the link tip displacement, are also required. However, a number of solutions which do not use the tip position feedback have been tried and these include strain gauges [10] and accelerometers at the link tip [11].

A wide range of control strategies have been proposed for flexible manipulators. Proportional-plus-integral-plus-derivative (PID) controllers have been used by some investigators. However, even though de Luca and Siciliano [12] showed that a simple proportional-derivative controller could asymptotically stabilize robots with flexible links, in most cases the performance is likely to be inadequate.

A linear quadratic Gaussian (LQG) approach was used by Cannon and Schmitz [2], and variants have been implemented by a number of researchers. To address the robustness problems associated with the LQG technique, $H^\infty$ controller designs have been proposed. Lenz et al. [13] developed a mixed-sensitivity $H^\infty$ controller that showed considerable promise. The majority of the work that has been conducted has concentrated on single payload loading conditions. However, Cashmore [14] used an uncertainty model to take into account some variation in payload conditions, thereby utilizing the robustness of the $H^\infty$ controller.

Recent work on the use of $H^\infty$ controllers with flexible manipulators has been published by Matsuno and Tanaka [15], Estiko et al. [16] and Landau et al. [17]. All these workers highlighted the difficulties associated with choosing the weighting functions for the controller design and this was emphasized by Jovik and Lennartson [18] where the choice of weighting functions was considered in detail. Linked to the choice of weighting function is the level of performance achieved from the controller. Estiko et al. [16] proposed the use of a feedforward controller in conjunction with the $H^\infty$ controller to counter the response overshoot experienced in previous studies [19].

In the present study, the aim is to investigate simultaneous motion in the horizontal and the vertical planes, including experimental verification of the results. Very little research has included gravitational effects; thus modelling and control in the vertical plane is of particular interest. In addition, the design of controllers which are robust to payload variations is also addressed. The experimental results are used to verify a mathematical model developed using the Newton–Euler method, and to compare an $H^\infty$ controller with a classical controller, concentrating on the vertical plane.

2 EXPERIMENTAL FLEXIBLE MANIPULATOR

2.1 Overview

An experimental single-link two-degree-of-freedom flexible manipulator has been constructed (Fig. 1). The two degrees of freedom allow the end effector to be moved in the horizontal and vertical planes. Figures 2 and 3 show the main components schematically.

The experimental manipulator can be divided into four main subsystems: the flexible link and payload, the motors and amplifiers, the sensor arrangement, and the computer and interfacing hardware. Table 1 contains the component specifications.

2.2 Flexible link and payload

The flexible link is a homogeneous cylindrical aluminium rod with constant properties along its length. The motion of the link is not artificially constrained in any plane. The link is symmetrical about its longitudinal axis, and this axis intersects both the horizontal and the vertical drive axes at the hub.

The ratio of the diameter to the length of the flexible link is approximately 0.01; this is small enough for the link to be considered slender [20]. This allows for the

Fig. 1 A photograph of the experimental flexible manipulator
a precision potentiometer ($\theta_{\text{flex}}$). This is combined with
the second component, the motion of the hub relative
to the base of the system, measured by an incremental
optical encoder ($\theta_{\text{hub}}$). The potentiometer measuring
$\theta_{\text{flex}}$ is part of a mechanical linkage which adds inertia
and weight to the manipulator and would not be suitable
for implementation as part of an industrial manipulator.
However, this arrangement gives accurate and reliable
results for the experimental set-up.

2.5 Computer and interfacing
A PC is used as the computing platform for controller
implementation, with A/D and encoder expansion cards
for interfacing. Figure 5 shows the interfacing between
the computer and the manipulator.

Controllers are implemented using Simulink®. A con-
troller can be designed in the Matlab environment, speci-
fied in terms of a Simulink block diagram (Fig. 6),
automatically converted to C code using the real-time code
generator and then used to control the manipulator. This
approach allows rapid controller prototyping and easy
comparison of real results with those predicted using
Simulink’s simulation capability. The continuous control-
ners designed in this study are converted to discrete time
using an approximate numerical integration technique.
This conversion is part of the automatic code generation.

3 MODELLING

3.1 Link element model
The Newton–Euler method for modelling transverse
vibrations in beams will be used. The approach broadly
follows that of Meirovich [21], with extensions to ac-
commodate gravitational effects in the vertical plane. A three-
The component of the link tip position

dimensional free-body diagram of a small element, of length $\Delta x$, of the flexible link is shown in Fig. 7.

In Fig. 7, the shear forces are denoted by $V_x$ and $V_y$, the bending moments by $M_x$ and $M_y$, and the torsional moment by $\tau$. Neglecting angular inertia and shear deformation of the element, and neglecting second-order effects, the following link element equation for horizontal motion can be derived:

$$ F_x(x, t) - EI \frac{\partial^4 Y(x, t)}{\partial x^4} = m \frac{\partial^2 Y(x, t)}{\partial t^2} \quad (1) $$

where $Y(x, t)$ is the horizontal displacement of the element in Earth axes. The angular motion of the link can be represented by linear motion of its elements if the angular deviation is small.

There is no distributed force in the horizontal plane; therefore equation (1) becomes

$$ -EI \frac{\partial^4 Y(x, t)}{\partial x^4} = m \frac{\partial^2 Y(x, t)}{\partial t^2} \quad (2) $$

Similarly, in the vertical plane, the link element motion is described by

$$ F_z(x, t) - EI \frac{\partial^4 Z(x, t)}{\partial x^4} = m \frac{\partial^2 Z(x, t)}{\partial t^2} \quad (3) $$

In this case the weight of the link acts as a distributed force; therefore equation (3) becomes

$$ -mg_x - EI \frac{\partial^4 Z(x, t)}{\partial x^4} = m \frac{\partial^2 Z(x, t)}{\partial t^2} \quad (4) $$

where $g_x$ is the gravitational component normal to the axis of the element.

Implicit in equations (2) and (4) is the assumption that the flexural rigidities $EI$ are the same in the horizontal and the vertical planes. Furthermore, it is assumed that the link is cylindrically symmetrical, so that the flexural rigidity remains constant despite link twisting. Twisting can occur owing to the combined horizontal and vertical deflection. This can be appreciated by considering the link element in torsion:

$$ \Delta \tau(x, t) + V_z(x, t) \Delta x \frac{\partial Y(x, t)}{\partial x} - V_y(x, t) \Delta x \frac{\partial Z(x, t)}{\partial x} = j_0 \Delta x \frac{\partial^2 \alpha(x, t)}{\partial t^2} \quad (5) $$

$$ K \frac{\partial^2 \alpha(x, t)}{\partial x^2} + V_z(x, t) \frac{\partial Y(x, t)}{\partial x} - V_y(x, t) \frac{\partial Z(x, t)}{\partial x} = j_0 \frac{\partial^2 \alpha(x, t)}{\partial t^2} $$

Table 1  Experimental manipulator specifications (A/D, analogue to digital; D/A, digital to analogue)

<table>
<thead>
<tr>
<th>Flexible link</th>
<th>Length $L$</th>
<th>1.00 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interchangeable payloads</td>
<td>Mass $M_g$</td>
<td>0.34 kg</td>
</tr>
<tr>
<td>Offset of root from motor axes $R$</td>
<td>0.06 m</td>
<td></td>
</tr>
<tr>
<td>Flexural rigidity $EI$</td>
<td>72.2 N m²</td>
<td></td>
</tr>
<tr>
<td>Hub</td>
<td>Mass $M_g$ and moment of inertia about hub</td>
<td></td>
</tr>
<tr>
<td>Payload 1</td>
<td>0.075 kg, 0.084 kg m²</td>
<td></td>
</tr>
<tr>
<td>Payload 2</td>
<td>0.414 kg, 0.465 kg m²</td>
<td></td>
</tr>
<tr>
<td>Payload 3</td>
<td>0.753 kg, 0.846 kg m²</td>
<td></td>
</tr>
<tr>
<td>Frameless DC torque motors (Kollmorgen Inland QT-6205)</td>
<td>Horizontal inertia $I_{ho}$</td>
<td>0.468 kg m²</td>
</tr>
<tr>
<td>Vertical inertia $I_{vo}$</td>
<td>0.249 kg m²</td>
<td></td>
</tr>
<tr>
<td>Incremental rotary encoders (Heidenhain ERO 1325)</td>
<td>Maximum torque</td>
<td>33 N m</td>
</tr>
<tr>
<td>Potentiometers (Novotechnik P2701A502)</td>
<td>Rotor inertia</td>
<td>0.013 kg m²</td>
</tr>
<tr>
<td>AD interface (Advantech PCL-814B/816-DA-1)</td>
<td>Electrical time constant</td>
<td>2.41 ms</td>
</tr>
<tr>
<td>Encoder interface card (Advantech PCL-833)</td>
<td>Torque constant</td>
<td>1.2 N m/A</td>
</tr>
<tr>
<td>Personal computer (PC) Intel® Pentium processor</td>
<td>Ratio of motor torque to amplifier volts</td>
<td>3.18 N m/V</td>
</tr>
</tbody>
</table>

Fig. 4 The components of the link tip position.

$\theta_{ma}$

$\theta_{na}$

$R$

$L$

$F_x$

$F_z$

$V_x$

$V_y$

$V_z$

$M_x$

$M_y$

$M_z$

$\tau$
where $\alpha$ is the link twist, $J_b$ is the polar moment of inertia about the longitudinal axis per unit length and $K$ is the torsional stiffness (the product of the modulus of rigidity and the polar moment of area). Thus a combination of horizontal and vertical deflections of the link can generate element twisting. An example scenario is horizontal motion of the link which has a static vertical deflection due to gravity.
3.2 Boundary conditions

Four boundary conditions can be derived for each direction of motion (horizontal and vertical). Considering horizontal motion, there is a geometric boundary condition at the root of the link:

\[ Y(x, t)|_{x=0} = R \frac{\partial Y(x, t)}{\partial x} \bigg|_{x=0} \]  

(6)

Taking moments about the hub axis gives a second boundary condition at the root of the link:

\[ T_y(t) - REI \frac{\partial^3 Y(x, t)}{\partial x^3} \bigg|_{x=0} + EI \frac{\partial^2 Y(x, t)}{\partial x^2} \bigg|_{x=0} = J_{hy} \frac{\partial^2 Y(x, t)}{\partial t^2} \bigg|_{x=0} \]  

(7)

where \( T_y \) is the motor torque and \( J_{hy} \) is the hub inertia in the horizontal plane.

Two boundary conditions are associated with the payload. The centre of gravity of the payload lies on the link axis at \( x = L \). Resolving forces acting on the payload horizontally gives

\[ EI \frac{\partial^3 Y(x, t)}{\partial x^3} \bigg|_{x=L} = M_p \frac{\partial^2 Y(x, t)}{\partial t^2} \bigg|_{x=L} \]  

(8)

where \( M_p \) is the mass of the payload. Finally, considering the moments acting about the centre of gravity of the payload gives

\[ -EI \frac{\partial^3 Y(x, t)}{\partial x^3} \bigg|_{x=L} = J_p \frac{\partial^2 Y(x, t)}{\partial t^2} \bigg|_{x=L} \]  

(9)

where \( J_p \) is the moment of inertia of the payload. Note that the payload is assumed to be cylindrically symmetrical about the longitudinal axis of the link, so that the link twist will not alter \( J_p \). Thus the payload moments of inertia are the same for horizontal and vertical motions.

The boundary conditions for vertical motion have the same form, except for the inclusion of payload weight, and a motor torque \( T_z \) required to support any out-of-balance hub weight:

\[ Z(x, t)|_{x=0} = R \frac{\partial Z(x, t)}{\partial x} \bigg|_{x=0} \]  

(10)

\[ T_z(t) - REI \frac{\partial^3 Z(x, t)}{\partial x^3} \bigg|_{x=0} + EI \frac{\partial^2 Z(x, t)}{\partial x^2} \bigg|_{x=0} = J_{hz} \frac{\partial^2 Z(x, t)}{\partial t^2} \bigg|_{x=0} \]  

(11)

\[ EI \frac{\partial^3 Z(x, t)}{\partial x^3} \bigg|_{x=L} - M_p g = M_p \frac{\partial^2 Z(x, t)}{\partial t^2} \bigg|_{x=L} \]  

(12)

\[ -EI \frac{\partial^3 Z(x, t)}{\partial x^3} \bigg|_{x=L} = J_p \frac{\partial^2 Z(x, t)}{\partial t^2} \bigg|_{x=L} \]  

(13)
3.3 Mode shapes

A set of mode shapes will be determined by considering the free response of the system. These will then be used to develop the solution to the forced vibration problem. In the horizontal plane, the free response is defined by the element equation (2) and the boundary conditions (6) to (9), with the motor torque set to zero in equation (7). A solution to these equations is assumed to have the form

\[ Y(x, t) = S_y(x)Q_y(t) \]  

(14)

Substituting this solution into equation (2) allows the variables to be separated:

\[
\frac{1}{Q_y(t)} \frac{d^2 Q_y(t)}{dt^2} = -\frac{EI}{m} \frac{1}{S_y(x)} \frac{d^4 S_y(x)}{dx^4}
\]  

(15)

For the two sides of the equation (15) to be equal for all values of \( x \) and \( t \), they must be constant; let this constant be \(-\omega_r^2\), where \( \omega_r \) is real. Hence

\[
\frac{d^2 Q_y(t)}{dt^2} + \omega_r^2 Q_y(t) = 0
\]  

(16)

\[
\frac{d^4 S_y(x)}{dx^4} - \frac{\omega_r^2 m}{EI} S_y(x) = 0
\]  

(17)

Equation (16) represents a second-order system with zero damping and natural frequency \( \omega_r \). Equation (17) describes the amplitude variation along the link, i.e. the mode shape. The boundary conditions for this differential equation are derived by substituting equation (14) into equations (6) to (9):

\[
S_y(x)|_{x=0} = R \frac{dS_y(x)}{dx} \bigg|_{x=0}
\]  

(18)

\[
REI \frac{d^3 S_y(x)}{dx^3} \bigg|_{x=0} + EI \frac{d^3 S_y(x)}{dx^3} \bigg|_{x=0} = -J_h \omega_r^2 \frac{dS_y(x)}{dx} \bigg|_{x=0}
\]  

(19)

\[
EI \frac{d^3 S_y(x)}{dx^3} \bigg|_{x=L} = -M_p \omega_r^2 S_y(x)|_{x=L}
\]  

(20)

\[
EI \frac{d^3 S_y(x)}{dx^3} \bigg|_{x=L} = J_r \omega_r^2 \frac{dS_y(x)}{dx} \bigg|_{x=L}
\]  

(21)

Solving equation (17) with these boundary conditions gives a set of mode shapes of arbitrary relative amplitude. For the experimental manipulator of Section 2, with payload 2 (\( M_p = 0.414 \) kg, see Table 1), the mode shapes are as shown in Fig. 8. The amplitudes of these modes have been set by normalization in line with the orthogonality condition derived in Appendix 1.

In the vertical plane, the free response without motor torque or gravitational forcing terms gives a completely analogous set of equations. The mode shapes are only different because of a difference between the vertical and the horizontal hub inertias.

3.4 Horizontal plane model

The general solution of equation (2) with boundary conditions (6) to (9) is a sum of modal terms:

![Fig. 8 The mode shapes (\( i = 1, \ldots, 6 \)) for the horizontal plane](image_url)
Thus the time responses for the individual modes are

\[ Y(x, t) = \sum_{i=0}^{\infty} S_i(x)Q_i(t) \]  

(22)

Substituting equation (22) into equation (2) gives

\[ \sum_{i=0}^{\infty} \left[ mS_i(x) \frac{d^2Q_i(t)}{dt^2} + EI \frac{d^4S_i(x)}{dx^4} Q_i(t) \right] = 0 \]  

(23)

and substituting equation (17) into equation (23) gives

\[ \sum_{i=0}^{\infty} \left[ \frac{d^2Q_i(t)}{dt^2} + \omega_i^2Q_i(t) \right] mS_i(x) = 0 \]  

(24)

\[ \sum_{i=0}^{\infty} \left[ \frac{d^2Q_i(t)}{dt^2} + \omega_i^2Q_i(t) \right] \int_0^L mS_i(x)S_j(x) \, dx = 0 \]  

(25)

Thus the time responses for the individual modes are independent. It is expected that in general, the higher the frequency, the less significant is the mode’s contribution to the overall response. Hence a finite number of the lower frequency modes, \( i = 0 \) to \( n \), will provide a good approximation. So equation (26) can be used to construct a state space model of finite dimension:

\[ \dot{x}_r = A_r x_r + B_r u_r \]  

(27)

where

\[ x_r = [Q_{r0} \ Q_{r0} \ Q_{r1} \ Q_{r1} \ \cdots \ Q_{rn} \ Q_{rn}]^T \]  

(28)

\[ A_r = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\omega_{r0}^2 & C_{r0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -\omega_{r1}^2 & C_{r1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\omega_{rn}^2 & C_{rn} \end{bmatrix} \]  

(29)

\[ B_r = \begin{bmatrix} \frac{S_{r0}(0)}{R} & 0 & \frac{S_{r1}(0)}{R} & \cdots & 0 & \frac{S_{rn}(0)}{R} \end{bmatrix}^T \]  

(30)

\[ u_r = T_r(t) \]  

(31)

and where \( C_{ri} \) are damping coefficients included to represent the low levels of structural and mechanical damping which will be present.

The following output equation defines two system outputs: the total angular deflection of the link tip and the hub angle respectively:

\[ y_r = C_r x_r \]  

(32)

\[ y_r = [\theta_{rot} \ \theta_{hub}]^T \]

\[ C_r = \begin{bmatrix} \frac{S_{r0}(0)}{R} & 0 & \frac{S_{r1}(0)}{R} & 0 & \cdots & \frac{S_{rn}(0)}{R} & 0 \end{bmatrix} \]  

(33)

Mode 0 can be interpreted as a rigid mode, i.e. the following solution to equations (17) to (21) obtained with \( \omega_{r0} = 0 \):

\[ S_{r0}(x) = \frac{1}{\sqrt{J_{Ty}}} (x + R) \]  

(34)

The ‘amplitude’ of the mode is determined by normalizing using the orthogonality condition; substituting equation (34) into equation (70) and putting \( i = j \) give

\[ J_{Ty} = \int_0^L m(x + R)^2 \, dx + J_{Hy} + M_p(L + R)^2 + J_p \]  

(35)

Thus \( J_{Ty} \) is the total inertia of the hub, link and payload in the horizontal plane referred to the hub. Note that the mode 0 terms in the \( B_r \) and \( C_r \) matrices can be simplified, as from equation (34)

\[ \frac{S_{r0}(0)}{R} = \frac{S_{r0}(0)}{R} = \frac{1}{\sqrt{J_{Ty}}} \]  

(36)

Thus, from the state space model, the rigid mode contribution to the angular deflection, \( \theta_{r0} \), can be found to be

\[ \frac{d^2\theta_{r0}}{dt^2} + C_{r0} \frac{d\theta_{r0}}{dt} = \frac{1}{J_{Ty}} T_r(t) \]  

(37)

Although the model has been developed using small-angle approximations, it can be seen that these rigid mode dynamics are valid for large movements. Hence the state space model is applicable throughout the manipulator work space.

### 3.5 Vertical plane model

The general solution of equation (4) with boundary conditions (10) to (13) is a sum of modal terms:

\[ Z(x, t) = \sum_{i=0}^{\infty} S_{zi}(x)Q_{zi}(t) \]  

(38)

Substituting equation (38) into equation (4) gives

\[ \sum_{i=0}^{\infty} \left[ mS_{zi}(x) \frac{d^2Q_{zi}(t)}{dt^2} + EI \frac{d^4S_{zi}(x)}{dx^4} Q_{zi}(t) \right] = -mg_n \]  

(39)
The following approximation will be used:

\[ u \text{ depends on the relative orientation of the gravity vector.} \]

The angle \( h \) is the payload weight which is by far the most significant. Even if this is not the case, the approximation will be good if the flexible displacement is small compared with the total range of manipulator movement.

If the centre of gravity of the hub is at an angle \( \theta_g \) above the hub angle and displaced \( L_g \) from the hub axis, then

\[ T_g = M_h g L_g \cos(\theta_{\text{hub}} + \theta_g) \quad (44) \]

Equation (42) can be expressed in the state space form entirely analogously to the horizontal plane model. Only the input terms are different. The \( B \) matrix now has extra columns to account for the effect of gravity acting on the payload:

\[
B_z = \begin{bmatrix}
0 & 0 & 0 \\
\frac{S_{z0}(t)}{R} & -M_h g L_g & \frac{S_{z0}(0)}{R} & -S_{z0}(L) M_p g & - \int_0^L mg S_{z0}(x) \, dx \\
0 & 0 & 0 \\
\frac{S_{z1}(t)}{R} & -M_h g L_g & \frac{S_{z1}(0)}{R} & -S_{z1}(L) M_p g & - \int_0^L mg S_{z1}(x) \, dx \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
\frac{S_{zn}(t)}{R} & -M_h g L_g & \frac{S_{zn}(0)}{R} & -S_{zn}(L) M_p g & - \int_0^L mg S_{zn}(x) \, dx
\end{bmatrix}
\]

The input vector is now

\[ u_z = \begin{bmatrix}
T_z(t) \\
\cos(\theta_{\text{hub}} + \theta_g) \\
\cos(\theta_{\text{tot}})
\end{bmatrix} \quad (46) \]

To illustrate the result of this modelling process, Appendix 2 contains models for the vertical plane motion for each of the three payloads. Note that three modes \((n = 0, 1, 2)\) are included in each case. The magnitude of the frequency response of the tip angle \( \theta_{\text{tot}} \) to the motor torque for payload 2 is shown in Fig. 9. The rapid roll-off with frequency indicates that the inclusion of two flexible modes should be sufficient. The other models exhibit similar trends in this respect.

4 MODEL VERIFICATION

Simulation and actual responses can be compared to verify that the models obtained in the previous section are accurate. As the open-loop plant is only marginally stable, a comparison of open-loop responses is im-

![Graph showing frequency response](Image)
practical. Hence closed-loop control is required, and a proportional-plus-integral-plus-velocity (PIV) feedback controller is used for this purpose. A comparison between simulated and experimental link tip position responses in the horizontal and vertical planes is shown in Figs 10 to 13. In each case three modes are simulated and payload 2 is used.

A good correlation between the simulated and experimental responses is seen. It is found that including more modes in the model gives no improvement in the simulated response [23], which further strengthens the argument that a three-mode model will be adequate for controller design. In the horizontal plane, it can be seen that the actual link tip comes to rest after the first over-

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![Fig. 10](image1.png)  
**Fig. 10** A comparison of the real and simulated step responses in the horizontal plane

![Fig. 11](image2.png)  
**Fig. 11** A comparison of the real and simulated step response control signals in the horizontal plane
shoot, whereas oscillation continues in the simulated response. This is a result of Coulomb friction, which is not included in the model. Coulomb friction also affects vertical plane motion, but because of the more oscillatory nature of the transient response its influence is less pronounced. However, a friction effect can be seen in Fig. 13 between 15 and 20 s, where the simulated steady state control signal is lower than at (say) 5–10 s because the change in angle reduces the gravity related torque, but the experimental control signal has arrested the motion when at a higher value.

It is an important observation that Coulomb friction is significant despite using direct drive torque motors for actuation in order to minimize friction. The use of low-backlash geared motors would be a more economical drive solution but would further increase any friction-related difficulties.

5 CONTROL

5.1 Gravity compensation

In the vertical plane model, the gravity-related terms in equations (45) and (46) will be neglected for the pur-
poses of linear controller design. Instead, an additional component will be added to the control signal to provide a torque which partially supports the static weight of the manipulator. This component is proportional to the vertical hub angle. The response of the PIV controller with and without this gravity compensation is seen in Fig. 14. The integral action in the PIV controller reduces the steady state error to zero much more rapidly when the weight compensation is present.

5.2 Mixed-sensitivity $H^\infty$ controller design

The PIV controller has been designed, by trial and error, to give the best achievable compromise between the speed of response, the settling time and the steady state error. However, it is apparent that the response, particularly in the vertical plane, is far from acceptable. Thus an $H^\infty$ controller has also been designed for this manipulator, with the objective of improving the vibration control performance while in addition guaranteeing stability despite the payload variation. The results are presented for the vertical plane.

A mixed-sensitivity $H^\infty$ controller is used. The structure of the controller is shown in Fig. 15. The cost function consists of the weighted sensitivity function, the weighted control sensitivity function and the weighted complementary sensitivity function. These three sensitivity functions are defined as follows:

Sensitivity function:

$$S(s) = [1 + K(s)G(s)]^{-1} \quad (47)$$

Control sensitivity function:

$$R(s) = K(s)[1 + K(s)G(s)]^{-1} \quad (48)$$

Complementary sensitivity function:

$$T(s) = K(s)G(s)[1 + K(s)G(s)]^{-1} \quad (49)$$

Thus the mixed-sensitivity $H^\infty$ controller is the controller $K(s)$ which stabilizes the system and minimizes $h$, where $h$ is the $H^\infty$ norm of the cost function:

$$h = \left\| \begin{bmatrix} W_1(j\omega)S(j\omega) \\ W_2(j\omega)R(j\omega) \\ W_3(j\omega)T(j\omega) \end{bmatrix} \right\|_{\infty} \quad (50)$$

and where $W_1(s)$, $W_2(s)$ and $W_3(s)$ are weighting functions. Suitable choices for the weighting functions in this case are

$$W_1(s) = \frac{4s + 1}{s^3(s + 4)} \quad (51)$$

$$W_2(s) = \frac{5 \times 10^{-4}(s + 20)}{s + 1} \quad (52)$$

$$W_3(s) = \frac{2s + 1}{s + 700} \quad (53)$$

The magnitudes of these weighting functions are plotted against frequency in Fig. 16. The sensitivity weighting function $W_1(s)$ is chosen to be high at low frequencies to give good low-frequency disturbance rejection. In fact by including two poles at $s = 0$ this forces the sensitivity function $S(s)$ to have two zeros at $s = 0$; the plant has one pole at $s = 0$ which appears as a zero in equa-

![Fig. 14 A comparison of the tip position response with and without gravity compensation](image)
Fig. 15 The controller structure

which leads to plant model $G_3(s)$, then the additive modelling errors when the other payloads are used are

$$\Delta_1(j\omega) = G_1(j\omega) - G_3(j\omega)$$

$$\Delta_2(j\omega) = G_2(j\omega) - G_3(j\omega)$$

The magnitudes of these additive modelling errors are plotted in Fig. 17, and it is shown that the control sensitivity weighting function has been chosen to over-bound these errors.

With the specified weighting functions, and the payload 3 model $G_3(s)$ shown in Appendix 2, the minimum $H_2$ norm [equation (50)] can be calculated to be $h_{\text{min}} = 0.96$. Hence the inequality (54) is satisfied, which guarantees robustness to the payload variation for which the system has been designed.

5.3 Experimental $H_\infty$ controller results for the vertical plane motion

Figure 18 presents the step response results with the actual plant being the same as the nominal. The results show that the system has a fast response and a small settling time. The performance of the controller is significantly better than that achieved using the PIV controller.
The presence of the integral action can be seen in the torque trace shown in Fig. 19. The torque increases to the level necessary to support the arm with zero steady state error at the tip.

The results presented in Figs 20 and 21 are the tip displacement and torque plots for the step response of the flexible arm across the range of payloads (1, 2 and 3). Although the $H^\infty$ controller exhibits stability robustness across this range as expected, there is distinct variation in the achieved level of performance.

### 5.4 Combined horizontal and vertical motion

The motions of the arm in the horizontal and the vertical planes should not interact if the modelling assumptions...
are accurate. The results presented in this section use the same mixed-sensitivity $H^\infty$ control approach with integral action that is described in Sections 5.2 and 5.3. Figures 22 and 23 show the step response of the arm in the horizontal and the vertical planes simultaneously. The reference signals for the two planes have been offset to ensure that any coupling effects are easily visible. An individual mixed-sensitivity $H^\infty$ controller with integral action is implemented for each plane. The coupling effect between the two planes of motion is very small, with only minor disturbances being visible in the vertical displacement trace caused by the step change in the horizontal plane. This is thought to be exaggerated at 25 s owing to stiction; the vertical displacement 'jumps' from one value to another due to a stick–slip phenomenon.
6 CONCLUSIONS

As shown in this work, the degree of sophistication required in the controller for a flexible manipulator is dramatically greater than its rigid counterpart. In turn, the modelling required to support the controller design is much more involved. Techniques which will successfully control general multiple-degree-of-freedom manipulators with flexible links are not yet available.

As part of the development towards a control scheme for general flexible manipulators, an experimental manipulator has been constructed, with a single link which can be driven in the horizontal and the vertical planes. The effect of gravity, almost always excluded from other studies, is included in the modelling of this system. A close correlation between the response of the model and that of the experimental system is found. Small discrepancies are attributable to friction, despite minimizing friction in the design of the manipulator by using direct drive torque motors for actuation.
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REFERENCES


APPENDIX I

Mode orthogonality

The orthogonality properties of the free vibration modes are required. Let $\phi_i(x)$, or simply $\phi_i$, represent a mode
shape, either $S_{zi}(x)$ (the horizontal plane) or $S_{zi}(x)$ (the vertical plane). If $\phi_i$ and $\phi_j$ are two modes with the associated frequencies $\omega_i$ and $\omega_j$, then commensurate with equation (17)

$$E\ell \phi''_i = \omega_i^2 m \phi_i$$

(58)

and

$$E\ell \phi''_j = \omega_j^2 m \phi_j$$

(59)

Multiplying equation (58) by $\phi_j$ and equation (59) by $\phi_i$ gives

$$\phi_j \phi''_j = \frac{\omega_j^2 m}{EI} \phi_i \phi_j$$

(60)

$$\phi_i \phi''_j = \frac{\omega_i^2 m}{EI} \phi_i \phi_j$$

(61)

Integrating equations (60) and (61) by parts gives the expressions

$$\frac{\omega_i^2 m}{EI} \int_0^L \phi_j \phi_i \, dx = \left[ \phi_j \phi_i'' - \phi_i \phi_j'' \right]_0^L + \int_0^L \phi_j' \phi_i' \, dx$$

(62)

and

$$\frac{\omega_j^2 m}{EI} \int_0^L \phi_i \phi_j \, dx = \left[ \phi_i \phi_j'' - \phi_j \phi_i'' \right]_0^L + \int_0^L \phi_i' \phi_j' \, dx$$

(63)

Subtracting equation (62) from equation (63) gives

$$\left( \omega_i^2 - \omega_j^2 \right) \frac{m}{EI} \int_0^L \phi_j \phi_i \, dx = \left[ \phi_j \phi_i'' - \phi_i \phi_j'' \right]_0^L - \left[ \phi_i \phi_j'' - \phi_j \phi_i'' \right]_0^L$$

(64)

Restating the boundary conditions (18) to (21),

$$\phi_i(0) = 0$$

(65)

$$-REI \phi''_i(0) + EI \phi''_i(0) = -J_h \omega_i^2 \phi_i(0)$$

(66)

$$EI \phi_i(L) = 0$$

(67)

$$EI \phi''_i(L) = J_p \omega_i^2 \phi_i(L)$$

(68)

Substituting in equation (64) for $\phi_i''(0)$ and $\phi_j''(0)$ using equations (66) and (65), and for $\phi_i''(L)$ and $\phi_j''(L)$ using equations (67) and (68), and similarly for $\phi_j$,

$$\left( \omega_i^2 - \omega_j^2 \right) \frac{m}{EI} \int_0^L \phi_j \phi_i \, dx = \left[ \phi_j(0) \phi_i''(0) + M_p \phi_i(L) \phi_j(L) \right.$$

$$\left. + J_p \phi_i(L) \phi_j(L) \right]$$

(69)

from which the orthogonality condition follows:

$$m \int_0^L \phi_i(x) \phi_j(x) \, dx + J_h \phi_i'(0) \phi_j'(0) + M_p \phi_i(L) \phi_j(L) + J_p \phi_i(L) \phi_j(L) = \delta_{ij}$$

(70)

where $\delta_{ij}$ is the Kronecker delta given by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

(71)

When $i = j$, the indeterminate condition in equation (69) allows the arbitrary choice in equation (70) of the left-hand side equalling unity. This choice normalizes, i.e. fixes the amplitude of the modes.

To prove separability of the mode time responses in equations (25) and (41), the orthogonality condition can be used, together with additional expressions for the mass and inertia terms. These expressions are derived below. Considering the vertical plane as the more general case, and substituting equation (38) into the boundary conditions (11) to (13),

$$T_z - T_g + \sum_{i=0}^{\infty} \left[ -REI \phi''_i(0) + EI \phi''_i(0) \right] Q_{zi}$$

(72)

$$\sum_{i=0}^{\infty} EI \phi''_i(L) Q_{zi} - M_p g_n = \sum_{i=0}^{\infty} M_p \phi_i(L) \tilde{Q}_{zi}$$

(73)

$$\sum_{i=0}^{\infty} -EI \phi''_i(L) Q_{zi} = \sum_{i=0}^{\infty} J_p \phi_i(L) \tilde{Q}_{zi}$$

(74)

Then substituting equation (66) into equation (72) gives

$$\sum_{i=0}^{\infty} J_h \phi_i'(0) \phi_j'(0) (\tilde{Q}_{zi} + \omega_i^2 Q_{zi}) = (T_z - T_g) \phi_j(0)$$

(75)

and substituting equation (67) into equation (73) gives

$$\sum_{i=0}^{\infty} M_p \phi_i(L) \phi_j(L) (\tilde{Q}_{zi} + \omega_i^2 Q_{zi}) = -M_p g_n \phi_j(L)$$

(76)

and finally substituting equation (68) into equation (74) gives

$$\sum_{i=0}^{\infty} J_p \phi_i(L) \phi_j(L) (\tilde{Q}_{zi} + \omega_i^2 Q_{zi}) = 0$$

(77)

The equations for the horizontal plane equivalent to equations (75) to (77) are of the same form but with the gravity-related terms set to zero:

$$\sum_{i=0}^{\infty} J_h \phi_i'(0) \phi_j'(0) (\tilde{Q}_{zi} + \omega_i^2 Q_{zi}) = T_y \phi_j(0)$$

(78)

$$\sum_{i=0}^{\infty} M_p \phi_i(L) \phi_j(L) (\tilde{Q}_{zi} + \omega_i^2 Q_{zi}) = 0$$

(79)

$$\sum_{i=0}^{\infty} J_p \phi_i(L) \phi_j(L) (\tilde{Q}_{zi} + \omega_i^2 Q_{zi}) = 0$$

(80)
APPENDIX 2

Plant models for the vertical plane

Model with payload 1

\[
A_{z1} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -3.4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -2530 & -0.9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -70700 & -0.9 \\
\end{bmatrix}
\]

\[
B_{z1} = \begin{bmatrix}
0 & 0 & 0 \\
1.37 & -5.47 & -1.06 \\
-1.33 & 5.30 & -1.47 \\
-0.320 & 1.27 & 1.11 \\
\end{bmatrix}
\]

\[
C_{z1} = \begin{bmatrix}
1.37 & 0 & 1.886 & 0 & -1.429 & 0 \\
1.37 & 0 & -1.305 & 0 & -0.311 & 0 \\
\end{bmatrix}
\]

Model with payload 2

\[
A_{z2} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -2.1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1440 & -0.9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -56800 & -0.9 \\
\end{bmatrix}
\]

\[
B_{z2} = \begin{bmatrix}
0 & 0 & 0 \\
1.07 & -4.27 & -4.61 \\
-1.64 & 6.55 & -3.40 \\
-0.357 & 1.42 & 1.73 \\
\end{bmatrix}
\]

\[
C_{z2} = \begin{bmatrix}
1.07 & 0 & 0.790 & 0 & -0.401 & 0 \\
1.07 & 0 & -1.657 & 0 & -0.370 & 0 \\
\end{bmatrix}
\]

Model with payload 3 (nominal plant)

\[
A_{z3} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1240 & -0.9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -54800 & -0.9 \\
\end{bmatrix}
\]

\[
B_{z3} = \begin{bmatrix}
0 & 0 & 0 \\
0.908 & -3.62 & -7.11 \\
-1.75 & 6.97 & -3.96 \\
-0.364 & 1.45 & 1.82 \\
\end{bmatrix}
\]

\[
C_{z3} = \begin{bmatrix}
0.908 & 0 & 0.506 & 0 & -0.232 & 0 \\
0.908 & 0 & -1.790 & 0 & -0.366 & 0 \\
\end{bmatrix}
\]

Note that the transfer function plant models relating motor torque to link tip position can be obtained thus:

\[
G_1(s) = C_{z1} (sI - A_{z1})^{-1} B_{z1}
\]

\[
G_2(s) = C_{z2} (sI - A_{z2})^{-1} B_{z2}
\]

\[
G_3(s) = C_{z3} (sI - A_{z3})^{-1} B_{z3}
\]

where \( B_{z1} \) and \( C_{z1} \) are the first column and first row of \( B_{z1} \) and \( C_{z1} \) respectively.