Inverse Laplace transforms via residue theory

The Laplace transform:

For a function (signal) $f(t)$ which is zero for $t < 0$, the Laplace transform is

$$ F(s) = \int_0^\infty f(t)e^{-st}dt $$

Here we use $s = \sigma + j\omega$ in place of $z = x + jy$, so we have

$$ F(s) = \int_0^\infty (f(t)e^{-\sigma t})e^{-j\omega t}dt $$

**Problem:** Given $F(s)$ how do we obtain $f(t)$?

The Fourier transform:

The Fourier transform of $f(t)$ is

$$ \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt $$

and the inverse transform is

$$ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{j\omega t}d\omega $$

In the Laplace transform let,

$$ \phi(t) = f(t)e^{-\sigma t}, \quad t \geq 0 $$

$$ = 0, \quad t < 0 $$

where $\sigma$ is a constant. Taking the Fourier transform of $\phi(t)$:

$$ \hat{\phi}(\omega) = \int_0^{\infty} \phi(t)e^{-j\omega t}dt = \int_0^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt = F(\sigma + j\omega) $$

So, taking the inverse Fourier transform:

$$ \phi(t) = \int_{\infty}^{\infty} F(\sigma + j\omega)e^{j\omega t}d\omega $$

$$ = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\sigma + j\omega)e^{j\omega t}e^{\sigma t}d\omega $$

Hence,

$$ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\sigma + j\omega)e^{j\omega t}e^{\sigma t}d\omega $$

Let $s = \sigma + j\omega, \; ds = j\omega \, dw$, then

$$ f(t) = \frac{1}{2\pi} \int_{s=\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds, \quad t \geq 0 $$

$$ = 0, \quad t < 0 $$
The inverse Laplace transform

The formula for the inverse Laplace transform was obtained in the previous section as:

\[ f(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \]

The relevant questions here are:

1. How do we choose the real parameter \( \sigma \)?
2. How do we evaluate the integral?

We already know that \( f(t) = 0, t < 0 \), and we shall see that this gives us an answer to (1).

![Integration contours](image)

Figure 1: Integration contours

First, consider the closed contour \( C_1 = C_{R_1} + L \) shown in Figure 1a. Then,

\[
\oint_{C_1} F(s)e^{st}ds = \int_L F(s)e^{st}ds + \int_{C_{R_1}} F(s)e^{st}ds
\]

\[
= \int_{\sigma-jR}^{\sigma+jR} F(s)e^{st}ds + \int_{C_{R_1}} F(s)e^{st}ds
\]

\[
= 2\pi j \sum \text{Res}[F(s)e^{st}]_{\text{poles in } C_1}
\]

Note that the residues are at the poles inside \( C_1 \). Now, as \( R \to \infty \), \( \int_{\sigma-jR}^{\sigma+jR} F(s)e^{st}ds \) becomes the integral we require, and we can show (Jordan’s lemma, see appendix) that for \( t > 0 \), \( \int_{C_{R_1}} F(s)e^{st}ds \to 0 \)
as $R \to \infty$. So, for $t > 0$,

$$\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = \sum_{\text{poles left of } \sigma} \text{Res}[F(s)e^{st}]$$

Next consider the contour $C_2 = C_{R_2} + L$ shown in Figure 1b. As before,

$$\int_{C_2} F(s)e^{st}ds = \int_{L} F(s)e^{st}ds + \int_{C_{R_2}} F(s)e^{st}ds$$

$$= \int_{\sigma-jR}^{\sigma+jR} F(s)e^{st}ds + \int_{C_{R_2}} F(s)e^{st}ds$$

$$= -2\pi j \sum_{\text{poles in } C_2} \text{Res}[F(s)e^{st}]$$

Note that the residues are at the poles inside $C_2$ and the minus sign is due to the fact that we are dealing with a clockwise contour. Again, using Jordan’s lemma, we have that for $t < 0$, $\int_{C_{R_2}} F(s)e^{st}ds \to 0$ as $R \to \infty$. So, for $t < 0$,

$$\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = -\sum_{\text{poles right of } \sigma} \text{Res}[F(s)e^{st}]$$

But we know that this must be zero, since $f(t) = 0$, $t < 0$. Hence $\sigma$ must be chosen such that $C_2$ does not contain any poles of $F(s)e^{st}$ (as $R \to \infty$), and thus $C_1$ must contain all poles of $F(s)e^{st}$. This is the answer to question (1). Note, finally, that since $e^{st}$ is analytic everywhere (i.e. has no poles), the poles of $F(s)e^{st}$ are the same as the poles of $F(s)$. This answers question (2) and we have

$$f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = \sum_{\text{all poles of } F(s)} \text{Res}[F(s)e^{st}]$$

**Example 1**

Find the inverse transform of $F(s) = \frac{1}{s+a}$.

The function $\frac{e^{st}}{s+a}$ has a simple pole at $s = -a$. Hence

$$\text{Res}_{s=-a}[F(s)e^{st}] = \lim_{s \to -a} \left[(s + a) \frac{e^{st}}{s + a}\right] = e^{-at}$$

and so,

$$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}, \quad t \geq 0$$

$$= 0, \quad t < 0$$

**Example 2**

Find the inverse transform of $F(s) = \frac{1}{(s+a)^2}$.

In this case the function $\frac{e^{st}}{(s+a)^2}$ has a pole of order 2 at $s = -a$. Remember that if a function $f(z)$ has a pole of order $n$, then its residue at this pole is given by

$$c_1 = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}}[(z - a)^n f(z)]$$
In our case the required residue is:

\[
\frac{1}{(2-1)!} \lim_{s \to -a} \frac{d}{ds} \left[ \left( s + a \right)^2 \frac{e^{st}}{(s+a)^2} \right] = \lim_{s \to -a} \left[ te^{st} \right] = te^{-at}
\]

and so,

\[
L^{-1} \left[ \frac{1}{(s+a)^2} \right] = te^{-at}, \quad t \geq 0
\]

\[
= 0, \quad t < 0
\]

**Example 3**

Find the inverse transform of \( F(s) = \frac{1}{(s+a)^2(s+b)} \).

In this case the function \( \frac{e^{st}}{(s+a)^2(s+b)} \) has a pole of order 2 at \( s = -a \) and a simple pole at \( s = -b \). The residue at \( s = -b \) is

\[
\lim_{s \to -b} \left[ \frac{e^{st}}{(s+a)^2} \right] = \frac{e^{-bt}}{(a-b)^2}
\]

The residue at \( s = -a \) is

\[
\frac{1}{(2-1)!} \lim_{s \to -a} \frac{d}{ds} \left[ \frac{e^{st}}{s+b} \right] = \lim_{s \to -a} \left[ \frac{te^{st}}{s+b} - \frac{e^{st}}{(s+b)^2} \right] = \frac{te^{-at}}{b-a} - \frac{e^{-at}}{(b-a)^2}
\]

and so

\[
L^{-1} \left[ \frac{1}{(s+a)^2(s+b)} \right] = \frac{e^{-bt}}{(a-b)^2} + \frac{te^{-at}}{b-a} - \frac{e^{-at}}{(b-a)^2}, \quad t \geq 0
\]

**Appendix: Jordan’s Lemma**

In the theory for inverting Laplace transforms using residue theory, we need the following result:

\[
\lim_{R \to \infty} \int_{C_R} F(s)e^{st} ds = 0 \tag{1}
\]

where \( t > 0 \) and \( C_R \) is the semicircular contour \( C_{R_1} \), shown in Figure 1a. Points on this contour are given by

\[
s = \sigma + Re^{j\theta}, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}
\]

Looking at the table of standard transforms it can be seen that most satisfy the conditions \( F(s) \to 0 \) as \( |s| \to \infty \) (e.g. \( \frac{1}{s}, \frac{1}{s+a} \), etc.). Therefore on \( C_R \) as \( R \to \infty, \) \( F(s) \to 0 \). This means that for any \( \epsilon > 0 \), an \( R \) can be found such that \( |F(s)| = |F(\sigma + Re^{j\theta})| < \epsilon \). For this \( R \) we have

\[
\left| \int_{C_R} F(s)e^{st} ds \right| \leq \epsilon \int_{C_R} |e^{st}| ds
\]

On \( C_R \),

\[
|e^{st}| = |e^{\sigma + Re^{j\theta}}| = |e^{[\sigma + R\cos \theta + jR\sin \theta]t}| = e^{[\sigma + R\cos \theta]t}
\]

and \( ds = jRe^{j\theta} d\theta \). Therefore,

\[
\epsilon \int_{C_R} |e^{st}| ds = \epsilon \Re e^{\sigma t} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} e^{Rt\cos \theta} d\theta = 2\epsilon \Re e^{\sigma t} \int_{0}^{\frac{\pi}{2}} e^{-Rt\sin \theta} d\theta
\]

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Plotting graphs of \( y = \sin \theta \) and the straight line \( y = \frac{2}{\pi} \theta \), we see that

\[
\sin \theta \geq \frac{2}{\pi} \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}
\]

Hence the integral is less than

\[
2\Re e^{\sigma t} \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi} \theta} d\theta = \frac{\epsilon \pi e^{\sigma t}}{t} (1 - e^{-Rt})
\]

For any \( t > 0 \) this last quantity goes to zero as \( R \to \infty \) (because \( \epsilon \) also goes to zero), and we have therefore proved (1).