1. The nonempty set $S$ of integers is closed under subtraction.

Prove that

(i) $S$ is closed under addition,

(ii) if $S$ contains more than one element then it contains a least positive element, $b$.

(iii) $S$ consists of all integer multiples of $b$.

Find an example of a nonempty set of integers which is closed under addition but which does not consist of all integer multiples of a least positive element.

Let $S$ be the set of all integers which are divisible by 10 and whose squares are divisible by 40. Show that $m \in S$ if and only if $m$ is divisible by $2^a5^b$ for $\alpha \geq 2$ and $\beta \geq 1$. Deduce that $S$ contains a least positive integer, which you should specify, and consists of all integer multiples of this.

2. The set of all square matrices of fixed order $k$ is denoted by $M_k$. A relation $\sim$ is defined on $M_k$ by

$$A \sim B \text{ if } A^r = B^s \text{ for some } r, s \in \mathbb{Z}_{>0}.$$  

Prove that $\sim$ is an equivalence relation on $M_k$.

Let $X$ and $Y$ be the following elements of $M_2$:

$$X = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}.$$  

Show that $X \sim I_2$ and $Y \sim I_2$ where $I_2$ is the $2 \times 2$ identity matrix. Prove by induction on $n \geq 1$ that

$$(XY)^n = (-1)^n \begin{pmatrix} 2n+1 & -2n \\ 2n & -2n+1 \end{pmatrix},$$

and deduce that $\sim$ is not compatible with matrix multiplication in $M_2$.

Show that $\sim$ is not compatible with matrix addition in $M_2$ either.
3. A binary operation of multiplication is defined on the set $Q$ of all quadruples of real numbers $(a, b, c, d)$ with $a \neq 0$ by

$$(a, b, c, d) (a', b', c', d') = (aa', ab' + b, ac' + c, d + d').$$

Prove that $Q$ is a group with respect to this multiplication.

Using the function $\phi : Q \to Q$ defined by

$$\phi (a, b, c, d) = (a, 0, 0, d),$$

or otherwise, prove that the subset

$$K = \{(1, b, c, 0) ; \ b, \ c \in \mathbb{R}\}$$

of $Q$ is a normal subgroup of $Q$.

Prove also that the quotient group $Q/K$ is abelian.

4. If $G$ and $H$ are groups, what is meant by saying that a function $h : G \to H$ is a homomorphism? Prove that $h(G)$ is a subgroup of $H$ for such a function.

Explain what is meant by the symmetric group $S(X)$ of a nonempty set $X$.

A proper, nonempty subset $Y$ of $X$ is given, and for $p \in S(Y)$ a function $p_X : X \to X$ is defined by

$$p_X(x) = \begin{cases} p(x) & \text{if } x \in Y \\ x & \text{if } x \in X - Y \end{cases}.$$  

Prove that $p_X \in S(X)$. Use the function $h : S(Y) \to S(X)$ defined by $h(p) = p_X$ for all $p \in S(Y)$ to show that $S(X)$ contains a subgroup isomorphic to $S(Y)$. 
5. If $G$ is any group and $H_1, H_2, H_3, \ldots$ are subgroups of $G$ such that $H_n < H_{n+1}$ for each $n$, prove that

(i) $H = \bigcup_{n=1}^{\infty} H_n$ is a subgroup of $G$.

(ii) $H$ cannot be generated by a finite number of elements.

Suppose $G$ is taken to be the group generated by the permutations $\alpha$ and $\beta$ of $\mathbb{R}$ defined by $\alpha(x) = x + 1$ and $\beta(x) = \frac{x}{2}$ for all $x \in \mathbb{R}$, the group operation being composition of functions. Let $\gamma_n = \beta^n \alpha \beta^{-n}$ and $H_n = \langle \gamma_n \rangle$ for $n = 1, 2, 3, \ldots$. Show that for all such $n$

(a) $\gamma_n(x) = x + \frac{1}{2^n}$ for all $x \in \mathbb{R}$,

(b) $\gamma_{n+1} \neq \gamma_n$,

(c) $\gamma_{n+1} \notin H_n$.

Deduce that $H_n < H_{n+1}$, for $n = 1, 2, 3, \ldots$, and hence determine a subgroup of $G$ which cannot be generated by a finite number of elements.