

NOTES

Edited by William Adkins

Erdős-Mordell-Type Inequalities in a Triangle

Razvan A. Satnoianu

The Erdős-Mordell triangle inequality states that, if P is a point in the interior of a triangle ABC whose distances are p, q, r from the vertices of the triangle and x, y, z from its sides, then

$$p + q + r \geq 2(x + y + z), \quad (1)$$

with equality if and only if the triangle is equilateral and P is its center. This inequality was conjectured by Erdős [1] and proved by Mordell and Barrow [2]. Oppenheim [3] established a number of additional inequalities relating the six distances $p, q, r, x, y,$ and z . Such an inequality will be referred to as an *Erdős-Mordell-type inequality*. A survey of some of these inequalities can be found in [4]. The aim of this note is to obtain a large family of Erdős-Mordell-type inequalities. To do so we first note that inequality (1) compares the elements $f(p), f(q), f(r)$ with $f(x), f(y), f(z)$ for the function $f(t) = t$. By applying one of the standard proofs for (1) (for example, as in [4]) and replacing f with a monotone multiplicatively convex function, a large class of Erdős-Mordell-type inequalities is obtained. Montel was among the first to consider multiplicative convexity [5]. A modern presentation is given by Niculescu [6]. We begin by recalling the definition of this concept.

Definition. Let $f : [0, d) \rightarrow [0, \infty)$ for $d > 0$. The function f is *multiplicatively convex* if

$$\sqrt{f(a)f(b)} \geq f(\sqrt{ab}) \quad (0 \leq a, b < d). \quad (2)$$

The following result, which can be found in [6, p. 158], gives a large class of functions that are simultaneously convex and multiplicatively convex.

Proposition. Let $f(x) = \sum_{n \geq 0} c_n x^n$ be an analytic function defined by a power series with radius of convergence $R > 0$. If all the coefficients c_n are nonnegative, then f is simultaneously convex and multiplicatively convex on any interval $[0, d)$ with d less than R . In particular, any polynomial with positive coefficients is a multiplicatively convex function.

We also need the following elementary result, which follows from Jensen's inequality. It was proved in a more general setting by Petrović [7].

Lemma. If $f : [0, d) \rightarrow [0, \infty)$ is a convex function, then

$$f(a + b) + f(0) \geq f(a) + f(b) \quad (3)$$

for all a and b satisfying $0 \leq a, b < d$ and $a + b < d$.

The main result is the following theorem.

Theorem. Let ABC be a triangle and let $f : [0, d] \rightarrow [0, \infty)$ be an increasing and multiplicatively convex function that satisfies inequality (3). If d is greater than the maximum side-length of ABC , then for every point P in the interior of ABC the following Erdős-Mordell-type inequalities hold:

$$1. f(p) + f(q) + f(r) + 3f(0) \geq 2(f(x) + f(y) + f(z)), \quad (4)$$

$$2. f(xp) + f(yq) + f(zr) + 3f(0) \geq 2(f(xy) + f(yz) + f(zx)). \quad (5)$$

Moreover, if $f(0) = 0$ then:

$$3. f(p)f(q)f(r) \geq 8f(x)f(y)f(z), \quad (6)$$

$$4. f(pq) + f(qr) + f(rp) \geq 4(f(xy) + f(yz) + f(zx)). \quad (7)$$

Equality is possible in any of the inequalities (4)–(7) if and only if the triangle ABC is equilateral and P is its center.

Proof. Let α, β, γ be the angles of ABC . To establish (4), we start from the following known relations that are given, for example, in [4]:

$$p \geq y \frac{\sin \beta}{\sin \alpha} + z \frac{\sin \gamma}{\sin \alpha}, \quad q \geq x \frac{\sin \alpha}{\sin \beta} + z \frac{\sin \gamma}{\sin \beta}, \quad r \geq y \frac{\sin \beta}{\sin \gamma} + x \frac{\sin \alpha}{\sin \gamma}, \quad (8)$$

$$p \geq z \frac{\sin \beta}{\sin \alpha} + y \frac{\sin \gamma}{\sin \alpha}, \quad q \geq z \frac{\sin \alpha}{\sin \beta} + x \frac{\sin \gamma}{\sin \beta}, \quad r \geq x \frac{\sin \beta}{\sin \gamma} + y \frac{\sin \alpha}{\sin \gamma}. \quad (9)$$

Relations (9) are the symmetric inequalities that are obtained from (8) by reflecting P in the angle bisectors of the angles α, β , and γ , respectively (see [3], [4]). Since f is increasing, we find using (9) that

$$f(p) \geq f\left(y \frac{\sin \gamma}{\sin \alpha} + z \frac{\sin \beta}{\sin \alpha}\right), \quad f(q) \geq f\left(x \frac{\sin \gamma}{\sin \beta} + z \frac{\sin \alpha}{\sin \beta}\right), \\ f(r) \geq f\left(y \frac{\sin \alpha}{\sin \gamma} + x \frac{\sin \beta}{\sin \gamma}\right). \quad (10)$$

But (3) then yields

$$f\left(y \frac{\sin \gamma}{\sin \alpha} + z \frac{\sin \beta}{\sin \alpha}\right) \geq f\left(y \frac{\sin \gamma}{\sin \alpha}\right) + f\left(z \frac{\sin \beta}{\sin \alpha}\right) - f(0) \quad (11)$$

and the two analogous inequalities resulting from (10). By adding the three relations (11) and grouping the terms two-by-two, we can apply the arithmetic-geometric mean (AM-GM) inequality to three pairs of terms, as in:

$$f\left(y \frac{\sin \gamma}{\sin \alpha}\right) + f\left(y \frac{\sin \alpha}{\sin \gamma}\right) \geq 2\sqrt{f\left(y \frac{\sin \gamma}{\sin \alpha}\right) f\left(y \frac{\sin \alpha}{\sin \gamma}\right)}. \quad (12)$$

Next, using the multiplicative convexity of f on each of the three inequalities (12) we obtain

$$\sqrt{f\left(y \frac{\sin \gamma}{\sin \alpha}\right) f\left(y \frac{\sin \alpha}{\sin \gamma}\right)} \geq f\left(\sqrt{\left(y \frac{\sin \gamma}{\sin \alpha}\right) \left(y \frac{\sin \alpha}{\sin \gamma}\right)}\right) = f(y) \quad (13)$$

and two similar inequalities. Finally, combining (10)–(14), we arrive at (4).

For (5) we use the inequalities (8). Since f is increasing, (8) and (3) yield the relation

$$\sum_{\text{cyclic}} f(xp) \geq \sum_{\text{cyclic}} \left(f \left(yz \frac{\sin \beta}{\sin \gamma} \right) + f \left(yz \frac{\sin \gamma}{\sin \beta} \right) \right) - 3f(0). \quad (14)$$

Further, by using the AM-GM inequality on the terms in parentheses on the right-hand side of (14) and applying the multiplicative convexity property of f , (5) follows.

We obtain (6) by multiplying the inequalities in (9), using the fact that f is increasing, and then invoking inequality (3), the AM-GM inequality, and the property of multiplicative convexity taken in that order.

For (7) we make use of the inequalities (8) and (9) together. The conclusion follows by using the properties of f as before.

Finally, by examining the requirements for equality in each case in turn, it becomes clear that in all cases equality is possible only when the triangle ABC is equilateral and P is its center. ■

Corollary. *If $f : [0, d) \rightarrow [0, \infty)$ is increasing, convex, and multiplicatively convex, then the Erdős-Mordell-type inequalities (4)–(7) are true. For example, this is the case if f is an analytic function satisfying the hypotheses of the proposition and, in particular, (4)–(7) hold whenever f is a polynomial function with positive coefficients.*

Many elementary functions or simple transformations of them are found to satisfy the corollary. A few examples are: $f_1(t) = e^t$, $f_2(t) = \sinh t$, $f_3(t) = \cosh t$, $f_4(t) = \tan t$, $f_5(t) = \sec t$, $f_6(t) = \operatorname{cosec} t$, $f_7(t) = (1 - t^m)^{-1}$ on $(0, 1)$ with $m > 1$, $f_8(t) = 1/t - \cotan t$ on $(0, \pi/2)$, $f_9(t) = \arcsin t$ on $[0, 1)$, $f_{10}(t) = -\log(1 - t)$ on $(0, 1)$, $f_{11}(t) = (1 + t)^m$ on $[0, \infty)$ with $m \geq 1$, and $f_{12}(t) = (1 - t)^{-m}$ on $(0, 1)$ with $m > 1$. Consequently, for any such function new Erdős-Mordell-type inequalities are obtained.

The classical Erdős-Mordell inequality (1) is recovered with the choice $g_1(t) = t$, whereas the functions $g_2(t) = t^k$ for $k \geq 1$ furnish all of the main Oppenheim inequalities from [3] (see also [4, pp. 313–314]). In fact, all four parts of the theorem generalize the results in [3]. For the function $g_3(t) = a^t$ ($t > 0$, $a \geq 1$), the corollary shows that in every triangle we have the inequality (15) (problem 10814 in this MONTHLY [8]):

$$a^p + a^q + a^r + 3 \geq 2(a^x + a^y + a^z). \quad (15)$$

REFERENCES

1. Paul Erdős, Proposed problem 3740, this MONTHLY **42** (1935) 396.
2. L. J. Mordell and D. F. Barrow, Solution to problem 3740, this MONTHLY **44** (1937) 252–254.
3. A. Oppenheim, The Erdős inequality and other inequalities for a triangle, this MONTHLY **68** (1961) 226–230.
4. D. S. Mitrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer, Dordrecht, 1989.
5. P. Montel, Sur les fonctions convexes et les fonctions souharmoniques, *J. Math.* **7** (1928) 29–60.
6. C. P. Niculescu, Convexity according to the geometric mean, *Math. Inequal. & App.* **3** (2000) 155–167.
7. M. Petrović, Sur une fonctionnelle, *Publ. Math. Univ. Belgrade* **1** (1932) 149–156.
8. R. A. Satnoianu, Proposed problem 10814, this MONTHLY **107** (2000) 567; solution to problem 10814, this MONTHLY **108** (2001) 572.

*Correspondence address: Department of Mathematics, City University, LONDON EC1V 0HB, UK.
E-mail: r.a.satnoianu@city.ac.uk*