Question 1

(a) Calculate the following limits:

(i) \( \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} \),
(ii) \( \lim_{x \to \infty} \frac{2x^2 + 3x + 1}{x^2 + 2x + 3} \),
(iii) \( \lim_{x \to 0} \frac{\sin x}{x} \)  

(10 marks)

(b) Find the domain and range of the function: \( f(x) = \frac{1}{\sqrt{x - 1}} \) and sketch its graph.

(10 marks)

Question 2

(a) Sketch the graph of the function \( f(x) = x^3 - x \) indicating clearly: (i) any points where the graph crosses the \( x \)-axis; (ii) any critical points (i.e., points where the function has a local maximum or minimum); and, (iii) any points of inflection.

(12 marks)

(b) Write down the Taylor series expansion of \( f(x) \) defined in part (a) (up to and including quadratic terms) around each critical point you identified in part (a)-(ii) and use it to justify your classification of the corresponding critical point as a local minimum or maximum.

(8 marks)

Question 3

(a) If \( w = f(ax + by) \), show that \( b \frac{\partial w}{\partial x} - a \frac{\partial w}{\partial y} = 0 \). Hint: Let \( z = ax + by \) and calculate \( \frac{\partial w}{\partial x} \) and \( \frac{\partial w}{\partial y} \).

(8 marks)

(b) Let \( u = f(x, y) \). Given \( x = e^t \cos t \) and \( y = e^t \sin t \) find expressions of \( \frac{\partial u}{\partial s} \) and \( \frac{\partial u}{\partial t} \) in terms of \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \). Hence prove that:

\[
\frac{\partial u}{\partial y} = e^{-t} \left( \sin t \frac{\partial u}{\partial s} + \cos t \frac{\partial u}{\partial t} \right)
\]

(12 marks)

Question 4

(a) Find all solutions of the equation \( z^3 = -8 \) and indicate their location in the complex plane.

(7 marks)
(b) By using de Moivre’s identity: 
\[ e^{in\theta} = \cos n\theta + i \sin n\theta \] (for \( n = 2 \)), establish the following two trigonometric identities: (i) \( \cos 2\theta = \cos^2 \theta - \sin^2 \theta \) and (ii) \( \sin 2\theta = 2 \sin \theta \cos \theta \).

(6 marks)

(c) Use the polar form of \( 1 + i \) and \( \sqrt{3} - i \) to evaluate:

\[ z = \frac{(1+i)^6}{(\sqrt{3} - i)^3} \]

(7 marks)

Question 5

(a) The position vector of a particle moving around a circular orbit of radius \( a \) with constant angular velocity \( \omega \) is given by:

\[ \mathbf{r}(t) = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j} \]

Calculate the velocity and acceleration vectors of the particle, \( \mathbf{v}(t) \) and \( \mathbf{a}(t) \), respectively. Hence show that \( \mathbf{v}(t) \cdot \mathbf{r}(t) = 0 \) and that \( \mathbf{a}(t) + \omega^2 \mathbf{r}(t) = \mathbf{0} \).

(10 marks)

(b) Find the equation of the plane \( P \) which passes through the point \( (x, y, z) = (1,1,1) \) and is perpendicular to the vector \( \mathbf{i} - \mathbf{j} + \mathbf{k} \). Find also the equation of the line representing the intersection of \( P \) with the plane \( z = 0 \).

(10 marks)

Question 6

(a) Consider the matrix:

\[ K = \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{pmatrix} \]

where \( k_1, k_2 \) and \( k_3 \) are real parameters.

(i) By calculating the determinant of the matrix (or otherwise), show that \( K \) is non-singular if all three parameters \( k_i \), \( (i = 1, 2, 3) \) are non-zero, but that \( K \) is singular if at least one of the three parameters is zero.

(10 marks)

(ii) Calculate the inverse of matrix \( K \) as a function of the three parameters \( k_1, k_2 \) and \( k_3 \) (assuming they are all non-zero).

(10 marks)
Question 7

(a) Calculate the following indefinite integrals: (i) \( I_1 = \int \sqrt{1 - 4x^2} \, dx \) \((|x| < 1/2)\) and (ii) \( I_2 = \int e^{2x} \cos x \, dx \). Hint for (i): Use the substitution \( 2x = \sin \theta \).

(10 marks)

(b) Find the length of the curve \( y = x^{3/2} \) between the points \((x, y) = (0, 0)\) and \((x, y) = (1, 1)\).

(10 marks)

Question 8

(a) Find the general solution of the following differential equation:

\[
\frac{dy}{dx} = \frac{y}{x} - \tan \left( \frac{y}{x} \right)
\]

Hint: Use the substitution \( u = y/x \).

(10 marks)

(b) Find the solution of the following differential equation subject to the indicated initial conditions:

\[
\frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 2y = e^x, \quad y(0) = \frac{dy(0)}{dx} = 0
\]

(10 marks)
FORMULAE IN THE DIFFERENTIAL AND INTEGRAL CALCULUS

Basic Derivatives

If \( y = x^n \), \[
\frac{dy}{dx} = nx^{n-1}
\]

If \( y = \sin \theta \), \[
\frac{dy}{dx} = \cos \theta
\]

If \( y = \cos \theta \), \[
\frac{dy}{dx} = -\sin \theta
\]

If \( y = \tan \theta \), \[
\frac{dy}{dx} = \sec^2 \theta
\]

If \( y = \cot \theta \), \[
\frac{dy}{dx} = -\csc^2 \theta
\]

If \( y = \sec \theta \), \[
\frac{dy}{dx} = \tan \theta \sec \theta = \frac{\sin \theta}{\cos^2 \theta}
\]

If \( y = \csc \theta \), \[
\frac{dy}{dx} = -\cot \theta \csc \theta = -\frac{\cos \theta}{\sin^2 \theta}
\]

If \( y = \sin^{-1} \frac{x}{a} \), \[
\frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}
\]

If \( y = \cos^{-1} \frac{x}{a} \), \[
\frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}
\]

If \( y = \tan^{-1} \frac{x}{a} \), \[
\frac{dy}{dx} = \frac{a}{a^2 + x^2}
\]

If \( y = \cot^{-1} \frac{x}{a} \), \[
\frac{dy}{dx} = -\frac{a}{a^2 + x^2}
\]

If \( y = \sec^{-1} \frac{x}{a} \), \[
\frac{dy}{dx} = -\frac{a}{x\sqrt{x^2 - a^2}}
\]

If \( y = \csc^{-1} \frac{x}{a} \), \[
\frac{dy}{dx} = -\frac{a}{x\sqrt{x^2 - a^2}}
\]

If \( y = e^x \), \[
\frac{dy}{dx} = e^x
\]

If \( y = e^{ax} \), \[
\frac{dy}{dx} = ae^{ax}
\]

If \( y = a^x \), \[
\frac{dy}{dx} = a^x \log a
\]

If \( y = \ln x \), \[
\frac{dy}{dx} = \frac{1}{x}
\]

Binomial Theorem

\[(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{1 \cdot 2} x^2 \pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \ldots\]
Maclaurin’s Theorem

\[ f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \]

Taylor’s Theorem

\[ f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!} f''(a) + \frac{(x - a)^3}{3!} f'''(a) + \ldots \]

Product

\[ y = uv, \quad \frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx} \]

Quotient

\[ y = \frac{u}{v}, \quad \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \]

Formula for Integration by Parts

\[ \int u dv = uv - \int v du \]

Basic Integrals

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1) \]
\[ \int \cos \theta \, d\theta = \sin \theta \]
\[ \int \sin \theta \, d\theta = -\cos \theta \]
\[ \int \sec^2 \theta \, d\theta = \tan \theta \]
\[ \int \csc^2 \theta \, d\theta = -\cot \theta \]
\[ \int \tan \theta \sec \theta \, d\theta = \sec \theta \]
\[ \int \cot \theta \csc \theta \, d\theta = -\csc \theta \]
\[ \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \]
\[ \int \frac{-dx}{\sqrt{a^2 - x^2}} = \cos^{-1} \frac{x}{a} \]
\[ \int \frac{adx}{a^2 + x^2} = \tan^{-1} \frac{x}{a} \]
\[ \int \frac{-adx}{a^2 + x^2} = \cot^{-1} \frac{x}{a} \]
\[
\int \frac{adx}{x\sqrt{x^2 - a^2}} = \sec^{-1} \frac{x}{a}
\]
\[
\int \frac{-adx}{x\sqrt{x^2 - a^2}} = \csc^{-1} \frac{x}{a}
\]
\[
\int e^x dx = e^x
\]
\[
\int e^{ax} dx = \frac{e^{ax}}{a}
\]
\[
\int a^x dx = \frac{a^x}{\ln a}
\]
\[
\int \frac{dx}{x} = \ln x
\]
\[
\int \sinh x dx = \cosh x
\]
\[
\int \cosh x dx = \sinh x
\]
\[
\int \sec h^2 x dx = \tanh x
\]
\[
\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a} \text{ or } \log \left( \frac{x + \sqrt{x^2 + a^2}}{a} \right)
\]
\[
\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} \text{ or } \log \left( \frac{x + \sqrt{x^2 - a^2}}{a} \right)
\]
\[
\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} \text{ or } \frac{1}{2a} \log \frac{a+x}{a-x}
\]

Trapezoidal Rule

\[
\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + \ldots + 2f_{n-1} + f_n)
\]

Simpson’s Rule

\[
\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}(f_0 + 4(f_1 + f_3 + \cdots) + 2(f_2 + f_4 + \cdots) + f_n)
\]

HYPERBOLIC FUNCTIONS

\[
\sinh x = \frac{1}{2}(e^x - e^{-x})
\]
\[
\cosh x = \frac{1}{2}(e^x + e^{-x})
\]
SOME TRIGONOMETRIC FORMULAE

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]
\[
\sin(A - B) = \sin A \cos B - \cos A \sin B
\]
\[
\cos(A + B) = \cos A \cos B - \sin A \sin B
\]
\[
\cos(A - B) = \cos A \cos B + \sin A \sin B
\]

FORMULAE FOR HYPERBOLIC FUNCTIONS

\[
\sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B
\]
\[
\sinh(A - B) = \sinh A \cosh B - \cosh A \sinh B
\]
\[
\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B
\]
\[
\cosh(A - B) = \cosh A \cosh B - \sinh A \sinh B
\]