Question 1

The pin-jointed plane frame shown in Figure Q1 comprises six bar members and four nodes. The diagonal members cross at the centre of the frame without being connected there. All members of the frame have the same value of axial rigidity, equal to 150,000 kN. The frame is held at nodes 1 and 2 against any movement in the X and Y directions and two loads of magnitude 100 kN each act at node 4, one in the positive X direction and the other in the positive Y direction. Using the finite element method of structural analysis, determine the displacements of nodes 3 and 4.

Figure Q1
Question 2

As shown in Figure Q2, the rigid-jointed plane structure has two identical beam elements connected at right angle. The supports at nodes 1 and 3 are fully fixed. Using the finite element method of structural analysis, determine the displacements at node 2 and obtain solutions by in turn considering and then neglecting axial deformation of the beams.

[25 marks]
Question 3

(a) State the main assumptions in the finite element method of structural analysis.

[8 marks]

(b) For the beam element shown in Figure Q3(b), the relationship in the local coordinate axes of the element loads and displacements is given by:

\[
\begin{bmatrix}
P_{xA} \\
P_{yA} \\
M_A \\
P_{xB} \\
P_{yB} \\
M_B
\end{bmatrix} = 10^7 \begin{bmatrix}
4 & 0 & 0 & -4 & 0 & 0 \\
0 & 3 & 3 & 0 & -3 & 3 \\
0 & 3 & 4 & 0 & -3 & 2 \\
-4 & 0 & 0 & 4 & 0 & 0 \\
0 & -3 & -3 & 0 & 3 & -3 \\
0 & 3 & 2 & 0 & -3 & 4
\end{bmatrix} \begin{bmatrix}
\delta_{xA} \\
\delta_{yA} \\
\theta_A \\
\delta_{xB} \\
\delta_{yB} \\
\theta_B
\end{bmatrix}
\]

(i) Calculate the values of the local axes forces and moments at the ends of the beam element if the global axes displacements and rotations at the ends of the beam element obtained from a finite element analysis are as follows

[15 marks]

\[
\begin{bmatrix}
\Delta_{xA} \\
\Delta_{yA} \\
\Theta_A
\end{bmatrix} = \begin{bmatrix}
0.1022 \\
-0.0164 \\
0.0012
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\Delta_{xB} \\
\Delta_{yB} \\
\Theta_B
\end{bmatrix} = \begin{bmatrix}
0.1032 \\
-0.0168 \\
0.0006
\end{bmatrix}
\]

Note that forces are in N, couples in Nm, displacements in m, and rotations in radians.

The global axes coordinates of the ends of the beam are:

\(X_A=15.8 \text{ m}, \ Y_A=4.7 \text{ m}, \ X_B=17.0 \text{ m} \) and \(Y_B=3.1 \text{ m}\)

![Figure Q3(b)](image)

(ii) Using the results of the previous analysis, perform a check on the equilibrium conditions of the beam element.

[2 marks]
Question 4

A straight and uniform aluminium tube forms a pin-ended strut of length 50.8 cm. The tube cross-section is hollow and circular with a wall thickness of 0.203 cm and mean wall diameter 3.61 cm.

The relationship between stress $\sigma$ and strain $\varepsilon$ for the alloy is given by

$$\varepsilon = \frac{\sigma}{E} + B \left( \frac{\sigma}{\sigma_R} \right)^n$$

where $E = 72.5 \text{ GN/m}^2$, $B$ and $n$ are constants and $\sigma_R$ is a reference stress which may be assumed to be 0.2% proof stress.

(a) Estimate the critical flexural buckling stress for the strut using the tangent modulus method when

$$\sigma_1 = 0.1\% \text{ proof stress} = 323.67 \text{ MN/m}^2$$

$$\sigma_2 = 0.2\% \text{ proof stress} = 338 \text{ MN/m}^2$$

(b) Determine the error that would occur if no account were taken of the deviation of the stress-strain curve from the slope at its origin. Express this error as a percentage of the lower of the two buckling stresses and comment on the results.

[20 marks]

[5 marks]
Question 5

A uniform thin-walled beam of anti-symmetrical cross-section has a length of 2m and wall thickness of 1.5mm. It is built-in at end A and is subjected to a concentrated vertical load of 100 N at the free-end B as shown Figure Q5(a). The cross section and other details of the beam are shown in Figure Q5(b).

The wall thickness may be considered to be small in comparison with other dimensions when calculating the cross-sectional properties of the beam.

(a) Calculate the maximum stresses in the cross section at the built-in end of the beam.

(b) Determine the neutral axis and sketch the stress distribution in the cross section.

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Figure Q5(a)

Figure Q5(b)

(Not to scale)
Question 6

In the two-dimensional stress system shown in Figure Q6, the normal stress on an inclined plane AC at 30° to the reference plane AB is 70 MN/m².

Determine the following:

(a) The normal stress $\sigma_y$ and the shear stress on the inclined plane AC. [8 marks]

(b) The magnitudes and directions of the principal stresses relative to AB. [8 marks]

(c) The magnitudes and directions of the normal stress at maximum shear stress plane relative to AB. [4 marks]

(d) The magnitudes of the principal strains that would occur in this two-dimensional stress field. [5 marks]

Take $E = 207 \times 10^3$ MN/m² and $\nu = 0.3$.

![Figure Q6](image-url)

$\sigma_y$

20 MN/m²

120 MN/m²

70 MN/m²

30°

120 MN/m²

20 MN/m²

$\sigma_y$
**Question 7**

Using a 60° strain gauge rosette in a uniform strain field, the strains $\varepsilon_a$, $\varepsilon_b$, and $\varepsilon_c$ are recorded to be 650, 150, and 250 $\mu\varepsilon$, respectively, as shown in Figure Q7.

(a) Determine the magnitudes and directions of the principal strains. [14 marks]

(b) What are the principal stresses associated with these strains and how do they act? [5 marks]

(c) If gauge [c] is set incorrectly with an error of $-10^\circ$, i.e. at $50^\circ$ from gauge [b], what strain would be recorded in gauge [c]? What would be the consequent percentage error for the principal strains if gauge [c] setting error is not detected? [6 marks]

Take $E = 207 \times 10^3$ MN/m$^2$ and $\nu = 0.3$

Examiners: Prof. J.R. Banerjee  
Dr. C.W. Cheung

External Examiner: Prof. M. Imregun
Information Sheet 1

(A) The stiffness matrix of a bar element in global co-ordinates is given by

\[
\begin{bmatrix}
\bar{k}_{11} & \bar{k}_{12} \\
\bar{k}_{21} & \bar{k}_{22}
\end{bmatrix}
\]

where

\[
\bar{k}_{11} = \bar{k}_{22} = -\bar{k}_{12} = -\bar{k}_{21} = \frac{EA}{L}\begin{bmatrix}
l^2 \\
lnm \\
ln m^2
\end{bmatrix}
\]

where \(EA\) and \(L\) are respectively the axial rigidity and the length of the bar; \(l\) and \(m\) are the direction cosines of the local co-ordinate axes with respect to the global co-ordinate axes.

(B) The stiffness matrix of a beam element in global co-ordinates is given by

\[
\begin{bmatrix}
\bar{k}_{11} \\
\bar{k}_{21}
\end{bmatrix}
\]

where

\[
\bar{k}_{11} = \begin{bmatrix}
F \\
G \\
H
\end{bmatrix}
, \quad \bar{k}_{22} = \begin{bmatrix}
F \\
G \\
-H
\end{bmatrix}
, \quad \bar{k}_{12} = \begin{bmatrix}
-G \\
-P \\
-H
\end{bmatrix}
, \quad \bar{k}_{21} = \begin{bmatrix}
-Q \\
-B \\
-H
\end{bmatrix}
\]

\[
\begin{bmatrix}
F & G & H \\
G & P & Q \\
H & Q & R
\end{bmatrix}
\]

\[
\begin{bmatrix}
F & G & -H \\
G & P & -Q \\
-H & -Q & R
\end{bmatrix}
\]

\[
\begin{bmatrix}
-F & -G & H \\
-G & -P & Q \\
-H & -Q & B
\end{bmatrix}
\]

\[
\begin{bmatrix}
-F & -G & -H \\
-G & -P & -Q \\
-H & -Q & B
\end{bmatrix}
\]

where \(F = \frac{EA}{L}l^2 + \frac{12EI}{L^3}m^2\), \(G = \frac{EA}{L}lnm - \frac{12EI}{L^3}ln m\), \(H = -\frac{6EI}{L^2}m\), \(P = \frac{EA}{L}m^2 + \frac{12EI}{L^3}l^2\), \(Q = \frac{6EI}{L^2}l\), \(R = \frac{4EI}{L}\), \(B = \frac{2EI}{L}\).

\(EA\) and \(EI\) are respectively the axial and bending rigidity of the beam; \(l\) and \(m\) are the direction cosines of the local co-ordinate axes with respect to the global co-ordinate axes and \(L\) is the length of the beam.
Information Sheet 2

The following relationships are given in the usual notation.

(i) Shear flow in a singly symmetric closed section

\[ q_s = q_o - \frac{S_y}{I_{XX}} \int_0^s t \ y \ ds \]

(ii) Rate of twist and torque are related as

\[ \frac{d \theta}{dz} = \frac{T}{GJ} \]

(iii) Torsion constant (\( J \)) for closed and open sections are respectively given by

\[ J_c = \frac{4A^2}{\int d s} \]
\[ J_o = \frac{l^3}{6} \]

(iv) Bredt-Batho formula

\[ T = 2 A q \]

(v) Bending stress due to asymmetric bending

\[ \sigma_z = \frac{M_x}{I_{XX}} y + \frac{M_y}{I_{YY}} x \]
or

\[ \sigma_z = \left[ \frac{M_y I_{XX} - M_x I_{YY}}{I_{XX} I_{YY} - I_{XY}^2} \right] y + \left[ \frac{M_x I_{XX} - M_y I_{XY}}{I_{XX} I_{YY} - I_{XY}^2} \right] x \]

(vi) Differential equations governing vertical and horizontal deflections due to asymmetric bending are respectively given by

\[ \frac{d^2 v}{dz^2} = - \frac{\overline{M}_x}{E I_{XX}} \]
\[ \frac{d^2 u}{dz^2} = - \frac{\overline{M}_y}{E I_{YY}} \]

where

\[ \overline{M}_x = \frac{M_x - M_y I_{XY} / I_{YY}}{1 - I_{XY}^2 / I_{XX} I_{YY}} \]
\[ \overline{M}_y = \frac{M_y - M_x I_{XY} / I_{XX}}{1 - I_{XY}^2 / I_{XX} I_{YY}} \]