Question 1

Consider the following equation, where \( u \) is the fluid velocity and \( x \) and \( y \) the directions of the Cartesian coordinate system:

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0
\]

a) Which conservation law is expressed by this equation and under which assumptions?

[ 5 marks ]

b) Express the same conservation principle expressed by the above equation in vector form.

[ 5 marks ]

c) Integrate the above equation using the finite-volume method on a uniform orthogonal Cartesian grid, using the following notation:

[ 15 marks ]
**Question 2**

a) State the Newton’s second law of a moving object. 

[5 marks]

b) Assume that the aerodynamic drag and gravity are the only forces acting on a spherical particle moving with velocity \( u \) in the direction of a gravitational field \( g \) with mass \( m \) not changing with time. Expand the Newton’s 2\(^{\text{nd}}\) law for that particle and state the condition of terminal velocity. 

[10 marks]

c) Derive a discrete formula of the Newton’s 2\(^{\text{nd}}\) law for the spherical particle of Question 2b, and provide the logical steps of an explicit numerical algorithm that can be used for the estimation of its trajectory. 

[10 marks]

**Question 3**

a) Describe the method of Reynolds decomposition for modelling turbulence. 

[10 marks]

b) Describe the concept of Eddy viscosity and Prandtl’s mixing length. 

[15 marks]
**Question 4**

The quasi-one-dimensional compressible flow within a converging diverging nozzle may be modelled by the following form of the Euler equations

\[
\frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} = s
\]

where “s” denotes the source term and “q” and “A” are defined as

\[
q \equiv \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}, \quad A \equiv \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1 / \rho \\ 0 & \gamma \rho & u \end{pmatrix}
\]

a) determine the eigenvalues of the system matrix “A” [5 marks]

b) for a supersonic flow throughout the duct, state the boundary conditions you would impose at the inlet and outlet [5 marks]

c) write down a first order implicit and stable finite difference discretisation of the equation system for a supersonic flow [5 marks]

d) synthetically formulate the discrete-system Jacobian and state how the numerical solution is eventually obtained for the linearised system. [5 marks]

**Question 5**

The quasi-one dimensional Euler equations may be expressed as

\[
\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = s
\]

a) briefly describe the concept of flux difference splitting [7 marks]

b) express the Euler system in non-discrete flux-difference-splitting form [8 marks]

c) write down a first order explicit stable discretization of the system for a subsonic flow in the positive x-axis direction [10 marks]
Question 6

With implied summation on repeated subscript indices, the compressible Navier-Stokes system may be expressed as:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial m_j}{\partial x_j} &= 0 \\
\frac{\partial m_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \frac{m_j}{\rho} - \rho \delta^j_i - \sigma^\mu_{ij} \right) &= 0 \\
\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} \left( \frac{m_j}{\rho} (E + p) - \frac{m_i}{\rho} \sigma^\mu_{ij} + q^F_j \right) &= m_i b_i - \rho g = 0
\end{align*}
\]

where the deviatoric stress tensor components $\sigma^\mu_{ij}$ and Fourier heat flux component $q^F_j$ are defined as

\[
\sigma^\mu_{ij} = \mu \left( \frac{\partial m_i}{\partial x_j} + \frac{\partial m_j}{\partial x_i} - \frac{2}{3} \frac{\partial m_i}{\partial x_k} \delta^k_j \right), \quad q^F_j = -k \frac{\partial T}{\partial x_j}
\]

a) write an equivalent integral statement for any suitable test function “$w$” [3 marks]

b) obtain a corresponding weak statement [5 marks]

c) state what terms in the weak statement provide effective venues for boundary-condition enforcement [7 marks]

d) state all the boundary conditions you would apply for the linear-momentum and energy equations at an adiabatic wall, a heated wall, an adiabatic subsonic outlet [10 marks]