SECTION A

Question 1

(a) In the usual notation derive the governing differential equation of motion in free
vibration for the single degree of freedom system shown in Figure Q1(a) by using
Newton's second law, Lagrange's equation and Hamilton's principle, respectively.

[9 marks]

(b) The undercarriage of the aircraft shown in Figure Q1(b) has been idealised by two
vertical springs. The nose undercarriage has a stiffness of 3k while the main
undercarriage has a total stiffness of 7k. The centre of gravity of the aircraft is at G,
its mass is M, and its pitching moment of inertia about its centre of gravity is \(6Ma^2\).
Treating the aircraft structure as rigid, calculate the two natural frequencies and the
 corresponding mode shapes of natural vibration.

[16 marks]
Question 2

(a) Derive the governing differential equation of motion of a bar with extensional rigidity $EA$, mass per unit length $\rho A$ and length $L$ undergoing free natural vibration in axial motion and then obtain its general solution. [6 marks]

(b) Determine the frequency equation of the bar when its one end is fixed and a mass $M$ is attached to the other, as shown in Figure Q2(b). [4 marks]

(c) Using the normal mode method of analysis, obtain an expression for the steady-state flexural response of a pinned-pinned beam subjected to a harmonic force $f(x,t) = f_0 \sin \Omega t$ applied at $x = a$ as shown in Figure Q2(c). [15 marks]

In the usual notation, the natural frequencies and mode shapes of a simply supported beam in flexural vibration are respectively given by

$$\omega_n = \frac{n^2 \pi^2}{L^2} \sqrt{\frac{EI}{\rho A}}$$

and

$$W_n(x) = \sin \frac{n\pi x}{L}$$

![Figure Q2(b)](image)

![Figure Q2(c)](image)
Question 3

(a) The stiffness and mass matrices of a 2-D bar element in the usual notation are given by

\[
[k] = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [m]_c = \frac{\rho Al}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad [m]_l = \frac{\rho Al}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

where the subscripts \( c \) and \( l \) to the mass matrix denote the consistent and lumped matrices, respectively.

Determine the fundamental natural frequency of the fixed-fixed bar using two elements (see Figure Q3(a)) and both consistent and lumped matrix representations and compare with the exact value \( \pi \sqrt{\frac{E}{\rho L^2}} \), where \( L \) is the total length of the bar.

Comment briefly on the differences amongst the results. [10 marks]

(b) The stiffness and mass matrices of a 2-D beam element in flexure are given by

\[
[k] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \quad \text{and} \quad [m] = \frac{\rho AL}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}
\]

Determine the natural frequencies of a simply supported beam using only one finite element and by comparing them with the ones given by the expression for \( \omega_n \) in Q2(c), estimate the errors. [15 marks]
SECTION B

Question 4

Figure Q4 shows the essential details of an aircraft wing. The wing itself is considered cantilevered with a length $l = 4.5$ m and its structural strength and stiffness in torsion may be considered to derive entirely from a torsion box made of aluminium. The flexural axis runs axially along the centre of the torsion box and the wing has a centre of lift axis $ec = 0.075$ m forward of the flexural axis as shown in Figure Q4(a). The wing chord tapers linearly from 1.5 m at the root to 0.75 m at the tip. The torsion box chord also tapers linearly from 0.6 m at the root to 0.3 m at the tip. The skin thickness of the torsion box is $t = 2$ mm. A cross section of the wing at the root is shown in Figure Q4(b).

Assume two-dimensional lift curve slope $a_i = 5.7$ and subsonic strip theory is applicable. Obtain the divergence speed for the wing using an assumed twist mode

$$\theta(x) = \left(\frac{x}{l}\right)q,$$

where $x$ is the distance measured along the span from the wing root and $q$ is the generalised coordinate. For air $\rho = 1.225$ kg/m$^3$ and aluminium $G = 2.6 \times 10^{10}$ N/m$^2$. Also $J = 4A^2t/s$ for a closed section, $A$ being the enclosed area, $s$ the peripheral length and $t$ the skin thickness of the torsion box.

[25 marks]
Question 5

For the purpose of investigating the dynamic behaviour of the wing of a high-wing aircraft, the wing is modelled as a uniform beam of length $s$, as shown in Figure Q5. The wing is pin-jointed at point O on the fuselage which is considered to be fixed. It is supported by a strut at mid-span which is represented by a spring from earth normal to the wing. The wing has flexural rigidity $EI$ and mass per unit length $\rho$ and the spring stiffness is $8EI/s^3$. The transverse deflection $z(y)$ of the wing is given in terms of assumed modes $q_1$ and $q_2$ where

$$z(y) = \left(\frac{y}{s}\right) q_1 + \left(\frac{y}{s}\right)^2 q_2$$

and $y$ is distance along the wing from O.

Obtain the equations of motion of $q_1$ and $q_2$ and hence find the two natural frequencies.

[19 marks]

Find also the corresponding normal modes and sketch their mode shapes, indicating any nodes.

[6 marks]

The strain energy of a beam in bending is given as

$$V = \frac{1}{2} \int_0^l EI \left[ \frac{d^2 z}{dy^2} \right]^2 dy$$

where the symbols have been defined above.
Question 6

Describe briefly the experimental apparatus and procedures required to carry out a simple resonance test on a small engineering structure in the laboratory. [5 marks]

An aircraft empennage (fin and tailplane) is attached to a massive and rigid fixture for the purpose of conducting a resonance test, as shown in Figure Q6. The mass distribution is modelled as three lumped masses of 10, 20 and 10 kg at points 1, 2 and 3 respectively. The measured modal modes for the displacements \( x_1, x_2 \) and \( x_3 \) at points 1, 2 and 3 as indicated in the figure and their corresponding modal frequencies are found to be

1\(^{st}\) Mode: \( \{0.62, 1.0, -0.60\} \) at 6 Hz

2\(^{nd}\) Mode: \( \{1.0, 0.05, 0.90\} \) at 14 Hz

3\(^{rd}\) Mode: \( \{1.0, -0.61, -0.95\} \) at 16 Hz

Calculate the generalised mass matrix in the measured normal modes and check if it is reasonable to discard the off-diagonal terms. [8 marks]

Also, obtain the generalised forcing vector if a sinusoidal force of \( 200 \sin \Omega t \) N acts as shown at point 2, where \( \Omega \) is the circular forcing frequency and \( t \) is time. [4 marks]

Additionally, formulate a set of uncoupled equations of motion for vibration of the empennage in the measured normal coordinates, given that the measured damping ratio in each normal mode is 1.5% and that the damping may be assumed viscous. [8 marks]

Examiners: Prof. J.R. Banerjee
Dr C.W. Cheung

External Examiner: Prof. D.I.A. Poll
**Information Sheet**

The strain energy $V$ of a thin-wall section beam of length $l$ and constant wall thickness $t$ in torsion $\theta(x)$ is given as

$$V = \frac{1}{2} \int_0^l G J(x) \left( \frac{d\theta}{dx} \right)^2 \, dx$$

where the torsional constant $J(x) = \frac{4t[A(x)]^2}{s(x)}$. $A(x)$ being the enclosed area of cross-section and $s(x)$ the section peripheral length.

The strain energy $V$ of the same beam in bending $z(x)$ is given as

$$V = \frac{1}{2} \int_0^l E I(x) \left( \frac{d^2 z}{dx^2} \right)^2 \, dx$$

where $I(x)$ is the section moment of area of cross-section.

The Langrange’s equations of motion with usual symbols may be expressed as

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where $Q_i$ is the vector of generalised forces including non-conservative forces.

The transformation from the structural mass and stiffness matrices, $M$ and $K$, to the generalised mass and stiffness matrices, $\overline{M}$ and $\overline{K}$ can be defined as

$$\overline{M} = R^T M R$$
$$\overline{K} = R^T K R$$

where $R$ is called the modal matrix.