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A Quantum Theoretical Explanation for Probability Judgment Errors

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A quantum probability model is introduced and used to explain human probability judgment errors including the conjunction and disjunction fallacies, averaging effects, unpacking effects, and order effects on inference. On the one hand, quantum theory is similar to other categorization and memory models of cognition in that it relies on vector spaces defined by features and similarities between vectors to determine probability judgments. On the other hand, quantum probability theory is a generalization of Bayesian probability theory because it is based on a set of (von Neumann) axioms that relax some of the classic (Kolmogorov) axioms. The quantum model is compared and contrasted with other competing explanations for these judgment errors, including the anchoring and adjustment model for probability judgments. In the quantum model, a new fundamental concept in cognition is advanced—the compatibility versus incompatibility of questions and the effect this can have on the sequential order of judgments. We conclude that quantum information-processing principles provide a viable and promising new way to understand human judgment and reasoning.

Keywords: quantum theory, conjunction fallacy, disjunction fallacy, order effects, inference

Nearly 30 years ago, Kahneman, Slovic, and Tversky (1982) began their influential program of research to discover the heuristics and biases that form the basis of human probability judgments. Since that time, a great deal of new and challenging empirical phenomena have been discovered including conjunction and disjunction fallacies, unpacking effects, and order effects on inference (Gilovich, Griffin, & Kahneman, 2002). Although heuristic concepts (such as representativeness, availability, and anchor adjustment) initially served as a guide to researchers in this area, there is a growing need to move beyond these intuitions and develop more coherent, comprehensive, and deductive theoretical explanations (Shah & Oppenheimer, 2008). In this article, we propose a new way of understanding human probability judgment using quantum probability principles (Gudder, 1988).

At first, it might seem odd to apply quantum theory to human judgments. Before we address this general issue, we would like to point out that we are not claiming the brain to be a quantum computer; rather, we only use quantum principles to derive cognitive models and leave the neural basis for later research. That is, we use the mathematical principles of quantum probability detached from the physical meaning associated with quantum mechanics. This approach is similar to the application of complexity theory or stochastic processes to domains outside physics.[1]

There are at least five reasons for doing so: (a) Judgment is not a simple readout from a pre-existing or recorded state; instead, it is constructed from the question and the cognitive state created by the current context. From this first point, it then follows that (b) drawing a conclusion from one judgment changes the context, which disturbs the state of the cognitive system, and the second point implies (c) changes in context and state produced by the first judgment affects the next judgment, producing order effects, so that (d) human judgments do not obey the commutative rule of Boolean logic. (e) Finally, these violations of the commutative rule lead to various types of judgment errors according to classic probability theory. If we replace "human judgment" with "physical measurement" and replace "cognitive system" with "physical system," then these are the same points faced by physicists in the 1920s that forced them to develop quantum theory. In other words, quantum theory was initially invented to explain noncommutative findings in physics that seemed paradoxical from a classical point of view. Similarly, noncommutative findings in cognitive psychology, such as order effects on human judgments, suggest that classical probability theory is too limited to provide a full explanation of all aspects of human cognition. So while it is true that

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[1] There is another line of research in which quantum physical models of the brain are used to understand consciousness (Hammeroff, 1998) and human memory (Pribram, 1993). We are not following this line; instead, we are using quantum models at a more abstract level, analogous to Bayesian models of cognition.

quantum probability rarely has been applied outside physics, a growing number of researchers are exploring its use to explain human cognition, including perception (Atmanspacher, Filk, & Romer, 2004), conceptual structure (Aerts & Gabora, 2005), information retrieval (van Rijsbergen, 2004), decision making (Franco, 2009; Pothos & Busemeyer, 2009), and other human judgments (Khrennikov, 2010).[2]

Thus, in this article, we had two major goals. An immediate goal was to use quantum probability theory to explain some paradoxical findings on probability judgment errors. But a larger goal was to blaze a new trail by which future applications of quantum probability theory could be guided to other fields of judgment research. The remainder of this article is organized as follows. First, we develop a psychological interpretation of quantum probability theory and compare it side by side with classic probability theory. Second, we use the quantum model to derive qualitative predictions for conjunction errors and disjunction errors and other closely related findings. Third, we examine the quantitative predictions of the quantum model for a probabilistic inference task and compare these predictions to a heuristic anchor-adjustment model previously used to describe order effects. Fourth, we briefly summarize other applications of quantum theory to cognition. Finally we discuss the main new ideas that quantum theory contributes and the issues that it raises about rationality.

Quantum Judgment Model

We applied the same quantum judgment model to two different types of probability judgment problems. Both types involve probability judgments about two or more events. The first type of problem is a single judgment about a combination of events such as the conjunction or disjunction of events. According to our quantum theory, judgments about event combinations require an implicit sequential evaluation of each component event. The second type of problem requires an explicit sequence of judgments about a hypothesis on the basis of evaluation of a series of events. We argue that judgment errors arise in both tasks from the sequential evaluation of events, because conclusions from earlier judgments change the context for later judgments.

Quantum theory requires the introduction of a number of new concepts to cognitive psychologists. First, we present these concepts in an intuitive manner that directly relates the ideas to psychological judgments. Later, we summarize the basic axioms of quantum probability more formally and compare these side by side with classic probability used in Bayesian models.

To introduce the new ideas, we first consider the famous "Linda" problem that has been used to demonstrate the conjunction fallacy. (Many different types of stories have been used in past research to study conjunction effects, but this story is the most famous of all). Judges are provided a brief story about a woman named Linda who used to be a philosophy student at a liberal university and who used to be active in an anti-nuclear movement. Then each judge is asked to rank the likelihood of the following events: that Linda is now (a) active in the feminist movement, (b) a bank teller, (c) active in the feminist movement and a bank teller, (d) active in the feminist movement and not a bank teller, and (e) active in the feminist movement or a bank teller. The conjunction fallacy occurs when Option C is judged to be more likely than Option B (even though the latter contains the former), and the disjunction fallacy occurs when Option A is judged to be more likely than Option E (again the latter contains the former).

State Representation

To apply quantum probability to this problem, our first postulate is that the Linda story generates a state of belief represented by a unit-length *state vector* that can be described by a high dimensional vector space. Each dimension of the vector space corresponds to a *basis vector.* Formally, a basis for a vector space is a set of mutually orthogonal and unit-length vectors that span the vector space. That is, any point in the space can be reached from a linear combination of the basis vectors. Psychologically, each basis vector represents a unique combination of properties or feature values, called a *feature pattern*, which is used to describe the situation under question. The state vector is a working-memory state (Baddeley, 1992) that represents the judge's beliefs about Linda regarding the feature patterns. On the one hand, our use of feature vectors to represent cognitive states follows other related cognitive research (e.g., on memory or categorization), whereby information is represented as vectors in high-dimensional spaces. On the other hand, our basis vectors and state vector are analogous to the elementary events and the probability function, respectively, used in classic probability theory.

In general, the feature space used to form the basis for the description of the state is constructed from long-term memory in response to both the story that is presented and the question that is being asked. To make this concrete, let us consider a very simple toy example. Initially focus on the Linda story and the question about whether or not Linda is a feminist and suppose this question calls to mind three binary features that are used to describe the judge's beliefs about Linda for this event: she may or may not be a feminist, she can be young or old, and she can be gay or straight. Then the vector space would have eight dimensions, and one basis vector would correspond to the feature pattern (feminist, young, gay), a second would correspond to the feature pattern (not feminist, young, straight), a third would correspond to the feature pattern (feminist, old, straight), and so forth. In classic probability theory, these eight feature patterns would represent the eight elementary events formed by the eight conjunctions of three binary events.

In actuality, there may be many more features, and each feature may have many values, all generated by the story and the question. In particular, if there are n individual features ($n = 3$ features in our example) that take on m different values ($m = 2$ in our example), then the dimension of the feature space is $N = n^m$. The problem of defining all the relevant features is not unique to quantum theory; it also arises in the specification of a sample space for a Bayesian model. Experimentally, one could devise artificial worlds in which the features are carefully controlled by instruction or training. For problems involving real-world knowledge, there is less control, and instead, one could ask judges to list all of the relevant features. For our toy example, we restricted our discussion to the three previously described binary features for simplicity. But our general theory does not require us to specify this a priori. In

[2] Also see the special issue on quantum cognition (Bruza, Busemeyer, & Gabora, 2009).

fact, one great advantage of the quantum model is that many qualitative predictions can be derived without these additional assumptions being imposed. However, later on when we present a quantitative test of the quantum model, we fully specify the feature space and its dependence on the story and the question.

To evaluate the question about feminism, the judge uses knowledge about the features based on the Linda story and other related past experience. In the state vector, the judge's beliefs about Linda are represented by a belief value, called an *amplitude*, assigned to each basis vector (feature pattern or combination of features), and the squared magnitudes of the amplitudes sum to one. In general, amplitudes can be complex numbers, but they can always be transformed to square roots of probabilities prior to a judgment, and only the latter is used to represent a belief that is available for reporting (see Appendix A). In our toy example, the amplitude assigned to the (feminist, young, gay) basis vector represents the judge's belief about this feature pattern. Usually the belief state has some amplitude assigned to each basis vector; in other words, the belief state is a linear combination of the basis vectors (called a *superposition state*). But a special case is one in which a belief state exactly equals a basis vector. In this special case, the belief state has an amplitude with magnitude equal to one assigned to a *single* basis vector and zeros everywhere else. This corresponds to the special case in which a person is certain about the presence of a specific feature pattern. In the section on qualitative tests, we derive predictions without assuming specific values for the amplitudes. However, the section on quantitative tests describes a specific way to assign these amplitudes.

Event Representation

An event refers to a possible answer to a question about features chosen from a common basis. For example, the answer "yes" to the feminism question is one event, and the answer "no" to the feminism question is the complementary event. Our second postulate is that each event is represented by a *subspace* of the vector space, and each subspace has a *projector* that is used to evaluate the event.

Consider once again our toy example with eight basis vectors. The event "yes" to the specific question "Is Linda is a feminist, young, gay person?" corresponds to the subspace spanned by the basis vector (feminist, young, gay), which is a single ray in the vector space. To evaluate this event, the judge maps (more formally projects) the belief state vector down onto this ray. This is analogous to fitting the belief state to this basis vector (feminist, young, gay) with simple linear regression. This fitting process is performed by a cognitive operator called the *projector* that is used to evaluate the fit of the feature pattern (feminist, young, gay). Thus the event "yes" to the question "Is Linda a feminist young gay person?" corresponds to a ray, and this ray has a projector that is used to evaluate its fit to the belief state.

Now consider a more general event such as saying "yes" to the question "Is Linda a feminist?" Note that the question about feminism concerns only one of the many possible features that are being considered. In our toy example, a yes answer to the feminism question is consistent with only four of the basis vectors: (yes feminist, young, gay), (yes feminist, young, straight), (yes feminist, old, gay), and (yes feminist, old, straight). The span of these four basis vectors forms a four-dimensional *subspace* within the eight-dimensional space, which represents the event "yes" to the feminist question. This is comparable to a union of these four elementary events in classic probability. To evaluate this event, the judge maps (more formally projects) the belief state down onto this four-dimensional subspace. This is analogous to fitting the belief state to the four basis vectors with multiple regression. Once again, the cognitive operator by which this mapping is performed is called the *projector* for the "yes" to the feminism question. In the event of a "no" answer to the feminism question, the complementary subspace is used, which is the subspace spanned by the remaining four basis vectors (not feminist, young, gay), (not feminist, young, straight), (not feminist, old, gay), and (not feminist, old, straight).

Projective Probability

Quantum theory provides a geometric way to compute probabilities. Our third postulate is that the judged probability of concluding that the answer to a question is yes equals the squared length of the projection of the state vector onto the subspace representing the question.

To make this clear, first let us consider the judged probability of concluding that a specific feature pattern, say (feminist, young, gay) from our toy example is true of Linda. To evaluate this event, the judge projects the belief state vector down onto the ray representing (feminist, young, gay), and the result of this fit is called the *projection*. In our toy example, the projection has zeros assigned to all basis vectors except the (feminist, young, gay) basis vector, and the basis vector (feminist, young, gay) is assigned a value equal to its original amplitude. Finally, the judged probability for a yes answer to this elementary event equals the squared length of this projection (the squared magnitude of the amplitude, which is analogous to the squared correlation). Psychologically speaking, the person evaluates how well each feature pattern fits the belief state, and the judged probability for that feature pattern equals the proportion of the belief state reproduced by the feature pattern.

Now consider the judged probability of a more general event. The judge evaluates the event of a yes answer to the feminism question by judging how well his or her beliefs about Linda are fit by the feminist feature patterns used to describe this event. The judge makes the projection for the yes response to the feminism question by mapping (projecting) the belief state vector down onto the subspace representing the yes answer to the feminism question. In our toy example, the amplitudes corresponding to (not feminist, young, gay), (not feminist, young, straight), (not feminist, old, gay), and (not feminist, old, straight) are set to zero, and only the remaining amplitudes previously assigned to (yes feminist, young, gay), (yes feminist, young, straight), (yes feminist, old, gay), and (yes feminist, old, straight) are retained. To continue with our example, the judged probability for a yes answer to the feminism question equals the square length of the projection onto the subspace corresponding to this event. This is analogous to the R^2 produced by the person's beliefs being fitted to the feminist basis vectors with multiple regression. In our toy example, the judged probability for a yes answer to the feminism question equals the sum of the squared magnitudes of the amplitudes assigned to the four basis vectors (yes feminist, young, gay), (yes feminist, young, straight), (yes feminist, old, gay), and (yes feminist, old, straight).

In classic probability, this is computed by summing the probabilities of elementary events that form the union.

The residual difference (between the original state vector and the projection on the yes answer to feminism) equals the projection on the complementary subspace corresponding to a no answer on the feminism question. Thus, the projection on the yes answer is orthogonal (i.e., uncorrelated) to the projection on the no answer to the feminism question. The judged probability for concluding that the answer to the feminism question is no is determined from the projection on the no subspace, so that the no probability equals 1 minus the probability of saying yes. If the vector lies entirely in a subspace, then the squared projection of the vector onto the subspace will be 1; if the vector is perpendicular to the subspace, then the squared projection will be 0. Note that two subspaces are orthogonal if they correspond to mutually exclusive states of affairs.

This scheme provides a precise way to express Tversky and Kahneman's (1983) representativeness proposal in judgment. Tversky and Kahneman suggested that the conjunction fallacy arises because participants consider Linda to be a representative case of feminists. However, previously, representativeness has been interpreted as an intuition of how much the belief about Linda based on the story matched the prototype of feminists in the question. Now the representativeness can be interpreted as the projection or fit of a belief state vector about Linda to the subspace corresponding to knowledge about feminists. The squared length of the projection corresponds to the proportion of the belief state reproduced by the subspace. This generalization of the concept of representativeness makes a critical difference in its application.

State Revision

Suppose the person concludes that an event is a true fact. Our fourth postulate is that the original state vector changes to a new *conditional state vector*, which is the projection onto the subspace representing the event that is concluded to be true but now is normalized to have unit length. This is called Lüder's rule (Niestegge, 2008), and it is analogous to computation of a conditional probability in classic theory. Now we need to expand on what it means for a person to conclude that an event is true.

First, suppose that the judge is simply informed that the answer to the feminism question is yes. On the basis of this information, the amplitudes corresponding to (not feminist, young, gay), (not feminist, young, straight), (not feminist, old, gay), and (not feminist, old, straight) are set to 0, and the remaining amplitudes previously assigned to (yes feminist, young, gay), (yes feminist, young, straight), (yes feminist, old, gay), and (yes feminist, old, straight) are now divided by the length of this projection. Thus, the new conditional state vector has unit length so that the squared magnitudes of the new amplitudes assigned by the conditional state vector sum to 1. This corresponds to the normalization used to form conditional probabilities in classic probability theory.

Second, consider an example related to an inference problem used in the section "Quantitative Predictions for Order Effects on Inference" for the second application of quantum theory presented in this article. Suppose a juror is evaluating guilt or innocence, which depends on whether positive or negative evidence is present. Before the evidence, the belief state has amplitudes assigned to four different patterns (guilty, positive), (guilty, negative), (not guilty, positive), and (not guilty, negative). Now suppose the prosecutor presents positive evidence. On the basis of this information, the amplitudes corresponding to (guilty, negative) and (not guilty, negative) are set to 0, and the remaining amplitudes are now divided by the length of the resulting vector so that the squared magnitude of the amplitudes of the revised state sum to 1. Again, this is analogous to how conditional probabilities are revised by evidence according to Bayes' rule.

The conditional state vector is then used to answer subsequent questions. For example, if the person who is judging concludes that Linda is a feminist, then that person uses the state conditioned on this conclusion to judge the probability that Linda is also a bank teller. Following the earlier principles, the judged probability for an answer of yes to this next question is determined by the judge projecting the conditional state vector onto the bank teller subspace and squaring this projection. In other words, the judged conditional probability for yes to the bank teller question, given that Linda is a feminist, equals the squared length of the projection of the conditional state (given a yes response to feminism) on the bank teller subspace. Alternatively, the judged probability that Linda is a bank teller, before any conclusions about feminism is made, is simply determined by the original belief state that was initially generated by the Linda story.

Compatibility

At this point, we have not yet defined the basis vectors used to describe the bank teller question. In our toy example, we started by considering the feminism question, which we assumed called to mind, in addition to the feminism feature, other related features such as age, and sexual orientation (and other related features not included for simplicity). However, for answers to this question, we did not rely on any features about bank tellers or other professional occupations. In other words, in considering the feminism question, we deliberately chose not to include these features, because we assumed that the judge never thought much about these unusual combinations of questions (feminism and bank teller) before. Thus, these have to be treated as two separate questions answered one at a time. The judge may have thought about professions and their relations to salaries and other occupational features but more likely than not, he or she never thought enough about feminism and professions together to form precise beliefs about these particular combinations. Therefore, in order to answer the question about the profession of Linda, the judge would need to view the problem from a different perspective and evaluate this question using knowledge about the combinations of a different set of features relating to professions. To continue with the toy example, suppose the judge considers four professions (e.g., bank teller, doctor, insurance agent, and computer programmer) along with two levels of salary (low or high), forming eight feature patterns (each combination of four professions and two salary levels), and the eight basis vectors corresponding to these feature patterns span an eight-dimensional vector space.[3] The key idea is that the set of feature patterns used to evaluate the professions is inconsistent with the set used to think about feminism, in which case we believe that the two questions are incompatible, and they must be answered sequentially.

[3] It is possible that some features, say gender or college major, are compatible with both feminism and bank teller.

In this toy example, we used only eight dimensions (e.g. four professions combined with two levels of salary) for simplicity. In a more realistic model, a much larger dimensional space could be used. For example, suppose we used $N = 100$ dimensions to represent the space. Then to answer the question about feminism, we could represent age by, say, 25 age levels (young vs. old then representing only two coarse categories of these 25 age levels) combined with two levels of feminism and two levels of sexual orientation. To answer the question about professions, we could use 10 professions combined with 10 salary levels (low vs. high then representing two coarse categories of the 10 salary levels). By increasing the dimensionality of the space, we could allow for more refined levels of the features, which then could be categorized in various ways.

We could formalize the concept of incompatibility by using a vector space representation of knowledge—the same vector space can be represented by many different sets of basis vectors (corresponding to different sets of feature patterns), and the same exact state (vector) can be defined by different sets of basis vectors. Each (orthonormal) set of basis vectors corresponds to a description of the situation with a particular set of features and their combinations. But different sets of basis vectors correspond to different descriptions, with different sets of features and combinations, representing *complementary* ways of thinking about events. Formally, we can apply a unitary operator to transform one set of basis vectors to another. This is analogous to rotating the axes in multidimensional scaling (Carrol & Chang, 1970; Shepard, 1962) or multivariate signal detection theory (Lu & Dosher, 2008; Rotello, Macmillan, & Reeder, 2004). Psychologically, this corresponds to consideration of different perspectives or different points of view for answering questions. For example, in the second application to inference, we argue that a juror has to view evidence from a prosecutor's point of view and then view the evidence from a defense point of view and that it is not possible to hold these two incompatible views in mind at the same time. Later on when we present our quantitative test of the quantum model, we provide a detailed description of this rotation process. However, qualitative tests of the quantum model can be derived without making these specific assumptions. So first we examine these qualitative properties of the theory, and later we examine a more specific model.

These ideas lead us to an important fifth postulate about compatibility. If one can answer two questions using a common basis (i.e. the same basis vectors corresponding to a common set of feature patterns), then the questions are said to be compatible. If one must answer two questions using different bases (i.e. using different sets of basis vectors corresponding to a different set of feature patterns), then the two questions are said to be incompatible. To continue with our toy example, a question about age is compatible with a question about feminism, and a question about salary is compatible with a question about profession, but a question about feminism is incompatible with a question about profession. In order to make questions about feminism, age, and sexual orientation compatible with questions about profession and salary, a person would need to utilize a $(2 \times 2 \times 2) \cdot (4 \times 2) = 64$ dimensional space, with each basis vector representing one of the feature patterns produced by a unique combination of these five features. This is also the number of elementary events that would be required to represent the sample space in classic probability theory. Instead the person could utilize a lower eight-dimensional space by representing questions about feminism, age, and sexual orientation in a way that is incompatible with questions about occupation and salary. Thus, compatibility requires use of a higher dimensional space to form all combinations, whereas incompatibility can make use of a lower dimensional representation by a change in perspectives. Incompatibility provides an efficient and practical means for a cognitive system to deal with all sorts and varieties of questions. But a person must answer incompatible questions sequentially.

Suppose the question about feminism were incompatible with the question about the bank teller (e.g., the basis vectors are related by a rotation). Then, the basis vectors used to represent the feminism question would not be orthogonal to the basis vectors used to represent the bank teller question. For example, the inner product (analogous to correlation) between the (feminist, old, straight) basis vector and the (bank teller, low salary) basis vector could be positive. More generally, the subspace for feminism lies at oblique angles with respect to the subspace for bank teller. To see the implications of using incompatible events, consider again the feminist–bank teller problem. Initially, it is very difficult from the details of the Linda story to imagine Linda as a bank teller, but once the judge concludes that Linda is a feminist, the state is projected onto the feminism subspace, which eliminates many specific details about Linda story. (In projecting onto the feminism subspace, only those elements of the original Linda story that are consistent with feminism would be retained). From this more abstract projection on the feminism subspace, the person can imagine all sorts of professions for feminists (e.g., some feminists who are bank tellers). Clearly, some professions remain more probable than others, given the original story, but when thinking about the more general category of feminists, the judge can entertain possibilities that were extremely unlikely for Linda herself. For example, if the projection of Linda onto the feminism subspace produces a state corresponding to (old, straight, feminist), then the judge may have some past experiences associating this type of feminist with bank clerks who receive low salaries. The associations do not have to be strong, but they make it easier to imagine Linda as a feminist and to imagine a feminist as a bank teller, even though it was initially (before the feminist question) very difficult to imagine Linda as a bank teller. In this way, quantum probability also incorporates ideas related to the popular availability heuristic (Kahneman et al., 1982). The answer to the first question can increase the availability of events related to a second question.

Order Effects

Incompatibility is a source of order effects on judgments, and here is the critical point at which quantum probabilities deviate from classic probabilities. To see how order effects can happen, consider the special simple case in which the judged probability of a feminist given bank teller equals the judged probability of a bank teller given feminist (a simple geometric example is shown in Appendix A). One order is to judge whether Linda is a bank teller, and given that she is a bank teller, whether she is also a feminist; this probability is obtained by the product of the probability that Linda is a bank teller and the conditional probability that she is a feminist, given that she is a bank teller. On the basis of the Linda story, the judged probability for a yes response to bank teller question is close to 0, and when this is multiplied by the probability of feminist given bank teller, it is even closer to 0. The other order is to judge whether Linda is a feminist, and given that she is a feminist, whether she is also is a bank teller; this probability is obtained by the product of the probability that Linda is a feminist and the conditional probability that she is a bank teller, given that she is a feminist. On the basis of the Linda story, the judged probability that Linda is feminist is very high, and when this is multiplied by the same (as assumed) conditional probability of bank teller given feminist, then the product produced by the feminist–bank teller order must be greater than the product produced by the bank teller–feminist order. This order effect cannot happen with classic probability theory (because these two orders produce the same joint probability), but in Appendix A, a very simple geometric and numerical example of this order effect is provided through quantum theory. In sum, the indirect path of thought from Linda to feminism to bank teller is a fair possibility even though the direct path from Linda to bank teller is almost impossible. In other words, asking first about feminism increases the availability of later thoughts about bank tellers.

What is the evidence for order effects, and is there any reason to think that quantum theory provides a good explanation for them? It is well established that presentation order affects human probability judgments (Hogarth & Einhorn, 1992). In the next section headed "Qualitative Predictions for Conjunction and Disjunction Questions," we present evidence for question-order effects on conjunction fallacies (Stolarz-Fantino, Fantino, Zizzo, & Wen, 2003), and we account for them with the quantum model. In the section headed "Quantitative Predictions for Order Effects on Inference," we successfully fit the quantum model to the results of a new study of order effects on inference (Trueblood & Busemeyer, in press). In the section headed "Other Applications and Extensions," we report some surprisingly accurate predictions of the quantum model for question order effects in attitude questionnaire research (Moore, 2002).

Theoretical Postulates

In this section, we summarize the five quantum postulates (von Neumann, 1932) more formally, and we compare them to the corresponding postulates of classic probability (Kolmogorov, 1933).[4] At a conceptual level, a key difference is that classic theory relies on a set theoretic representation, whereas quantum theory relies on a geometric representation.

1. Classic theory begins with the concept of a sample space, which is a set that contains all the events. Suppose (for simplicity) the cardinality of this sample space is N so that the sample space is comprised of N elementary events or points. Classic theory defines the state of a system (e.g., all of a person's beliefs) by a probability function $\boldsymbol{p}$ in which a probability is assigned (a real number between 0 and 1 inclusive) to each elementary event, and the probabilities assigned by $\boldsymbol{p}$ sum to 1. If E_i is an elementary event, then $\boldsymbol{p}(E_i)$ is the probability assigned to this event.

Quantum theory uses an N dimensional vector space to contain all the events. The vector space is described by a set of N (orthonormal) basis vectors, and each basis vector corresponds to an elementary event. Quantum theory defines the state of a system (e.g., a person's belief state) by a state vector, denoted $|\psi\rangle$, which assigns an amplitude to each basis vector, and the state vector has unit length. The amplitude assigned to a basis vector, such as the basis vector $|E_i\rangle$, equals the inner product between the basis vector and state vector, denoted $\langle E_i|\psi\rangle$.

2. Classic theory defines a general event as a subset of the sample space. The event A is defined by the union of the elementary events that it contains: $A = \cup_{i \in A} E_i$.

Quantum theory defines a general event as a subspace of the vector space, and each subspace corresponds to a projector. The projection of a state onto a ray spanned by basis vector $|E_i\rangle$ equals $\boldsymbol{P}_i|\psi\rangle = |E_i\rangle \cdot \langle E_i|\psi\rangle$, where $\langle E_i|\psi\rangle$ is an inner product. The projector for this ray equals $\boldsymbol{P}_i = |E_i\rangle \cdot \langle E_i|$, which is an outer product. The projector for event A spanned by a subset $\{|E_1\rangle, \ldots, |E_k\rangle\}$ of orthonormal basis vectors equals $\boldsymbol{P}_A = \Sigma_{i \in \mathrm{A}} \boldsymbol{P}_i$.

3. In classic theory, the probability of an event equals the sum of the probabilities assigned to the elementary events contained in the subset. If A is an event, then $\boldsymbol{p}(A) = \Sigma_{i \in A}\, \boldsymbol{p}(E_i)$, where E_i is an elementary event.

In quantum theory, the probability of event A equals the squared length of the projection of the state onto the corresponding subspace. If $\boldsymbol{P}_A$ is the projector for subspace A, then $\boldsymbol{P}_A|\psi\rangle$ is the projection, and the probability of event A equals $\|\boldsymbol{P}_A|\psi\rangle\|^2 = \Sigma_{i \in A}|\langle E_i|\psi\rangle|^2$.

4. Suppose that event A is concluded to be a true. Given this fact, classic theory changes the original probability function $\boldsymbol{p}$ into a new conditional probability function $\boldsymbol{p}_A$ by the classic rule $\boldsymbol{p}_A(B) = \boldsymbol{p}(A \cap B)/\boldsymbol{p}(A)$. This conditional probability is more commonly written as $\boldsymbol{p}(B|A)$.

Quantum theory changes the original state $|\psi\rangle$ into a new conditional state $|\psi_\mathrm{A}\rangle$ by what is known as Lüder's rule: $|\psi_\mathrm{A}\rangle = \boldsymbol{P}_A|\psi\rangle/\|\boldsymbol{P}_A|\psi\rangle\|$. The probability of event B given event A is known to be true equals $\|\boldsymbol{P}_B|\psi_\mathrm{A}\rangle\|^2 = \|\boldsymbol{P}_B\boldsymbol{P}_A|\psi\rangle\|^2/\|\boldsymbol{P}_A|\psi\rangle\|^2$.

5. Classic probability assumes a single common sample space from which all events are defined. In other words, all events are compatible. Two events from the sample space can always be intersected to form a single event in the sample space.

According to quantum theory, the same exact state can be represented by more than one basis. This allows for two kinds of events: compatible versus incompatible. If two events A and B can be described by a common basis, then they are compatible, and the projectors commute ($\boldsymbol{P}_B\boldsymbol{P}_A = \boldsymbol{P}_A\boldsymbol{P}_B$). When two events are compatible, the two subspaces can be combined to form a single event represented by a single projector. If these two events cannot be described by a common basis, then they are incompatible, and the projectors do not commute ($\boldsymbol{P}_B\boldsymbol{P}_A \neq \boldsymbol{P}_A\boldsymbol{P}_B$). Formally, the basis vectors used to describe event A are a unitary transformation of the basis vectors used to describe event B. If event A is incompatible with event B, then the pair of events cannot be represented by a single projector, and they have to be evaluated sequentially.

If all events are compatible, then quantum theory is equivalent to classic theory (Gudder, 1988). Thus, incompatibility is a key new idea that distinguishes quantum and classic theories.

A short and simple tutorial of the quantum postulates appears in Appendix A (see also Busemeyer & Bruza, in press). These same five quantum postulates are consistently used in both of applications presented in this article.

[4] Both theories are applicable to the continuum, but for simplicity, we will limit this presentation to the finite case.

Implications

From these postulates, we can also derive new implications for both classic theory and quantum theory. First, classic theory defines the negated event, $\sim A$, as the complement of the subset for A, and its probability equals $p(\sim A) = 1 - p(A)$. Quantum theory defines the negation of an event as the subspace orthogonal to the event A, represented by the projector $P_{\sim A} = I - P_A$, where I is the identity operator ($I \cdot |\psi\rangle = |\psi\rangle$). Then the probability of $\sim A$ equals $||P_{\sim A}|\psi\rangle||^2 = 1 - ||P_A|\psi\rangle||^2$.

Classic theory defines the probability of the conjunction A and B as the probability $p(A) \cdot p(B|A) = p(A \cap B)$, but because $p(A \cap B) = p(B \cap A)$, this is also equal to $p(B \cap A) = p(B) \cdot p(A|B)$, which equals the probability of B and A. Thus, order does not matter, and it makes sense to consider this a conjunction of events A and B without regard to order. In quantum theory, order does matter, and the events in question have to be evaluated as a sequence (Franco, 2009): According to Lüder's rule, the probability of event A and then event B equals $||P_A|\psi\rangle||^2 \cdot ||P_B|\psi_A\rangle||^2 = ||P_B P_A|\psi\rangle||^2$. If the questions are compatible, so that the projectors commute, then $||P_B P_A|\psi\rangle||^2 = ||P_A P_B|\psi\rangle||^2$, order does not matter, and the conjunction can be interpreted in the same way as in classic theory. But if the events are incompatible, then the projectors do not commute, and $||P_B P_A|\psi\rangle||^2 \neq ||P_A P_B|\psi\rangle||^2$. In other words, asking a sequence of two incompatible questions corresponds to the person's starting from his or her initial belief state, projecting onto the subspace corresponding to the answer to the first question, and then projecting the resulting state onto the subspace corresponding to the answer to the second question. Reversing the order of these projections can lead to different results. Psychologically, such order effects can be interpreted in the sense that the first statement changes a person's viewpoint for evaluating the second statement. Given the prevalence of order effects on human probability judgments (Hogarth & Einhorn, 1992), this is an important advantage for quantum theory.

The classic probability for the disjunction of two events A and B is the probability assigned to the union of the two subsets representing the two events, which equals

$$\begin{aligned} p(A \cup B) &= p(A \cap B) + p(A \cap \sim B) + p(\sim A \cap B) \\ &= p(A) + p(\sim A \cap B) \\ &= 1 - p(\sim A \cap \sim B) \end{aligned}$$

The last form, $1 - p(\sim A \cap \sim B)$, is commonly used because it extends most easily to disjunctions involving more than two events. It is clear that $p(A \cup B) = p(B \cup A)$ so the order does not matter for classic theory, and so it makes sense to define this as a disjunction of events A and B. Quantum theory assigns a probability to the sequence A or then B equal to

$$||P_B P_A|\psi\rangle||^2 + ||P_{\sim B} P_A|\psi\rangle||^2 + ||P_B P_{\sim A}|\psi\rangle||^2 = \\ ||P_A|\psi\rangle||^2 + ||P_B P_{\sim A}|\psi\rangle||^2 = 1 - ||P_{\sim B} P_{\sim A}|\psi\rangle||^2.$$

Again, we use the form $1 - ||P_{\sim B} P_{\sim A}|\psi\rangle||^2$ because this extends most easily to disjunctions involving more than two events. This form also makes it is clear that order does matter for quantum theory when the events are incompatible.

The classic probability rule for inferring a hypothesis on the basis of new evidence is Bayes' rule, which is essentially derived from the definition of a conditional probability. A quantum analogue of Bayes' rule is obtained from Postulate 4, which is known as Lüder's rule. In the section on quantitative tests, we provide a more detailed description of the quantum model applied to inference problems.

Clearly, the sequential order in which questions are considered is a major aspect of the application of quantum probability to human judgments. Any application of quantum theory must specify this order. In the section on quantitative tests, we present an experiment in which we directly manipulated this order. However, in other problems, the order of processing is not controlled, and the individual is free to choose an order. Sometimes, a causal order is implied by the questions that are being asked. For example, when asked to judge the likelihood that "the cigarette tax will increase and a decrease in teenage smoking will occur," it is natural to assume that the causal event "increase in cigarette tax" is processed first. But for questions with no causal order, such as feminist and bank teller, we assumed that individuals tend to consider the more likely of the two events first. Note that a person can easily rank order the likelihood of individual events (being a feminist vs. being a bank teller) before going through the more extensive process of estimating the probability of a sequence of events (being a feminist and then being a bank teller conditioned on the answer to the question about feminism). There are several ways to justify the assumption that the more likely event is processed first. One is that the more likely event matches the story better and so these features are more quickly retrieved and available for consideration. A second reason is that individuals sometimes conform to a confirmation bias (Wason, 1960) and seek questions that are likely to be confirmed first. Finally, our assumption of considering the more likely event first is analogous to the assumption that most important cues are considered first in probability inferences (Gigerenzer & Goldstein, 1996). For more than two events, the same principle applies, and the events are processed in rank order of likelihood.

Summary of the Quantum Judgment Model

When given a story, the judge forms a belief state that is represented by a state vector in a possibly high-dimensional (feature) vector space. An answer to a question about an event is represented by a subspace of this vector space. The judged probability of an answer to a question equals the squared projection of the belief state onto the subspace representing the question. Two questions are incompatible if the two subspaces require the use of different sets of basis vectors. If the events involved in conjunction and disjunction questions are incompatible, then they must be processed sequentially, and the more likely of the two questions is processed first. In the latter case, the conclusion from the first question changes the state and affects the second question, producing order effects which in turn cause conjunction and disjunction errors. Judgments about hypotheses are revised according to Lüder's rule, in which the normalized projection is used to update the state on the basis of the observed evidence. If the sequence of evidence involves incompatible events, then the inference judgments exhibit order effects.

Now we are prepared to apply the quantum judgment model to conjunction and disjunction errors and related phenomena. Later we present a quantitative test for order effects on inference. The qualitative tests are important because they do not require specific assumptions regarding the dimension of the feature space, the amplitudes assigned to the initial state, or the relations between the incompatible features. The quantitative test is important to describe how to make these specifications as well as to examine the capability of the model to make precise predictions in comparison with previous models.

Qualitative Predictions for Conjunction and Disjunction Questions

Conjunction and Disjunction Fallacies

A large body of empirical literature establishes the findings of both conjunction (Gavanski & Roskos-Ewoldsen, 1991; Sides, Osherson, Bonini, & Viale, 2002; Stolarz-Fantino et al., 2003; Tversky & Kahneman, 1983; Wedell & Moro, 2008) and disjunction fallacies (Bar-Hillel & Neter, 1993; Carlson & Yates, 1989; Fisk, 2002). These findings are very robust and occur with various types of stories (e.g., female philosophy students who are now feminist bank tellers, high-pressure business men who are older than 50 and have heart disease, Norwegian students with blue eyes and blond hair, state legislatures that increase cigarette taxes and reduce teenage smoking), and various types of response measures (e.g., choice, ranking, probability ratings, monetary bids) (Sides et al., 2002; Wedell & Moro, 2008). These fallacies are not simply the result of misunderstanding the meaning of probability, because they even occur with bets in which the word *probability* never appears. For example, Sides et al. (2002) found that participants preferred to bet on the future event "cigarette tax will increase and teenage smoking will decrease" over betting on the single event "teenage smoking will decrease."

Moreover, both fallacies have been observed to occur at the same time (Morier & Borgida, 1984). For example, Morier and Borgida (1984) used the Linda story and found that the mean probability judgments were ordered as follows, where $J(\mathrm{A})$ denotes the mean judgment for event: J(feminist) = .83 > J(feminist or bank teller) = .60 > J(feminist and bank teller) = .36 > J(bank teller) = .26 (N = 64 observations per mean, and all pair wise differences are statistically significant). These results violate classic probability theory, which is the reason they are called fallacies.

The quantum model starts with a state vector $|\psi\rangle$ that represents the participant's belief state after reading the Linda story; the event "yes to the feminist question" is represented by a subspace corresponding to the projector $\boldsymbol{P}_F$; the event "yes to the bank teller question" is represented by an incompatible subspace corresponding to the projector $\boldsymbol{P}_B$; and finally, the event "no to the feminist question" is represented by an orthogonal subspace corresponding to the $\boldsymbol{P}_{\sim F}$ so that $\boldsymbol{P}_F + \boldsymbol{P}_{\sim F} = \boldsymbol{I}$. Our key assumption is that the projector $\boldsymbol{P}_F$ does not commute with the projector $\boldsymbol{P}_B$ (Franco, 2009). When considering a conjunction, the more likely event is considered first, and because "yes to feminist" is more likely than "yes to bank teller," the judged probability of the event "feminist and bank teller" equals $||\boldsymbol{P}_F|\psi\rangle||^2 \cdot ||\boldsymbol{P}_B|\psi_F\rangle||^2 = ||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2$.

For the conjunction fallacy, we need to compare the probability for the single event $||\boldsymbol{P}_B|\psi\rangle||^2$ with the probability for the conjunction $||\boldsymbol{P}_B\boldsymbol{P}_{\mathrm{F}}|\psi\rangle||^2$, and a conjunction fallacy is predicted when $||\boldsymbol{P}_B\boldsymbol{P}_{\mathrm{F}}|\psi\rangle||^2 > ||\boldsymbol{P}_B|\psi\rangle||^2$. To do this comparison, we decompose the quantum probability of the bank teller event by expanding this event as follows:

$$\begin{aligned}||\boldsymbol{P}_B|\psi\rangle||^2 &= ||\boldsymbol{P}_B \cdot \boldsymbol{I}|\psi\rangle||^2 = ||\boldsymbol{P}_B(\boldsymbol{P}_F + \boldsymbol{P}_{\sim F}) \cdot |\psi\rangle||^2 = \\ &||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle + \boldsymbol{P}_B\boldsymbol{P}_{\sim F}|\psi\rangle||^2 = \\ &||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2 + ||\boldsymbol{P}_B\boldsymbol{P}_{\sim F}|\psi\rangle||^2 + \langle\psi_{B,\sim F}|\psi_{B,F}\rangle + \langle\psi_{B,F}|\psi_{B,\sim F}\rangle,\end{aligned} \tag{1}$$

where $|\psi_{B,F}\rangle = \boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle$ and $|\psi_{B,\sim F}\rangle = \boldsymbol{P}_B\boldsymbol{P}_{\sim F}|\psi\rangle$. The last term on the right hand side of Equation 1, denoted $\delta_B = \langle\psi_{B,\sim F}|\psi_{B,F}\rangle + \langle\psi_{B,F}|\psi_{B,\sim F}\rangle$, is called the interference term for the bank teller event.[5] There is another interference, $\delta_{\sim B}$, corresponding to the probability $||\boldsymbol{P}_{\sim B}|\psi\rangle||^2$, but the two interferences must sum to 0, so that $(\delta_B + \delta_{\sim B}) = 0$ (see Appendix B). Thus, one of these interference terms must be negative, and we argue that $\delta_B < 0$, because this makes it less likely to judge that Linda is a bank teller. Also, the story suggests that the probability $||\boldsymbol{P}_B\boldsymbol{P}_{\sim F}|\psi\rangle||^2$ that Linda is not a feminist and is a bank teller is small. Under these conditions, the interference can be sufficiently negative so that $\delta_B < -\,||\boldsymbol{P}_B\boldsymbol{P}_{\sim F}|\psi\rangle||^2$, and consequently $(||\boldsymbol{P}_B\boldsymbol{P}_{\sim F}|\psi\rangle||^2 + \delta_B) < 0$, which implies $||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2 > ||\boldsymbol{P}_B|\psi\rangle||^2 = ||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2 - |(||\boldsymbol{P}_B\boldsymbol{P}_{\sim F}|\psi\rangle||^2 + \delta_B)|$ as required to explain the conjunction fallacy.

The interference, δ_B, is determined by the inner product of two projections: One is the projection $|\psi_{B,F}\rangle$ of the initial state first on the "she is a feminist" subspace and then onto the "she is a bank teller" subspace; the second is the projection $|\psi_{B,\sim F}\rangle$ of the initial state first onto the "she is not a feminist" subspace and then onto the "she is a bank teller" subspace. Recall that the inner product is analogous to the correlation between two vectors. For many judges, the features matching "feminist bank teller" may be negatively correlated (pointing in a dissimilar direction) with the features matching "not feminist bank teller," thus producing negative interference.

Next, consider the disjunction probability, in which the person judges the probability of saying no to "Linda is neither a bank teller nor a feminist." First note that when one is processing the two events "Linda is *not* a bank teller" versus "Linda is *not* a feminist," the former is more likely than the latter, and so the former is processed first. In this case, we need to compare the single event $||\boldsymbol{P}_F|\psi\rangle||^2 = 1 - ||\boldsymbol{P}_{\sim F}|\psi\rangle||^2$ with the probability for the disjunction $1 - ||\boldsymbol{P}_{\sim F}\,\boldsymbol{P}_{\sim B}|\psi\rangle||^2$, and disjunction fallacy is predicted when $||\boldsymbol{P}_F|\psi\rangle||^2 = 1 - ||\boldsymbol{P}_{\sim F}|\psi\rangle||^2 > 1 - ||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\sim B}|\psi\rangle||^2$, or equivalently when $||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\sim B}|\psi\rangle||^2 > ||\boldsymbol{P}_{\sim F}|\psi\rangle||^2$. To do this, we mathematically decompose the quantum probability that Linda is not a feminist as follows:

$$\begin{aligned}||\boldsymbol{P}_{\sim F}|\psi\rangle||^2 = ||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\sim B}|\psi\rangle||^2 + ||\boldsymbol{P}_{\sim F}\boldsymbol{P}_B|\psi\rangle||^2 +& \\ \langle\psi_{\sim F,B}|\psi_{\sim F,\sim B}\rangle + \langle\psi_{\sim F,\sim B}|\psi_{\sim F,B}\rangle&.\end{aligned} \tag{2}$$

In this case, the interference is $\delta_{\sim F} = \langle\psi_{\sim F,B}|\psi_{\sim F,\sim B}\rangle + \langle\psi_{\sim F,\sim B}|\psi_{\sim F,B}\rangle$. Once again, there is a corresponding interference δ_F for $||\boldsymbol{P}_F|\psi\rangle||^2$, and these two interferences must sum to 0 (δ_F +

[5] The interference equals an inner product plus its conjugate, and so it is a real number. Cross-product interference terms also arise in other applications of decision theory (Luce, Ng, Marley, & Aczel, 2008).

$\delta_{\sim F}) = 0$ (see Appendix B). Thus, one of these two interference terms must be negative, and we argue that $\delta_F > 0$, because this makes it more likely that Linda is a feminist. If the interference for $\delta_{\sim F}$ is sufficiently negative so that $(||\boldsymbol{P}_{\sim F}\boldsymbol{P}_B|\psi\rangle||^2 + \delta_{\sim F}) < 0$, then $||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\sim B}|\psi\rangle||^2 > ||\boldsymbol{P}_{\sim F}|\psi\rangle||^2 = ||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\sim B}|\psi\rangle||^2 - |(||\boldsymbol{P}_{\sim F}\boldsymbol{P}_B|\psi\rangle||^2 + \delta_{\sim F})|$ as required to explain the disjunction fallacy.

The interference, $\delta_{\sim F}$, is determined by the inner product of two projections: One is the projection $|\psi_{\sim F,\ \sim B}\rangle$ of the initial state first on the "she is a not a bank teller" subspace and then onto "she is a not a feminist" subspace; the second is the projection $|\psi_{\sim F,\ B}\rangle$ of the initial state first onto "she is a bank teller" subspace and then onto "she is not a feminist" subspace. For many judges, the features matching "not a bank teller and not a feminist" may be negatively correlated (pointing in a dissimilar direction) with the features matching "bank teller and not a feminist," thus producing negative interference.

To complete the analysis of conjunction and disjunction fallacies, we must check to see what the quantum model predicts for the remaining ordinal relations reported by Morier and Borgida (1984). The quantity $||\boldsymbol{P}_B|\psi_F\rangle||^2$ is a probability so that $1 \geq ||\boldsymbol{P}_B|\psi_F\rangle||^2 \geq 0$, and it mathematically follows that

$$||\boldsymbol{P}_F|\psi\rangle||^2 \cdot 1 \;\geq\; ||\boldsymbol{P}_F|\psi\rangle||^2 \cdot ||\boldsymbol{P}_B|\psi_F\rangle||^2 = ||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2. \quad (3)$$

Therefore, the quantum model must predict that the event 'Linda is a feminist' is judged at least as likely as the conjunction.

Now consider the order of the conjunction versus disjunction. The Linda story is designed so that the probability $||\boldsymbol{P}_{\sim B}\,\boldsymbol{P}_F|\psi\rangle||^2$ corresponding to the "Linda is a feminist and she is not a bank teller" conjunction is more likely than the probability $||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\sim B}|\psi\rangle||^2$ corresponding to "Linda is not a bank teller and she is not a feminist" conjunction.[6] This design implies that

$$||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\sim B}|\psi\rangle||^2 < ||\boldsymbol{P}_{\sim B}\,\boldsymbol{P}_F|\psi\rangle||^2 + ||\boldsymbol{P}_{\sim F}|\psi\rangle||^2,$$

but it is also true that

$$||\boldsymbol{P}_{\sim B}\boldsymbol{P}_F|\psi\rangle||^2 + ||\boldsymbol{P}_{\sim F}|\psi\rangle||^2 = 1 - ||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2$$
$$\rightarrow ||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2 < 1 - ||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\sim B}|\psi\rangle||^2, \quad (4)$$

and Equation 4 implies that the conjunction is less likely than the disjunction. This last prediction is important because even though human judgments tend to satisfy this constraint, there is no requirement for them to do so. Therefore, if both the conjunction and disjunction fallacies occur, then the quantum model must produce the order reported by Morier and Borgida (1984). This is not true of theoretical explanations that we present later, which are free to produce consistent or inconsistent orderings of disjunction and conjunction events depending on free parameters.

Now the quantum model is forced to make another strong qualitative prediction. Both conjunction and disjunction effects require the events to be incompatible; for if the events are compatible, then there is no interference (see Appendix B). But incompatible events produce order effects. To simultaneously explain both the conjunction and disjunction fallacies, the model requires the following order constraint (see Appendix B): $||\boldsymbol{P}_F\boldsymbol{P}_B|\psi\rangle||^2 < ||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2$. This constraint exactly fits our psychological explanation of order effects that we presented earlier—the first likely event increases availability of the second unlikely event. In other words, processing the likely event first facilitates retrieval of relevant thoughts for the second event, which then increases the likelihood of the conjunction. By contrast, if the unlikely event is processed first, it is hard to imagine any thoughts at all in favor of this unlikely event from the very beginning, which lowers the probability of the conjunction.

Order Effects

The quantum explanation for conjunction and disjunction errors must predict that order of processing is a critical factor for determining whether or not the fallacy will occur. One effective way to manipulate this order is to ask people to judge the conjunction first or last when judging the likelihood of events. For example, after hearing a story, a person could be asked to judge the unlikely event U first and then judge the conjunction U and L, or the person could be asked these questions in the opposite order. The quantum model predicts smaller effects when the conjunction is presented last, because in this case, the person evaluates the probability $||\boldsymbol{P}_U|\psi\rangle||^2$ for the unlikely event first and so is encouraged to use this probability estimate to determine the conjunction probability for U and L. But in the latter case, it must be predicted that $||\boldsymbol{P}_U|\psi\rangle||^2 \cdot ||\boldsymbol{P}_L|\psi_U\rangle||^2 = ||\boldsymbol{P}_L\boldsymbol{P}_U|\psi\rangle||^2$, and mathematically it follows that $||\boldsymbol{P}_U|\psi\rangle||^2 \cdot 1 \geq ||\boldsymbol{P}_U|\psi\rangle||^2 \cdot ||\boldsymbol{P}_L|\psi_U\rangle||^2$; therefore, no conjunction error can occur. This reduction does not happen in the reverse order when the conjunction is evaluated first, because in this case, the "start with the higher probability event first" rule applies, and the conjunction is always computed from the opposite order $||\boldsymbol{P}_L|\psi\rangle||^2 \cdot ||\boldsymbol{P}_U|\psi_L\rangle||^2 = ||\boldsymbol{P}_U\boldsymbol{P}_L|\psi\rangle||^2$, which produces conjunction errors as given by Equation 1.

In fact, conjunction errors are significantly larger when the conjunction is rated first as opposed to being rated last (Gavanski & Roskos-Ewoldsen, 1991; Stolarz-Fantino et al., 2003). In the study by Stolarz-Fantino et al. (2003), when the single judgment for the unlikely event was made first, the mean judgment for the unlikely event was $J(U) = .14$ compared with $J(U \text{ and } L) = .17$ for the conjunction ($N = 105$, not significantly different), but when the conjunction was rated first, the mean judgment for the conjunction was $J(U \text{ and } L) = .26$ compared with $J(U) = .18$ for the unlikely event ($N = 102$, significantly different). Similar robust and large effects of order were reported by Gavanski and Roskos-Ewoldsen (1991). This order effect also explains why ratings produce fewer errors than rank orders (Wedell & Moro, 2008)—the latter procedure does not require any estimates of the constituent events ahead of time.

Averaging Type Errors

One of the earliest explanations for conjunction and disjunction fallacies is that these judgments are based on the average of the likelihoods of the individual events (Abelson, Leddo, & Gross, 1987; Fantino, Kulik, & Stolarz-Fantino, 1997; Nilsson, 2008; Wyer, 1976). For example, if one averages the likely event L with an unlikely event U, then the average must lie between these two likelihoods. If one assumes that more weight is placed on the

[6] In fact, the empirical results are that $||\boldsymbol{P}_{\sim B}\boldsymbol{P}_F|\psi\rangle||^2 = .47 > ||\boldsymbol{P}_{\sim F \sim B}|\psi\rangle||^2 = .40$.

unlikely event for the conjunctive question and that more weight is placed on the likely event for the disjunction question, then this model can accommodate both fallacies at the same time.

An important source of support for the averaging model is another fallacy called the *averaging error* (Fantino et al., 1997). This finding involves a story followed by questions that are judged to be unlikely (U), moderately likely (M), and very likely (L) to be true on the basis of the story. These questions produce the following reversal in the order for the mean judgments: $J(U) < J(U$ and $M)$ but $J(M$ and $L) < J(L)$, which again violates classic probability theory.

This finding also rules out an additive model in which judgments are assumed to be made by adding (rather than averaging) the signed evidence of individual events (Yates & Carlson, 1986). According to an additive model, if $J(M$ and $L) < J(L)$, then signed evidence for M is negative, but if this is true, then $J(U) > J(U$ and $M)$ should be found, but the opposite occurs.

For these unlikely (U), moderately likely (M), and very likely (L) type of questions, the quantum model must always predict the order $||\boldsymbol{P}_L|\psi\rangle||^2 > ||\boldsymbol{P}_L|\psi\rangle||^2 \cdot ||\boldsymbol{P}_U|\psi_L\rangle||^2 = ||\boldsymbol{P}_U\boldsymbol{P}_L|\psi\rangle||^2$, which satisfies the second inequality that forms the averaging error. The first inequality in the averaging error is simply a conjunction fallacy, $||\boldsymbol{P}_U\boldsymbol{P}_M|\psi\rangle||^2 > ||\boldsymbol{P}_U|\psi\rangle||^2$, which we have already explained using negative interference (see Equation 1). Thus, the quantum model also explains this averaging error.

Event Likelihoods

In general, the interference term, δ, will depend on both the story and the question. For the Linda story, the event "Linda is a feminist" was designed to seem likely (producing a large projection for the likely event L, denoted $||\boldsymbol{P}_L|\psi\rangle||^2$), whereas the event "Linda is a bank teller" was designed to be unlikely (producing a small projection for the unlikely event U, denoted $||\boldsymbol{P}_U|\psi\rangle||^2$). From Equation 3, it follows that the size of the conjunction error is bounded by

$$||\boldsymbol{P}_L|\psi\rangle||^2 \geq ||\boldsymbol{P}_U\boldsymbol{P}_L|\psi\rangle||^2 \geq ||\boldsymbol{P}_U|\psi\rangle||^2, \tag{5}$$

and it shrinks to 0 if $||\boldsymbol{P}_L|\psi\rangle||^2 = ||\boldsymbol{P}_U|\psi\rangle||^2$. In fact, researchers find that both fallacies depend on the difference between the likelihoods of the two events (Gavanski & Roskos-Ewoldsen, 1991; Wells, 1985; Yates & Carlson, 1986). For example, the mean estimates reported by Gavanski and Roskos-Ewoldsen (1991) were $J(A) = .28$, $J(B) = .19$, and $J(A$ and $B) = .18$ when both events (A, B) were unlikely; $J(A) = .77$, $J(B) = .23$, and $J(A$ and $B)=.38$ when event A was unlikely and event B was likely; and $J(A) = .76$, $J(B) = .69$, and $J(A$ and $B) = .67$ when both events (A, B) were likely. The mean estimates reported by Fisk (2002) were $J(A) = .36$, $J(B) = .14$, and $J(A$ or $B) = .27$ when both events (A, B) were unlikely; $J(A) = .23$, $J(B) = .73$, and $J(A$ or $B) = .59$ when event A was unlikely and event B was likely; and $J(A) = .80$, $J(B) = .62$, and $J(A$ or $B) = .75$ when both events (A, B) were likely.

The constraint on the judgments imposed by Equation 5 implies another strong prediction of the quantum model. Only a single conjunction error can occur—that is, when the conjunction is judged more likely than the lower likelihood event. In examinations of the mean or median of probability estimates, this prediction is generally supported (Gavanski & Roskos-Ewoldsen, 1991). Furthermore, it is also generally found that single conjunction errors are overwhelmingly most frequent (Yates & Carlson, 1986). Double conjunction errors are infrequent, but they occasionally occur with two highly likely events, and the latter easily could be caused by judgments errors when all the events are rated almost equally high (Costello, 2009).

An averaging model also predicts that conjunction and disjunction errors are larger for the (unlikely, likely) combination of events and that only single conjunction errors can occur. But the quantum and averaging models make distinct predictions for the extreme case of complementary events A and not A. For complementary events, the quantum model must predict that the probability of the conjunction is 0 ($||\boldsymbol{P}_{\sim A}\boldsymbol{P}_A|\psi\rangle||^2 = 0$), and the probability of the disjunction is 1 ($1 - ||\boldsymbol{P}_{\sim A}\boldsymbol{P}_A|\psi\rangle||^2 = 1-0$). Thus, the quantum model must predict no conjunction or disjunction errors for this extreme case, except those errors produced accidently by random error (Costello, 2009). However, an averaging model must predict that these effects remain as large as ever for this extreme condition because the average must always fall between the likelihood of A and the likelihood of not A. For example, if A is highly likely to be true, then $\sim A$ is highly likely to be false, and the averaging model predicts that the conjunction will fall between these two mutually exclusive events. In fact, conjunction and disjunction errors are greatly reduced when the events are mutually exclusive (Wolfe & Reyna, 2010).

Event Dependencies

The quantum model makes another strong prediction concerning the effect of dependencies between events on the conjunction fallacy. In classic theory, if $\boldsymbol{p}_L(U) > \boldsymbol{P}(U)$ so that knowledge of event L increases the probability of event U, then there is a positive dependency of event L on event U. According to the quantum model, an event L has a positive dependency on an event U if $||\boldsymbol{P}_U|\psi_L\rangle||^2 > ||\boldsymbol{P}_U|\psi\rangle||^2$. To produce a conjunction fallacy, the quantum model requires

$$||\boldsymbol{P}_U\boldsymbol{P}_L|\psi\rangle||^2 = ||\boldsymbol{P}_L|\psi\rangle||^2 \cdot ||\boldsymbol{P}_U|\psi_L\rangle||^2 \geq ||\boldsymbol{P}_U|\psi\rangle||^2 \rightarrow$$

$$||\boldsymbol{P}_U|\psi_L\rangle||^2 \geq ||\boldsymbol{P}_U|\psi\rangle||^2/||\boldsymbol{P}_L|\psi\rangle||^2 \geq ||\boldsymbol{P}_U|\psi\rangle||^2. \tag{6}$$

Thus, the quantum model is forced to predict that conjunction errors occur only when there is a positive dependency of the unlikely event on the likely event. For example, according to the quantum model, knowing that Linda is a feminist increases the likelihood that she is a bank teller. In fact, the presence of dependencies between events A and B has been shown to affect the rate of conjunction fallacies—a positive conditional dependency generally increases the frequency of conjunction errors (Fisk, 2002).

Both classic and quantum theories predict that dependencies between events strongly influence the probability judgment for a sequence of events. This property is important because the averaging model, which simply averages the likelihoods of the individual events, fails to consider event dependencies. Not surprisingly, human judgments are strongly influenced by event dependencies, as cleverly shown by Miyamoto, Gonzalez, and Tu (1995). In their design, judges evaluated four conjunctions of events: A and X, A and Y, B and X, and B and Y. Contrary to an averaging model, violations of independence were observed: $J(A$ and $X) > J(B$ and $X)$ but $J(A$ and $Y) < J(B$ and $Y)$. According to the averaging model, the common event (X) in the first comparison cancels out, and so the first inequality implies that event A is more likely than event B; similarly, the common event (Y) in the second comparison cancels out, and so the order established by the first comparison should be maintained for the second comparison (but it is not). According to both the classic and quantum models, the probability of event A conditioned on the state X is larger than event B, but the opposite occurs conditioned on the state Y.

Event Relationships

One of the major criticisms of the representativeness heuristic concerns the effect of manipulation of the relatedness between the two events. Suppose two stories are told, one about the liberal college student named Linda and another about an intellectual but somewhat boring man named Bill. After hearing both stories, the judge could be asked two *related* questions concerning the same person such as "Is Linda a feminist, and is Linda a bank teller?" Alternatively, the judge could be asked two *unrelated* questions such as "Is Linda a feminist, and does Bill play jazz for a hobby?" It turns out that the conjunction fallacy is almost equally strong for related and unrelated questions (Gavanski & Roskos-Ewoldsen, 1991; Yates & Carlson, 1986). This finding has been interpreted as evidence against the representativeness heuristic and evidence for a simple averaging rule. But this is not a problem for the quantum interpretation of the representativeness heuristic.

The quantum model predicts that conjunction errors only occur when there is interference, and interference can only occur when the two projectors do not commute. Thus, the key question is whether the projectors commute; that is, whether the subspaces are based on a compatible set of basis vectors representing a common set of features.

Having already considered the case of related questions, let us now consider the case of unrelated questions (e.g., is Linda a feminist, and does Bill play jazz for a hobby?). According to the quantum model, the knowledge obtained from the two stories is represented by a state vector $|\psi\rangle$ that now must contain knowledge about features of both Linda and Bill. The projector $\boldsymbol{P}_{LF}$ represents the question "Is Linda a feminist," and another projector $\boldsymbol{P}_{BJ}$ represents the question "Does Bill play jazz for a hobby?" The key question is whether these two projectors commute. Given that the judge never heard of these two people before and given that the judge is unlikely to know anything about the co-occurrences of women who are feminists and men who are jazz players, the judge cannot form a compatible representation that combines all these features in a consistent representation. Instead, the judge must fall back on a simpler incompatible representation in which one set of features is used to evaluate the Linda question and a different set of features is used to evaluate the Bill question. Thus, we expect these two projectors to be noncommutative. This is exactly the property required to produce the conjunction error.

Given that that the projectors for the two unrelated questions are incompatible, then the probability for the conjunction is obtained first by projection of the belief state onto the "Linda is a feminist" subspace followed by the projection onto the "Bill plays jazz for a hobby" subspace. The interference effect produced by this incompatible representation depends on the particular stories and questions. In this particular example, negative interference implies that thoughts evoked by thinking about a woman who is not a feminist are negatively correlated with thoughts about a man who plays jazz for a hobby.

Further support for the idea that the unrelated questions are answered by incompatible subspaces comes from the finding of order effects found in the same studies by Gavansky and Roskos-Ewoldsen (1991). Conjunction errors were found to be more frequent and significantly larger when the conjunction question (e.g. Linda is a feminist, and Bill plays jazz for a hobby) was presented first as opposed to being presented last.

Unpacking Effects

A finding that is closely related to the disjunction error is the implicit unpacking effect (Rottenstreich & Tversky, 1997; Sloman, Rottenstreich, Wisniewski, Hadjichristidis, & Fox, 2004).[7] In this case, a person is asked to rank order the likelihood of the same logical event when it is described in the "packed" form B versus in the "unpacked" form (B and A or B and $\sim A$). When an event (e.g., death by murder) is unpacked into a likely cause (murder by a stranger) and an unlikely cause (murder by an acquaintance), then the unpacked event is judged to be more likely than the packed event, which is called *subadditivity* (Rottenstreich & Tversky, 1997). But if an event (e.g. death by disease) is unpacked into an unlikely cause and a residual (death from pneumonia or other diseases), then the packed event is judged to be more likely than the unpacked event (Sloman et al., 2004). Support theory (Tversky & Koehler, 1994) was designed to explain the first (subadditivity), but it cannot explain the second (superadditivity).

This effect is especially interesting because it provides an example where both positive and negative interference is required to explain the opposing results. According to the quantum model, the probability for the unpacked event B can be decomposed as follows:

$$\begin{aligned}||\boldsymbol{P}_B|\psi\rangle||^2 &= ||\boldsymbol{P}_B\cdot(\boldsymbol{P}_A+\boldsymbol{P}_{\sim A})|\psi\rangle||^2\\ &= ||\boldsymbol{P}_B\boldsymbol{P}_A|\psi\rangle||^2+||\boldsymbol{P}_B\boldsymbol{P}_{\sim A}|\psi\rangle||^2+\delta,\end{aligned} \tag{7}$$

where δ is the interference term.[8] The probabilities for the two unpacked events sum to $||\boldsymbol{P}_B\boldsymbol{P}_A|\psi\rangle||^2+||\boldsymbol{P}_B\boldsymbol{P}_{\sim A}|\psi\rangle||^2$.

In general, the interference, δ, can be positive or negative, depending on the inner product between the projection $\boldsymbol{P}_B\boldsymbol{P}_A|\psi\rangle$ and $\boldsymbol{P}_B\boldsymbol{P}_{\sim A}|\psi\rangle$. In all of the previous examples, we assumed that this inner product was negative, producing negative interference, resulting in a conjunction and disjunction effect. To account for subadditivity, we again needed the interference to be negative, but if we were to account for the opposite superadditive effect, the interference had to become positive. The quantum model agrees with the intuition provided by Sloman et al. (2004) that when

[7] The unpacking effect refers to a comparison between the sum of judgments of individual events versus the judgment of the union of these events. However, these findings are affected by the response scale used to make judgments, as well as judgment errors produced by judging individual events. We focus on the implicit unpacking effect which simply asks a person to order the likelihood of two events.

[8] Bordley (1998) first pointed out that quantum theory provides an alternative explanation for unpacking effects.

unpacking an event into an unlikely event and a residual, the indirect retrieval paths produced by unpacking make it difficult to reach the conclusion, and it is easier to reach the conclusion directly from the packed event. The positive interference implies that the projection of the initial state first onto pneumonia and then onto death is positively correlated (pointing in a similar direction) with the projection of the initial state first onto the residual (diabetes, cirrhosis, and so forth) and then onto death.

Conditional Versus Conjunction Probabilities

Both classic and quantum probability models make a strong prediction concerning the comparison of the probability of a conjunction with the conditional probability involving the same events. According to classic probability theory, $\boldsymbol{p}_L(U) \geq \boldsymbol{p}(L) \cdot \boldsymbol{p}_L(U) = \boldsymbol{p}(L \cap U)$ and similarly the quantum model must obey

$$||\boldsymbol{P}_U|\psi_L\rangle||^2 \geq ||\boldsymbol{P}_L|\psi\rangle||^2 \cdot ||\boldsymbol{P}_U|\psi_L\rangle||^2 = ||\boldsymbol{P}_U\boldsymbol{P}_L|\psi\rangle||^2. \quad (8)$$

A conditional fallacy occurs when the probability of a conjunction strictly exceeds the conditional probability.

The evidence regarding this fallacy is mixed. Tversky and Kahneman (1983) reported a study involving an unlikely (*U*) event "dying from heart attack" and a likely (*L*) causal event "age over 50." The mean judgment for the conditional probability equaled $J(U \text{ given } L) = .59$; the mean judgment for the conjunction probability equaled $J(U \text{ and } L) = .30$; and the mean judgment for the unlikely event equaled $J(U) = .18$. Thus, the conditional event exceeded the conjunction event, but the conjunction exceeded the single event. Hertwig, Bjorn, and Krauss (2008) found no differences between the conditional and conjunction probabilities and used this to argue that people confuse or misinterpret these two types of questions. Miyamoto, Lundell, and Tu (1988) investigated the conditional fallacy using four different stories. In one of the stories, the conditional exceeded the conjunction; in another, the conditional equaled the conjunction; and in two other stories, the conjunction exceeded the conditional. The largest fallacy occurred with a story based on rain and temperature in Seattle, which produced the results ($N = 150$): $J(L) = .71 > J(L \text{ and } U) = .61 > J(U) = .49 > J(U \text{ given } L) = .47$ (the difference between the means for the conjunction and the conditional was statistically significant). However, there was little difference between the conditional probability $J(U$ given $L)$ and the single event probability $J(U)$, and so it is possible that the participants ignored the conditioning event *L* when judging the conditional *U* given *L*. More research is needed on this important question.

Conjunction of Three Events

The quantum model also makes clear predictions for conjunctions involving two and three constituent events. According to the quantum model, the judgment for the conjunction of unlikely (*U*), moderately likely (*M*), and likely (*L*) events must be lower than the conjunction for a moderately likely (*M*) and likely (*L*) event. This follows from the fact that

$$||\boldsymbol{P}_L|\psi\rangle||^2 \cdot ||\boldsymbol{P}_M|\psi_L\rangle||^2 \geq ||\boldsymbol{P}_L|\psi\rangle||^2 \cdot ||\boldsymbol{P}_M|\psi_L\rangle||^2 \cdot ||\boldsymbol{P}_U|\psi_{M,L}\rangle||^2. \quad (9a)$$

The quantum model predicts a higher judgment for a conjunction of an unlikely (*U*) event, a likely (*L1*) event, and another likely (*L2*) event as compared with an unlikely (*U*) event and likely (*L2*) event under the following condition (for simplicity, suppose *L2* is more likely than *L1*):

$$||\boldsymbol{P}_{L2}|\psi\rangle||^2 \cdot ||\boldsymbol{P}_U|\psi_{L2}\rangle||^2 \leq ||\boldsymbol{P}_{L2}|\psi\rangle||^2 \cdot ||\boldsymbol{P}_U\boldsymbol{P}_{L1}|\psi_{L2}\rangle||^2 \rightarrow ||\boldsymbol{P}_U|\psi_{L2}\rangle||^2 \leq ||\boldsymbol{P}_U\boldsymbol{P}_{L1}|\psi_{L2}\rangle||^2.$$

Expanding the left-hand term (as we did in Equation 1) produces the following expression:

$$||\boldsymbol{P}_U|\psi_{L2}\rangle||^2 = ||\boldsymbol{P}_U \cdot (\boldsymbol{P}_{L1} + \boldsymbol{P}_{\sim L1})|\psi_{L2}\rangle||^2 = ||\boldsymbol{P}_U\boldsymbol{P}_{L1}|\psi_{L2}\rangle||^2 + ||\boldsymbol{P}_U\boldsymbol{P}_{\sim L1}|\psi_{L2}\rangle||^2 + \delta. \quad (9b)$$

It follows that the required inequality, $||\boldsymbol{P}_U|\psi_{L2}\rangle||^2 \leq ||\boldsymbol{P}_U\boldsymbol{P}_{L1}|\psi_{L2}\rangle||^2$, will hold if the interference is sufficiently negative so that $\delta < -||\boldsymbol{P}_U\boldsymbol{P}_{\sim L1}|\psi_{L2}\rangle||^2$. In this case, the judgment for a conjunction of three events is judged more likely than a conjunction of two events.

In fact, both of these predicted results have been experimentally obtained. Judgments for the conjunction of an unlikely event, a moderately likely event, and a likely event were found to be lower than judgments for the conjunction of the same moderately likely event and likely event (Stolarz-Fantino et al., 2003; Nilsson, Winman, Juslin, & Hansson, 2010). Furthermore, judgments for an unlikely, likely, and second likely event were found to be higher than judgments for the conjunction of the same unlikely event and likely event (Nilsson et al., 2010). Previously, these results have been explained by an averaging model, but they are also consistent with the quantum model.

Comparison of Explanations

The classic (Kolmogorov) probability model fails to explain conjunction and disjunction fallacies because when a story *S* and two uncertain events *U* and *L* are given, $\boldsymbol{p}(U \cap L|S) \leq \boldsymbol{p}(U|S)$ and $\boldsymbol{p}(U \cup L|S) \geq \boldsymbol{p}(L|S)$ are required. However, it is possible that people evaluate the conditional in the wrong direction (Gigerenzer & Hoffrage, 1995). Classic probability theory does allow $\boldsymbol{p}(S|U \cap L) > \boldsymbol{p}(S|U)$ and $\boldsymbol{p}(S|U \cup L) < \boldsymbol{p}(S|L)$. This explanation fails to predict any ordering for $\boldsymbol{p}(S|U \cap L)$ versus $\boldsymbol{p}(S|L)$, nor does it predict any ordering for $\boldsymbol{p}(S|U \cup L)$ versus $\boldsymbol{p}(S|U \cap L)$. A more serious problem is that this idea cannot explain why the fallacy occurs for a conjunction of future events that entail the current state. For example, given the current cigarette tax and teenage smoking rate, people prefer to bet on the event "an increase in cigarette tax from the current rate and a decrease in teenage smoking from the current rate" rather than the event "a decrease in teenage smoking from the current rate" (Sides et al., 2002). In this case, if we let *S* represent the current state of the world, then we are asked to compare $\boldsymbol{p}(S \cap U \cap L|S) = \boldsymbol{p}(U \cap L|S)$ versus $\boldsymbol{p}(S \cap U|S) = \boldsymbol{p}(U|S)$. If the conditional is reversed, then we have $\boldsymbol{p}(S|S \cap U \cap L) = \boldsymbol{p}(S) = \boldsymbol{p}(S|S \cap U)$, which fails to explain the findings.

Support theory (Narens, 2009; Rottenstreich & Tversky, 1997; Tversky & Koehler, 1994) proposes that unpacking an event into its component parts increases the availability of the components, and thus the unpacked event is judged to be more likely than the logically identical packed event. This theory provides an account of unpacking effects when they are subadditive but not when they are superadditive (Macchi, Osherson, & Krantz, 1999). Tversky and Kohler (1994) also explained conjunction errors as an effect of unpacking an unlikely event (e.g., bank teller). So far, however, support theory, has not provided an explanation of disjunction errors. This may be difficult because a packed event (e.g., being a feminist) is judged greater than the explicit disjunction of this same event with another event (e.g., being a feminist or bank teller).

The most popular models for both conjunction and disjunction fallacies are the averaging (Wyer, 1976) and adding (Yates & Carlson, 1986) models. These models seem especially plausible when conjunction errors are obtained without presenting any story, and judges are simply given numerical likelihoods on which to base their judgments (Gavanski & Roskos-Ewoldsen, 1991). In the latter case, it is hard to see how one could use a representativeness type heuristic that is reliant on feature descriptions when there are no features to use. The averaging model assumes that each item is assigned a likelihood value (from 0 to 1), and the judgment for a conjunction or disjunction question equals the weighted average of these likelihoods. The adding model assumes each item is assigned a signed value of evidence (negative one to positive one), and the judgment for a conjunction or disjunction question equals the weighted sum of evidence. Different weights must be assigned to the unlikely and likely events to explain both the conjunction and disjunction errors. The averaging model turns out to be superior to the adding model, because the latter is ruled out by averaging type errors. But the averaging model also has some serious deficiencies. One of the most important is that it fails to account for interdependence among events. An item is assigned a likelihood value independent of the other items with which it is paired. This independence assumption is falsified by empirical violations of independence. Also, this model fails to account for the effect of event dependencies on the size and rate of conjunction errors, and it fails to explain the reduction in conjunction and disjunction errors when mutually exclusive events are used. Finally, the averaging model cannot account for double conjunction errors and the conditional fallacy, but these findings are still open to question.

A probability judgment model based on memory retrieval has also been used to explain conjunction errors (Dougherty, Gettys, & Odgen, 1999). Two specific types of models were proposed, one for judgments based on stories (vignettes) and the other for judgments based on training examples. All of the studies in our review were based on stories (vignettes), and so our discussion is limited to the first model. According to the vignette memory model, information about the story is stored in a memory trace (column) vector. A single question is represented by a probe vector of the same length with values assigned to features related to both the question and the story, and zeros otherwise. A conjunctive question is represented by a single conjunctive probe, which is the direct sum of the two vectors, one vector representing each separate item. Retrieval strength (echo intensity) to a question is determined by the inner product between the memory trace vector and the question probe vector, and relative frequency judgments are proportional to echo intensity. In Appendix C, we show that the vignette memory predicts the same order as an averaging model and thus shares many of the same advantages and disadvantages of the averaging model. Like the averaging model, the vignette memory model has no explicit mechanism for explaining event dependencies on conjunction errors. The latter problem arises from the fact that the conjunctive probe is simply the direct sum of the separate item probes.

The quantum judgment model provides a common simple explanation for both conjunction and disjunction errors as well as unpacking effects and averaging errors. More important, it also makes a number of strong, testable, a priori predictions that are supported by the empirical results. This includes (a) the ordering of the most likely event compared with either disjunction or disjunction events (Equation 3), (b) the ordering of judgments for conjunction and disjunction events (Equation 4), (c) the effect of event dependency on the conjunction fallacy (Equation 5), (d) the effect of event likelihood on conjunction fallacy (Equation 6), (e) the order of a conditional versus a conjunction (Equation 8), (f) the effect of event order on the conjunction fallacy, (g) the occurrence of conjunction fallacies for three events (Equation 9), and (h) conjunction errors for unrelated events. Overall, the predictions of the quantum judgment model agree with all of the well-established empirical findings. The quantum model has some difficulty with double conjunction errors and the conditional fallacy, but the empirical status of these latter two findings remains weak. So far we have relied on evidence based on qualitative properties that provide tests of general principles. Next, we turn to a more specific quantitative comparison of the averaging model and the quantum model.

Quantitative Predictions for Order Effects on Inference

Inference tasks provide an ideal paradigm for testing the quantum model. The hypotheses and different types of evidence can be controlled to manipulate the feature space, and the order in which evidence is presented is easy to manipulate. Also, one of the oldest and most reliable findings regarding human inference is that the order in which evidence is presented affects the final inference (Hogarth & Einhorn, 1992). Consider the following example from a medical inference task (Bergus, Chapman, Levy, Ely, & Oppliger, 1998). Physicians ($N = 315$) were initially informed about a particular women's health complaint, and they were asked to estimate the likelihood that she had an infection on the basis of (a) the patient's history and findings of the physical exam and (b) laboratory test results, presented in different orders. For one order, the physicians' initial estimate started out at .67; after they had seen the patient's history and findings of the physical exam, the estimate increased to .78; and then after they had also seen the lab test results, it decreased to .51. For the other order, the initial estimate again started at .67; after they had seen the lab test results, the estimate decreased to .44; and then after they also had seen the history and findings of the physical exam, it increased to .59. This is called a *recency effect*, because the same evidence had a larger effect when it appeared at the end as opposed to the beginning of a sequence. Recency effects are commonly observed in inference tasks whenever a sequence of judgments is made, one after each new piece of evidence (Hogarth & Einhorn, 1992). One might suspect that these order effects arise from memory recall failures, but it turns out that memory recall is uncorrelated with order effects in sequential judgment tasks (Hastie & Park, 1986).

Order effects are problematic for a Bayesian model for the following reason. Suppose we have two abstract events A and B and a hypothesis H; then

$$p(H|A \cap B) = p(H|A) \cdot \frac{p(B|H \cap A)}{p(B|A)}$$

$$= p(H|B) \cdot \frac{p(A|H \cap B)}{p(A|B)} = p(H|B \cap A),$$

and the order used to evaluate these two events does not matter because the events commute $A \cap B = B \cap A$. For a Bayesian model to account for order effects, presentation order would need to be introduced as another piece of information (e.g., event O_1, that A is presented before B, and event O_2, that B is presented before A), so that we obtain $\boldsymbol{p}(H|A \cap B \cap O_1) > \boldsymbol{p}(H|A \cap B \cap O_2)$. But without specification of $\boldsymbol{p}(H) \cdot \boldsymbol{p}(O_i|H) \cdot \boldsymbol{p}(A|H \cap O_i) \cdot \boldsymbol{p}(B|H \cap O_i \cap A)$, this approach simply redescribes the empirical result, and such a specification is not known at present. One difficulty that arises for this approach is that presentation order is randomly determined, and order information is irrelevant.

To explain order effects, Hogarth and Einhorn (1992) proposed an anchor-adjust model in which order is not simply another piece of information, but rather evidence is accumulated one step at a time with a weight that depends on serial position. Recently, Trueblood and Busemeyer (in press) developed a quantum inference model in which order is an intrinsic part of the process of sequentially evaluating information represented by incompatible perspectives. However, the previous studies provided too few data points to provide a sufficiently strong test of the two competing models. Therefore Trueblood and Busemeyer (2010) conducted a larger study of order effects to compare these two models. First, we summarize this study and its basic findings. Then we describe the details of fitting both the anchor-adjust model and the quantum model to the results. Finally, we summarize the comparison of fits produced by the two competing models.

Order Effects on Criminal Inference

The Trueblood and Busemeyer (2010) study included total of 291 undergraduates from Indiana University. Each of these students participated in a computer-controlled experiment in which they read fictitious criminal cases (robbery, larceny, or burglary) and made judgments of guilt or innocence on a probability scale ranging from 0 to 1. A sequence of three judgments was made for each case: one before any presenting any evidence, and two more judgments after presentations of evidence by a prosecutor and a defense. For a random half of the cases, the prosecution was presented before the defense, and for the other half, the defense was presented first. For example, in one case, participants read a short story (one short paragraph) about a burglarized warehouse, made an initial judgment that was based on no information, read a strong prosecution (described by three sentences), made a second judgment that was based only on this prosecution, read a weak defense (described by one sentence), and made a third judgment that was based on both the prosecution and the defense. Altogether, each person was presented with eight cases based on the experimental design shown in Table 1. Each case was different for each person, and the assignment of cases to orders was counterbalanced across participants, which produced approximately 38 participants per order condition (see Trueblood & Busemeyer, in press, for details). Note that different groups of participants are needed to produce different orders of evidence, and as far as we know, 12 order conditions is the largest existing study of order effects on inference. The main results are shown in Table 1, which shows the mean judgment, averaged over participants and across the eight cases.

Trueblood and Busemeyer (in press) provided more detailed results separately for each of the eight separate cases, but the results were consistent across cases, and so here we only present a summary. The initial judgment (prior to any information) produced a mean probability equal to .459 (this is shown in the table notes). This small bias against guilt reflects the

Table 1
Estimation of Guilt Following Presentation of Evidence

	Observed results		Averaging version of anchor–adjust model		Quantum inference model	
Type and strength of evidence	After 1st evidence	After 2nd evidence	After 1st evidence	After 2nd evidence	After 1st evidence	After 2nd evidence
WP	.651		.578		.647	
WPWD		.516		.552		.502
WPSD		.398		.436		.407
SP	.805		.748		.870	
SPWD		.687		.587		.689
SPSD		.540		.437		.527
WD	.390		.499		.390	
WDWP		.619		.589		.639
WDSP		.779		.747		.758
SD	.278		.401		.275	
SDWP		.495		.568		.487
SDSP		.690		.756		.702

Note. Initial value equals .46. W = weak evidence, S = strong evidence, P = Prosecution, D = Defense.

instruction to assume innocence at the beginning. The first column of Table 1 shows the effect of the first piece of information, which demonstrates a clear effect produced by manipulation of the evidence. The second column shows the judgment after both pieces of evidence, which provide four tests for order effects. The strongest example is strong prosecution and strong defense: SPSD = .54 < SDSP = .69 (which is a recency effect equal to .15); the other three recency effects were approximately equal to .10. All four tests for order effects produced strong and statistically significant recency effects (all with $p < .001$; see Trueblood & Busemeyer, in press, for details).

Anchor–Adjust Inference Model

Hogarth and Einhorn (1992) proposed a heuristic model of inference in which a new state of belief equals the previous (anchor) state plus an adjustment:

$$C_n = C_{n-1} + w_n \cdot [s(E_n) - R_n], \tag{10}$$

where C_n is the participant's new belief state after observing n pieces of information, C_{n-1} is the participant's previous belief state after observing n–1 pieces of information, $s(E_n)$ is the evidence provided by the nth piece of information, w_n is a weight and R_n is a reference point for this serial position. Furthermore, Hogarth and Einhorn (1992) proposed the following principle for setting the serial position weight: if $[s(E_n) - R_n] > 0$, then $w_n = (1 - C_{n-1})$, and if $[s(E_n) - R_n] < 0$, then $w_n = C_{n-1}$.

Different versions of the model can be formed by imposition of assumptions on the evidence $s(E_n)$ and the reference point R_n. One can form an important variation, the averaging model, by assuming that $0 \leq s(E_n) \leq 1$ and setting $R_n = C_{n-1}$.[9] Hogarth and Einhorn (1992) proved that the averaging model is guaranteed to produce recency effects, which is found in all tests shown in Table 1. Another important version, the adding model, is formed by assuming $-1 \leq s(E_n) \leq 1$ and setting $R_n = 0$. As pointed out by Hogarth and Einhorn (1992), the adding model is not guaranteed to produce recency effects.

Recall that the averaging model provides a better explanation than the additive model for conjunction and disjunction errors. In fact, the adding model was ruled out because of the averaging type errors discussed earlier. We think it is important for a model to be consistent across both probability judgment paradigms, the conjunction/disjunction and inference paradigms. Therefore, we focused here on the averaging model. Of course, this is only one version of the anchor and adjust model, and one could always construct more complex versions by relaxing the assumptions about the serial position weight and the reference point. But the averaging model is one of the primary models proposed by Hogarth and Einhorn (1992) for recency effects, and it is also one of the primary models for explaining conjunction and disjunction fallacies. Trueblood and Busemeyer (2010, in press) presented more model comparisons including averaging and adding models and even more complex anchor–adjust models, but the conclusions we reached remain the same.

The averaging model cannot make any predictions for the first judgment (before presentation of any evidence), and so we used this mean (.459) to initiate the averaging process, $C_0 = .459$, and then we fitted the model to the remaining 12 conditions on the basis of the second and third judgments. The averaging model requires estimation of four parameters to fit the 12 conditions in Table 1. These four parameters represent the four values of $s(E)$ corresponding to the four types of evidence: weak defense (WD), strong defense (SD), weak prosecution (WP), and strong prosecution (SP). We fit the four parameters by minimizing the sum of squared errors (SSE) between the predicted and observed mean probability judgments for each of the 12 conditions, which produced a SSE = .0704 (standard deviation of the error = .0766, $R^2 = .9833$). The predicted values are shown under the two columns labeled anchor–adjust in Table 1. The model correctly predicts the recency effects, but despite the high R^2 values, the model fit is only fair. For example, the model severely overestimates the recency effect for the SDSP versus SPSD comparison (predicted effect = .319, observed effect = .15). Also, the model fails to reproduce the correct ordering across all the conditions. For example, the averaging model predicts that SDSP = .756 > WDSP = .747, when in fact SDSP = .69 < WDSP = .779. There are many other substantial quantitative prediction errors, which illustrate that even when the model is designed to produce recency effects, it still remains a challenge to fit these order effects.

Quantum Inference Model

Before introducing the quantum model proposed by Trueblood and Busemeyer (in press), let us first think about how a classic Bayesian model would be set up for this task. A simple classic probability model would be based on a sample space containing eight elementary events formed by combining two types of prosecutor evidence with two types of defense evidence and two hypotheses. A quantum model could be set up in the same manner on a single basis formed by eight basis vectors, one corresponding to each of these eight elementary events. Then the events would all be compatible, and the quantum model would make the same predictions as the classic Bayesian model. But this model would not produce any order effects. Instead, Trueblood and Busemeyer (in press) proposed a quantum model that was designed to be as simple as possible for application to this criminal inference task.[10] The basic idea is that the judge evaluates two types of evidence (positive vs. negative) regarding two hypotheses (guilty vs. innocent) from three points of view: a naïve point of view, the prosecutor's point of view, and the defense's point of view. In this basic idea, only a four-dimensional vector space is required. (In the following presentation, a superscript $^{\mathrm{T}}$ is used to represent a transpose of a matrix, and a superscript † is used represent a conjugate transpose of a matrix. In particular, [row vector]$^{\mathrm{T}}$ is a column vector.).

The judgment process begins with a description of this four-dimensional space in terms of four basis vectors used to make a judgment from the naïve point of view: $\{|N_{G+}\rangle, |N_{G-}\rangle, |N_{I+}\rangle, |N_{I-}\rangle\}$, representing (guilty, positive), (guilty, negative), (innocent, positive), and (innocent, negative), respectively. The initial

[9] In this case, $C_n = C_{n-1} + w_n \cdot [s(E_n) - C_{n-1}] = (1 - w_n) \cdot C_{n-1} + w_n \cdot s(E_n)$ and for $n = 2$

$C_2 = (1 - w_{n-2})(1 - w_{n-1}) \cdot C_0 + w_{n-2} \cdot (1 - w_n) \cdot s(E_1) + w_n \cdot s(E_n)$.

[10] Trueblood and Busemeyer (in press) used this same quantum inference model to fit the results from the Bergus et al.'s (1998) medical inference study and McKenzie et al.'s (2002) criminal inference study.

state equals $|\psi\rangle = n_{G+} \cdot |N_{G+}\rangle + n_{G-} \cdot |N_{G-}\rangle + n_{I+} \cdot |N_{I+}\rangle + n_{I-} \cdot |N_{I-}\rangle$.

For example, the third coordinate, n_{I+} represents the amplitude $\langle N_{I+}|\psi\rangle$ initially assigned to the basis vector $|N_{I+}\rangle$. To be concrete, we represented $|N_{G+}\rangle$ by the column vector $[1, 0, 0, 0]^T$, represented $|N_{G-}\rangle$ by the column vector $[0, 1, 0, 0]^T$, represented $|N_{I+}\rangle$ by the column vector $[0, 0, 1, 0]^T$, and represent $|N_{I-}\rangle$ by the column vector $[0, 0, 0, 1]^T$. Thus, the initial state vector $|\psi\rangle$ assigns a column vector of amplitudes $\boldsymbol{n} = [n_{G+}, n_{G-}, n_{I+}, n_{I-}]^T$ to the four basis vectors. We start with $n_{G+} = n_{G-} = (1/\sqrt{2})(\sqrt{.459})$ and $n_{I+} = n_{I-} = (1/\sqrt{2})(\sqrt{.541})$. The positive or negative sign of the evidence has no meaning at this point because the judge has no idea what the evidence is about (we labeled it positive or negative for convenience, but at this stage, it represents only two possible types of evidence). Equating the amplitudes for the two types of unknown evidence is analogous to using a uniform prior in a Bayesian model when nothing is known. The amplitude for guilt is lower because the instructions inform the person to assume innocence until guilt is proven, and the .459 is chosen to reproduce the observed value of the first judgment (before any evidence is presented). This is also the same initial state used for the averaging model. The probability of guilt from this naïve perspective is obtained first by projection of this initial state onto the subspace for guilt. The projector for guilty equals $\boldsymbol{P}_G = |N_{G+}\rangle\langle N_{G+}| + |N_{G-}\rangle\langle N_{G-}|$, which is represented by a 4×4 diagonal matrix with 1s in the diagonals of the first two rows and 0s elsewhere. The projection equals $\boldsymbol{P}_G \cdot \boldsymbol{n} = [(1/\sqrt{2})(\sqrt{.459}), (1/\sqrt{2})(\sqrt{.459}), 0, 0]^T$, and so the probability of guilt from the naïve judgment point of view equals $||\boldsymbol{P}_G \cdot \boldsymbol{n}||^2 = .459$. This initial state was chosen to reproduce the observed mean judgment of guilt equal to .459, slightly favoring not guilty, before any information is provided.

Next, suppose the prosecutor presents positive evidence favoring guilt followed by a likelihood judgment. This requires an evaluation according to a different set of basis vectors, $\{|P_{G+}\rangle, |P_{G-}\rangle, |P_{I+}\rangle, |P_{I-}\rangle\}$, which again represent (guilty, positive), (guilty, negative), (innocent, positive), and (innocent, negative) but now represent the prosecutor's viewpoint. The initial state can be expressed in this basis as $|\psi\rangle = p_{G+} \cdot |P_{G+}\rangle + p_{G-} \cdot |P_{G-}\rangle + p_{I+} \cdot |P_{I+}\rangle + p_{I-} \cdot |P_{I-}\rangle$.

For example, the first coordinate, p_{G+} represents the amplitude $\langle P_{G+}|\psi\rangle$ initially assigned to the basis vector $|P_{G+}\rangle$. Note that the amplitudes assigned according to the naïve perspective are different than those assigned according to the prosecutor's perspective because the latter reflect the prosecutor's arguments for guilt. The four prosecutor basis vectors can be represented by a 4×4 unitary matrix denoted U_{np}, with the first column representing $|P_{G+}\rangle$, the second column representing $|P_{G-}\rangle$, the third column representing $|P_{I+}\rangle$ and the fourth column representing $|P_{I-}\rangle$. Later, we will show exactly how we computed the unitary matrix U_{np}, but at this point, we assume it was known and continue with the evaluation of the prosecutor's evidence. First, we consider how to revise the initial state on the basis of the prosecutor's positive evidence. The projector for the positive evidence is denoted $\boldsymbol{P}_+$ and is spanned by $\{|P_{G+}\rangle, |P_{I+}\rangle\}$. According to Postulate 4, $|\psi_+\rangle = \boldsymbol{P}_+|\psi\rangle/||\boldsymbol{P}_+|\psi\rangle||$, and

$$\boldsymbol{P}_+|\psi\rangle = |P_{G+}\rangle\langle P_{G+}|\psi\rangle + |P_{I+}\rangle\langle P_{I+}|\psi\rangle =$$

$$p_{G+} \cdot |P_{G+}\rangle + 0 \cdot |P_{G-}\rangle + p_{I+} \cdot |P_{I+}\rangle + 0 \cdot |P_{I-}\rangle,$$

and therefore

$$|\psi_+\rangle = \frac{p_{G+}}{(|p_{G+}|^2 + |p_{I+}|^2)^{.5}} \cdot |P_{G+}\rangle + 0 \cdot |P_{G-}\rangle + \frac{p_{I+}}{(|p_{G+}|^2 + |p_{I+}|^2)^{.5}} \cdot |P_{I+}\rangle + 0 \cdot |P_{I-}\rangle.$$

The revised state $|\psi_+\rangle$ is now represented on the prosecutor basis by the column vector of amplitudes $\boldsymbol{p}_+ = [p_{G+}, 0, p_{I+}, 0]^T/(|p_{G+}|^2 + |p_{I+}|^2)^{.5}$.

Next, we consider how to determine the probability of guilt after being presented with the prosecutor's positive evidence. The projector for guilty is denoted $\boldsymbol{P}_G$ and is spanned by $\{|P_{G+}\rangle, |P_{G-}\rangle\}$. According to Postulate 3, $||\boldsymbol{P}_G|\psi_+\rangle||^2 = ||(\boldsymbol{P}_{G+} + \boldsymbol{P}_{G-})\,|\psi_+\rangle||^2$, and because $\boldsymbol{P}_{G+}$ and $\boldsymbol{P}_{G-}$ are orthogonal projections, it follows that

$$||\boldsymbol{P}_{G+}|\psi_+\rangle + \boldsymbol{P}_{G-}|\psi_+\rangle||^2 = ||\boldsymbol{P}_{G+}|\psi_+\rangle||^2 + ||\boldsymbol{P}_{G-}|\psi_+\rangle||^2,$$

and because the evidence is positive, we have $||\boldsymbol{P}_{G-}|\psi_+\rangle||^2 = 0$ so that

$$||\boldsymbol{P}_G|\psi_+\rangle||^2 = |\langle P_{G+}|\psi_+\rangle|^2 = |p_{G+}|^2/(|p_{G+}|^2 + |p_{I+}|^2). \quad (11)$$

Equation 11 provides a simple formula for computing the probability of guilt following the positive evidence by the prosecutor. All that is needed is the column vector of amplitudes $\boldsymbol{p} = [p_{G+}, p_{G-}, p_{I+}, p_{I-}]^T$ assigned to the four prosecutor basis vectors. These are related to amplitudes for the naïve basis by the unitary transformation $\boldsymbol{p} = U_{np}{}^{\dagger} \cdot \boldsymbol{n} = U_{pn} \cdot \boldsymbol{n}$, which is described later. At this point, we will continue with the evaluation of the defense evidence.

Finally, suppose the defense presents negative evidence after the positive evidence given by the prosecutor. Now the judge needs to view the two hypotheses and two types of evidence from the defense perspective. This entails a change of perspective to the defense basis, which requires an evaluation according to the four basis vectors $|D_{G+}\rangle$, $|D_{G-}\rangle$, $|D_{I+}\rangle$, and $|D_{I-}\rangle$. The revised state can be re-expressed in terms of this basis as

$$|\psi_+\rangle = d_{G+} \cdot |D_{G+}\rangle + d_{G-} \cdot |D_{G-}\rangle + d_{I+} \cdot |D_{I+}\rangle + d_{I-} \cdot |D_{I-}\rangle.$$

For example, the second coordinate, $d_{G-,}$ represents the amplitude $\langle D_{G-}|\psi_+\rangle$ assigned to the basis vector $|D_{G-}\rangle$ at this point. Note that the amplitudes for the defense differ from the amplitudes for the prosecutor because the defense tries to persuade the judge to view the evidence from a different perspective, which weakens the prosecution and strengthens the defense. The four defense basis vectors can be represented by a 4×4 unitary matrix, denoted U_{nd}, with the first column representing $|D_{G+}\rangle$, the second column representing $|D_{G-}\rangle$, the third column representing $|D_{I+}\rangle$, and the fourth column representing $|D_{I-}\rangle$. Later, we will show exactly how we computed the unitary matrix, U_{nd}, but at this point, we assume it is known and continue with the evaluation of the defense evidence. Now, consider how to revise the state based on the defense negative evidence. The projector for the negative evidence is denoted $\boldsymbol{P}_-$ and is spanned by $\{|D_{G-}\rangle, |D_{I-}\rangle\}$. According to Postulate 4, $|\psi_{+,\,-}\rangle = \boldsymbol{P}_-|\psi_+\rangle/||\boldsymbol{P}_-|\psi_+\rangle||$, and

$$\boldsymbol{P}_-|\psi_+\rangle = \boldsymbol{P}_{G-}\,|\psi_+\rangle + \boldsymbol{P}_{I-}|\psi_+\rangle = |D_{G-}\rangle\langle D_{G-}|\psi_+\rangle + |D_{I-}\rangle\langle D_{I-}|\psi_+\rangle =$$

$$0 \cdot |D_{G+}\rangle + d_{G-} \cdot |D_{G-}\rangle + 0 \cdot |D_{I+}\rangle + d_{I-} \cdot |D_{I-}\rangle.$$

Finally, we consider how to determine the probability of guilt after being presented with the prosecutor's positive evidence and the defense's negative evidence. The projector for guilty is denoted $\boldsymbol{P}_G$ and is spanned by $\{|D_{G+}\rangle, |D_{G-}\rangle\}$. According to Postulate 3, we obtain

$$||\boldsymbol{P}_G|\psi_{+,-}\rangle||^2 = |d_{G-}|^2/(|d_{G-}|^2 + |d_{I-}|^2). \qquad (12)$$

In sum, Equation 12 provides a simple formula for a judge to compute the probability of guilt following presentation of the positive evidence by the prosecutor and then negative evidence by the defense. All that is needed for this formula is the vector of amplitudes $\boldsymbol{d}_+ = [d_{G+}, d_{G-}, d_{I+}, d_{I-}]$ assigned to the four defense basis vectors. These amplitudes are related to amplitudes for the prosecutor basis by the unitary transformation $\boldsymbol{d}_+ = U_{nd}^{\dagger} U_{np} \cdot \boldsymbol{p}_+ = U_{dn}U_{np} \cdot \boldsymbol{p}_+$, which is described next.

It is time to return to the question about how to specify the unitary matrices. A unitary matrix is one that satisfies $U \cdot U^{\dagger} = I = U^{\dagger} \cdot U$, and this is necessary for the quantum model in order to preserve lengths and inner products of the basis vectors (Nielsen & Chuang, 2000). The model has three different bases: one for the naïve point of view, one for the prosecution point of view, and one for the defense point of view. This in turn implies three unitary matrices that relate the amplitudes of the three bases: U_{pn} that transforms amplitudes of the naïve basis into amplitudes of the prosecutor basis; U_{dn} that transforms amplitudes of the naïve basis into amplitudes of the defense; and U_{dp} that transforms amplitudes of the prosecutor into amplitudes of the defense. However, the last one is derived from the first two by the relation $U_{dp} = U_{dn}U_{np}$, with $U_{np} = U_{pn}^{\dagger}$, and furthermore $U_{pd} = U_{dp}^{\dagger}$, and so we only need to describe how to construct U_{dn} and U_{pn} and all the rest are determined from just these two.[11] Note that these unitary transformations are used independently of the particular belief state, and the same transformation from one set of coordinates to another is used for initial belief states as well as revised belief states. In short, the transformations are only used to change the coordinate system that represents the current belief state.

Any unitary matrix can be constructed from a Hermitian matrix, $H = H^{\dagger}$, by the complex matrix exponential transformation $U = \exp(-i \cdot x \cdot H)$; see Nielsen & Chuang, 2000), where x is a parameter.[12] Trueblood and Busemeyer (in press) used a Hermitian matrix that was previously developed for two earlier psychological applications involving four dimensional vector spaces (see Busemeyer, Wang, & Lambert-Mogilianksy, 2009, and Pothos & Busemeyer, 2009). In these previous applications, the Hermitian matrix H is constructed from two components, $H = H_1 + H_2$, defined by

$$H_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \qquad (13)$$

The purpose of H_1 is to rotate amplitudes to favor either the presence of positive evidence or negative evidence; and the purpose of the second of H_2 is to rotate beliefs toward guilt when positive evidence is present and to rotate beliefs toward innocence when negative evidence is present. Together, these two matrices coordinate beliefs about evidence and hypotheses. The parameter x determines the degree of rotation, and this is a free parameter in the model. We allowed a different parameter value of x for U_{pn} versus U_{dn}. We also allowed a different parameter value of x for strong and weak evidence. Altogether this produces four free parameter values for x, one for each combination of the four types of evidence WP, SP, WD, and SD This way of constructing the unitary matrices was chosen because it was the same as used in previous applications, and it is as simple as we can make it. Just as with the anchor-adjust model, more complex models are possible (for more details on this topic, see Trueblood and Busemeyer, in press).

To summarize, we started with the naïve initial state $\boldsymbol{n} = \sqrt{.5} \cdot [\sqrt{.459}\ \sqrt{.459}\ \sqrt{.541}\ \sqrt{.541}]^{\mathrm{T}}$, and computed the two unitary matrices $U_{pn} = \exp(-i \cdot x_p \cdot H)$ and $U_{dn} = \exp(-i \cdot x_d \cdot H)$ with H defined by Equation 13. First, the prosecutor–defense order was considered. We transformed $\boldsymbol{p} = U_{pn} \cdot \boldsymbol{n}$, set $\boldsymbol{p}_+ = [p_{G+}, 0, p_{I+}, 0]^{\mathrm{T}}/(|p_{G+}|^2 + |p_{I+}|^2)^{.5}$, and then took the squared magnitude of the first coordinate of $\boldsymbol{p}_+$ to obtain the probability of guilt following the first positive evidence. Next, we transformed to $\boldsymbol{d}_+ = U_{dn}U_{np} \cdot \boldsymbol{p}_+$, set $\boldsymbol{d}_{+,-} = [0, d_{G-}, 0, d_{I-}]^{\mathrm{T}}/(|d_{G-}|^2 + |d_{I-}{}^2)^{.5}$, and take the squared magnitude of the second coordinate of $\boldsymbol{d}_{+,-}$ to obtain the probability of guilt following the second negative evidence. Next, we considered the defense–prosecutor order. We transformed $\boldsymbol{d} = U_{dn} \cdot \boldsymbol{n}$, set $\boldsymbol{d}_- = [0, d_{G-}, 0, d_{I-}]^{\mathrm{T}}/(|d_{G-}|^2 + |d_{I-}|^2)^{.5}$ and then took the squared magnitude of the second coordinate of $\boldsymbol{d}_-$ to obtain the probability of guilt following the first negative evidence. Next, we transformed to $\boldsymbol{p}_- = U_{pn}U_{nd} \cdot \boldsymbol{d}_-$, set $\boldsymbol{p}_{-,+} = [p_{G+}, 0, p_{I+}, 0]^{\mathrm{T}}/(|p_{G+}|^2 + |p_{I+}|^2)^{.5}$ and took the squared magnitude of the first coordinate of $\boldsymbol{p}_{-,+}$ to obtain the probability of guilt following presentation of the second positive evidence. Recency effects occur because the two operations of (a) unitary transformation used to change the point of view followed by (b) projection on type of evidence do not commute. This causes the judgments after each piece of evidence to be order dependent, and the last point of view has the greatest impact.

The quantum model requires fitting four parameters, a pair ($x_{p,s}$, $x_{d,s}$) for strong evidence and another pair ($x_{p,w}$, $x_{d,w}$) for weak evidence, to the 12 conditions in Table 1. We fit the four parameters by minimizing the SSE between the predicted and observed mean probability judgments for each of the 12 conditions plus the initial judgment, which produced a SSE = .0058 (*SD* of the error = .022, $R^2 = .9986$). The predicted values are displayed in the last two columns of Table 1. This quantum model provides a very accurate fit, and it is a clearly better fit than the averaging model. Note that the quantum model correctly predicts all of the recency effects, and it also correctly reproduces ordering of the probabilities across all conditions. The only place where the model makes a noticeable error is for the SP condition where it overestimates the strength of this evidence.

[11] The relation between $U_{dp} = U_{dn}U_{np}$ follows from the fact that $U_{nd} \cdot \boldsymbol{d} = \boldsymbol{n} = U_{np} \cdot \boldsymbol{p}$ and so $\boldsymbol{d} = U_{dn}U_{np} \cdot \boldsymbol{p}$.

[12] This matrix exponential is the solution to the Schrödinger equation. It is a function that is commonly available in matrix programming languages.

Summary of the Quantitative Test

We had three purposes for this quantitative test of the quantum model. One was to extend the quantum model from the conjunction–disjunction paradigm to the inference paradigm. The second was to provide a detailed example of how to construct a vector space and unitary transformations relating different incompatible bases. The third was to provide a quantitative test that compares the quantum model with another heuristic model, the averaging model, to explain order effects on inference. The averaging model was chosen for comparison because it was the strongest candidate to explain conjunction–disjunction errors, and it was also designed specifically to explain recency effects observed in inference tasks.

Both the quantum model and the averaging model used the same initial belief, and both models were allowed to fit a separate parameter to the SP, WP, SD, and WD types of evidence. Thus both models had the same number of parameters (although the relative complexity of these models remains unknown). The models were fit to 12 different conditions in Table 1, which provides a challenging data set with strong recency effects. It is not so easy to fit these 12 conditions, because the averaging model did not even succeed in reproducing the correct ordering across all the conditions. The quantum model succeeded in producing a very accurate fit to all 12 conditions.

The quantitative test reported here was based on the average across eight individual criminal cases presented to the participants. Trueblood and Busemeyer (2010, in press) provided a more thorough analysis of each of the eight cases, and they showed that the quantum model continues to fit better than the averaging model for all eight cases. Trueblood and Busemeyer (2010, in press) also compared the quantum model to the additive model (again with both using four parameters), and the quantum model continued to fit better than the additive model. More important, Trueblood and Busemeyer (2010, in press) derived an important qualitative prediction from the quantum model that distinguished the quantum model from the additive model. This property was based on the fact that the additive model is insensitive to the interdependence of evidence, whereas the quantum model is sensitive to this interdependence. Trueblood and Busemeyer (2010, in press) reported the results of a second experiment designed to test this property, and the predictions strongly supported the quantum model over the additive model. Finally, Trueblood and Busemeyer (in press) compared the quantum model to a more complex version of the anchor–adjust model (one in which the reference R was allowed to be a free parameter and a logistic response function was used, which entailed more parameters than the quantum model). They compared the two models using a challenging set of order effects on inference reported by McKenzie, Lee, and Chen (2002), and the quantum model continued to produce a better fit than the anchor–adjust model.

We do not claim that we have proven the quantum model to be the correct explanation for recency effects on inference. Nor have we proven that the quantum model is always better than the anchor–adjust model. Much more research is needed to establish these facts. What we have concluded is that this quantitative test makes a convincing case for the quantum model to be considered a viable new candidate for modeling human inference, and it deserves to enter the model testing fray.

Other Applications and Extensions

The quantum model presented here has been successfully applied to several other interesting areas, which demonstrates the generality of the theory. Now we briefly summarize three of these other applications. We also point out third area that needs further theoretical and experimental research.

Attitude Questions

Question order effects are ubiquitous in survey research (Moore, 2002), and quantum theory provides a natural explanation for these effects. In one example of a Gallup poll (N = 1002) reported in Moore (2002), half the participants were asked the pair of questions: "Is Clinton honest and trustworthy?" and then "Is Gore honest and trustworthy?"; half were asked the same pair of questions in the opposite order. Clinton received 50% agreement when he was asked about first and 57% when asked about second; Gore received 68% when he was asked about first and 60% when asked about second. (This is called an *assimilation effect* because the candidates become more similar after the first question). In another example of a Gallup poll (N = 1015) presented by Moore (2002), half the participants were asked "Is Gingrich honest and trustworthy?" and then "Is Dole honest and trustworthy?"; the other half were asked the same in the opposite order. Gingrich received 41% agreement when he was asked about first and 33% when asked about second; Dole received 60% agreement when he was asked about first and 64% when asked about second (which is called a *contrast effect* because the candidates become more different on the second question). Two other kinds of order effects, called *additive effects* and *subtractive effects*, are also found (Moore, 2002). In all of the studies reviewed by Moore (2002), order effects were found so that $p(\text{AyBn}) \neq p(\text{BnAy})$ and $p(\text{AnBy}) \neq p(\text{ByAn})$ was observed, where, for example, $p(\text{AyBn})$ is the probability of a yes response to question A followed by a response no to question B.

Z. Wang and Busemeyer (2010) assumed that answers to back-to-back questions such as those reviewed in Moore (2002) are made with a sequence of projectors. For example, $p(\text{AyBy}) = \|\boldsymbol{P_B P_A}\ |\psi\rangle\|^2$ and $p(\text{ByAy}) = \|\boldsymbol{P_A P_B}\ |\psi\rangle\|^2$. If the projectors are noncommuting, then the sequence of projections produces order effects. This is the same assumption that we used to predict conjunction and disjunction errors. Z. Wang and Busemeyer (2010) were able to derive all of the order effects reported in Moore (2002) from this simple model; but more important, they derived the following parameter free prediction from the model: If questions are answered back to back and no new information is presented in between questions, then

$$\mathbf{q} = [p(\text{AyBn}) + p(\text{AnBy})] - [p(\text{ByAn}) + p(\text{BnAy})] = 0,$$

Surprisingly, for the three data sets that satisfied the test requirement, the observed results produced an average $q = .008$ (average z test statistic $= .44$, $N \approx 1000$), which is a highly accurate prediction.[13] These results provide strong evidence that the quantum model can make precise and accurate predictions regarding order effects on judgment.

Decision Making

The more specific quantum model described in the previous section also has been used in two of our earlier applications in decision making. Pothos and Busemeyer (2009) used this model to explain a phenomenon called the *disjunction effect* (Shafir & Tversky, 1992). Researchers have studied this most frequently using the prisoner dilemma paradigm, which is a two-player game in which each player can choose to defect or cooperate. The disjunction effect refers to the surprising fact that the probability of defecting when the move of the opponent is unknown turns out to be less than the probability of defecting when either of the opponent's moves is known. The quantum model used a four-dimensional vector space to represent the four combinations of beliefs about the opponent's move (opponent defects or not), and actions by the player (player defects or not). This quantum model was compared to a Markov model which used the same four states, and while the quantum model provided a highly accurate description of the disjunction effect, the Markov model failed to do so.

Busemeyer et al. (2009) used the same quantum model to explain a phenomenon called the *interference of categorization on decision making*. This phenomenon has been studied in a categorization–decision task in which participants are shown a face, and then they are asked to (a) categorize the face as good or bad or (b) make a decision to act friendly or defensive or (c) categorize the face and decide on an action. The interference effect refers to the surprising fact that the probability of attacking was higher when no categorization was made as compared to when the action was preceded by a categorization. Once again, the quantum model used a four-dimensional vector space to represent the combinations of categorizations (good, bad) and actions (friendly, defensive). As before, the quantum model was compared to a Markov model which used the same four states, and while the quantum model provided an accurate description of the results, the Markov model failed to so.

Following our initial applications of quantum theory to conjunction fallacies and the disjunction effect, several other physicists have formulated alternative quantum models for these phenomena (Aerts, 2009; Conte, et al., 2009; Khrennikov, 2010; Yukalov & Sornette, 2010). However, all of these variations rely on the common use of interference to account for these results.

Quantum Judgment Process

This article presents a theory of probability judgments, where the judged probabilities are based on the postulates previously described. There are at least two important questions that we still need to address. How are these judgments made, and how does one judgment affect a later judgment?

The first question is what cognitive mechanism is used to produce a probability judgment? In physics, it is not possible to ask an electron to judge the probability that it is in an excited as opposed to a ground state. The physicist can only force the particle by a measurement interaction to resolve into a definite yes or no answer. Humans, however, are capable of making judgments. As in the case with many Bayesian judgment models, in our quantum judgment model we remain agnostic about the exact mechanism used to generate these judgments. But if we were forced to speculate, then one idea is that beliefs in a quantum judgment model are assessed in the same way as familiarity in a memory recognition model. With regard to this idea, it is useful to compare the quantum model with a memory process model for probability judgments (MINERVA-DM; Dougherty et al., 1999). According to the memory model, probability judgments are determined by an *echo intensity*, which equals the sum of the cubed inner products between vectors representing the memory for the story and a vector representing the question. According to the quantum model, probability judgments are determined by a *squared projection*, which equals the sum of the squared inner products between each basis vector entailed by a question and a belief state based on the story. In short, the squared projection from quantum theory is analogous to the echo intensity from MINVERA-DM.

The second question is how does one judgment affect a later judgment? According to our Postulate 4, the belief state is updated when the judge concludes that a new event has occurred or a new fact is true. This is the same principle that is used to update conditional probabilities in classic probability theory. The two probability theories only differ when incompatible events are involved in the judgment. Now we examine the two types of judgment problems reviewed in the sections "Qualitative Predictions for Conjunction and Disjunction Questions" and "Quantitative Predictions for Order Effects on Inference."

Let us start with the juror inference task in which evidence is presented followed by a probability judgment of guilt. The presentation of new evidence causes the state to be revised by projection of the state onto the subspace consistent with the evidence. This is the same assumption that would be used in a Bayesian updating model or the averaging model. After this update, the person judges the probability of guilt. The belief state used to make this judgment contains the square roots of the judged probabilities for guilt and innocence. This judgment does not require the juror to resolve his or her uncertainty about guilt (i.e., the juror does not have to conclude whether the defendant is definitely guilty or not). Therefore the judgment about guilt leaves the juror in the same indefinite and uncertain state regarding guilt as before judgment. If instead we ask the juror to resolve all uncertainty and make a firm decision (definitely decide guilty vs. not guilty), then the conclusion that the juror finally reaches about guilt could change the juror's state of belief from an indefinite to a definite state (and affect later punishment judgments).

Finally, consider the probability judgment for the conjunction task. If the person is asked to judge the probability that Linda is a feminist bank teller, then the person first judges the probability that feminism is true of Linda; second, the person projects the state onto the subspace for the feminism event in order to judge the conditional probability of bank teller given that Linda is truly a feminist. The person only judges the probability that Linda is a bank teller at this point and

[13] If new information is inserted between questions, thus violating a key assumption used to derive the prediction, then we find strong and statistically significant deviations (see Z. Wang and Busemeyer, 2010).

is not required to reach any firm conclusions. Therefore, the state remains indefinite about the bank teller question after the probability judgment about bank teller, and the final state immediately after this sequence equals the projection on the feminism event. Now suppose another question about Linda is asked afterwards. One hypothesis is that the person remains passively in the state left over from the previous judgment (the normalized projection of Linda on feminism). However, people are not passive entities like particles in physics, and instead they are capable of actively changing their own state (by reading information or retrieving new thoughts). A more plausible hypothesis is that the person refers back to the Linda story before another judgment is made and thereby resets the state to one that is based on the original Linda story.[14]

Contribution of Quantum Ideas to Psychology and Rationality

Quantum probability theory introduces a new concept to the field of psychology–that is, the concept of compatibility between events. More accurately, we should say that this distinction is "re-introduced" because Niels Bohr (one of the founding fathers of quantum theory) actually borrowed the idea of complementarity (Bohr's term for incompatibility) from William James (one of the founding fathers of psychology). Quantum theory also raises some questions about the rationality of human judgments. Is this probability system rational, and if not, then why would people use this system? These two issues are addressed in the following sections.

Compatibility

The key new principle that distinguishes classic and quantum probabilities is the concept of compatibility. According to classic probability, all events are subsets of a common sample space, S; that is, all events are based on a common set of elementary events. Questions about different events, A and B, must refer to this same common space S, which makes the two questions compatible. In the present application, each of the elementary events represents a combination of feature values, and so a classic representation requires one to assign probabilities to all of the combinations for all of the features. If there are a lot of features, then this involves a large number of elementary events, resulting in a very complex probability function. To simplify this probability function, Bayesian theorists often impose strong conditional independence assumptions, which may or may not be empirically valid.

Quantum theory allows a person to use an incompatible representation. In other words, a person is not required to use a single (but very large) common set of features and their combinations. Instead, one set of features and their combinations could be used to answer a question A, and another set of features and their combinations could be used to answer another question B. The features can be selected to answer a specific question. The person does not have to assign probabilities to all the combinations from both questions A and B. Moreover, forming all combinations for answering all possible questions could easily exceed a person's knowledge capabilities. This is especially true if one considers all the various sorts of questions that a person might be asked. It is more practical and efficient for a person to use an incompatible representation, because one only needs to assign probability amplitudes to the set of feature patterns needed to answer a specific question. Quantum theory achieves this efficiency with different basis vectors used to represent different questions within the same vector space. Quantum theory retains coherence among these different incompatible questions by relating them through a (unitary) rotation of the basis vectors. In other words, one question might require viewing the problem from a first perspective, but then a second question might require viewing the problem from a different perspective. The two perspectives are complementary in the sense that they are systematically related by a rotational transformation.

An important question for any quantum model of cognition is the following: when will two questions rely on a compatible versus an incompatible representation? We argue that a compatible representation may be formed under two circumstances. The first is when the judge has received a sufficiently extensive amount of experience with the combinations of feature values to form a belief state over all of these combinations. If an unusual or novel combination of events is presented, and the person judging has little or no experience with such combinations, then the person may not have formed a compatible representation and must rely on incompatible representations of events that use the same small vector space but require the person to take different perspectives. In fact, conjunction errors disappear when individuals are given direct training experience with pairs of events (Nilsson, 2008), and order effects on abductive inference also decrease with training experience (H. Wang, Todd, & Zhang, 2006). A second way to facilitate the formation of a compatible representation is to present the required joint frequency information in a tabular format (Wolfe, 1995; Wolfe & Reyna, 2009; Yamagishi, 2003). Instructions to use a joint frequency table format would encourage a person to form and make use of a compatible representation that assigns amplitudes to the cells of the joint frequency tables.

Quantum Rationality

Both classic (Kolmogorov) and quantum (von Neumann) probability theories are based on a coherent set of principles. In fact, classic probability theory is a special case of quantum probability theory in which all the events are compatible. So why use incompatible events, and isn't this irrational? In fact, the physical world obeys quantum principles, and incompatible events are an essential part of nature. Nevertheless, there are clear circumstances in which everyone agrees that the events should be treated classically (such as random selection of balls from urns or dice throwing). Perhaps in these circumstances a person uses a quantum representation because he or she is willing to trade some accuracy for a simpler (lower dimensional) representation of uncertainty. Furthermore, it remains an empirical question whether quantum or Bayesian methods are more useful for modeling probabilities of very complex sequences when the joint probabilities are largely unknown.[15] Also, incompatible events may be essential for understanding commonly occurring but nevertheless very complex human interactions. For instance, when one is trying to judge something as uncertain as winning an argument with another person, the likeli-

[14] One can go on asking how this is done using quantum computing and information principles, and one answer is to use if–then types of control U gates (see Nielsen and Chuang, 2000), but this is getting too far into the realm of speculation with respect to the data at hand.

[15] In this case, a Bayesian model must approximate with conditional independence assumptions that could be false.

hood of success may depend on using incompatible representations that allows viewing the same facts from different perspectives. As another example, judges or jurors in a courtroom setting must adopt both prosecution and defense perspectives for viewing the same facts (Trueblood & Busemeyer, in press).

Human judges may be capable of using either compatible or incompatible representations, and they are not constrained or forced to use just one. The use of compatible representations produces judgments that agree with the classic laws of probability, whereas the use of incompatible representations produces violations. But the latter may be necessary to deal with deeply uncertain situations (involving unknown joint probabilities), where one needs to rely on simple incompatible representations to construct sequential probabilities coherently from quantum principles. In fact, both types of representations, compatible and incompatible, may be available to the judge, and the context of a problem may trigger the use of one or the other (Reyna & Brainerd, 1995). More advanced versions of quantum probability theory (using a Fock space, which is analogous to a hierarchical Bayesian type model) provide principles for combining both types of representations (Aerts, 2009).

Concluding Comments

During the 19th century, it was hard for scientists to imagine that there could be any geometry other than Euclidean geometry; but non-Euclidean geometries eventually became essential for many important scientific applications. During the 20th century, it was equally hard for scientists to imagine that there could be any probability theory other than classic probability, but quantum probability became essential to physics. Its importance for psychology is beginning to be recognized as well (Shiffrin, 2010).

Quantum theory is one of the most elegant and internally consistent creations of the human mind. It was developed by several ingenious physicists as a way to assign probabilities to physical events. In this article, we have explored its potential for assigning probabilities to psychological events, specifically in the context of human judgment. In fact, we have utilized the basic axioms of quantum probability theory and simply augmented them with an additional postulate, regarding the order in which multiple questions are evaluated. On the basis of uncontentious assumptions regarding the relatedness of different pieces of information and the similarity between different instances, we showed how it is possible to account for many of the basic findings in human probabilistic judgment. The main aspect of quantum theory that makes it successful relates to order effects in probability computations. Order effects arise in quantum theory because it is a geometric theory of probabilities: probabilities are computed from projections to different subspaces. But as we have shown, the order with which these projections occur typically can affect the eventual outcome. Empirical findings on human judgment indicate strong order effects as well, and it is for this reason that quantum theory appears to provide an intuitive and parsimonious explanation for such findings. We conclude that quantum information processing principles provide a viable and promising new way to understand human judgment and reasoning.

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Appendix A

The first part of this appendix provides a simple geometric and numerical example of an order effect based on the vectors shown in Figure 1 (visual display limits this to three dimensions). Our example expresses all the vectors in terms of coordinates with respect to the standard X, Y, Z basis in Figure 1. In this figure, one basis is generated by the $X = [1, 0, 0]$, $Y = [0, 1, 0]$, and $Z = [0, 0, 1]$ basis vectors. The blue X, Y, Z basis could represent three mutually exclusive and exhaustive answers to an X or Y or Z question. A second basis is generated by the $U = [1/\sqrt{2}, 1/\sqrt{2}, 0]$, $V = [1/2, -1/2, 1/\sqrt{2}]$, and $W = [-1/2, 1/2, 1/\sqrt{2}]$ basis vectors. The orange U, V, W basis could represent three mutually exclusive and exhaustive answers to another incompatible U or V or W question. The initial state is represented by the black vector $S = [-.6963, .6963, .1741]$ in the figure.

To become familiar with the quantum method of calculating probabilities, let us first compute the probabilities for saying yes to question X (squared length of the projection of S onto the ray spanned by X), as well as the probability of saying yes to question W (squared length of the projection of S on the ray spanned by W). In general, the projection of a state onto a ray is determined by the inner product of the state and the basis vector that spans the ray. The inner product between a vector T with coordinates $[t_1, \ldots, t_N]$ and another vector S with coordinates $[s_1, \ldots, s_N]$ is defined (with Dirac bracket notation) as $\langle T|S\rangle = \Sigma(t_i^* \cdot s_i)$. (Here t_i^* is the conjugate of t_i, but in this example, all of the coordinates are real and so $t_i^* = t_i$). First, consider the probability of an individual choosing X when asked question X or Y or Z from state S. The event of responding yes to X is represented by a ray spanned by the basis vector X. The inner product between X and S equals $\langle X|S\rangle = (1) \cdot (-.69631) + (0) \cdot (.69630) + 0 \cdot (.17410) = -.6963$, the projection of S onto X equals the point labeled $A = (-.6963) \cdot X$ in the figure, and the probability of choosing this answer equals $\|-.6963 \cdot X\|^2 = |-.6963|^2 \cdot \|X\|^2 = (.6963)^2 \cdot 1 = .4848$. Note that it is arbitrary whether the basis vector X or $X^* = (-X)$ is used to span the ray representing question X, because they both span the same ray. In the latter case, the inner product equals $\langle X^*|S\rangle = .6963$, yet the projection is exactly the same $A = (.6963) \cdot X^* =$

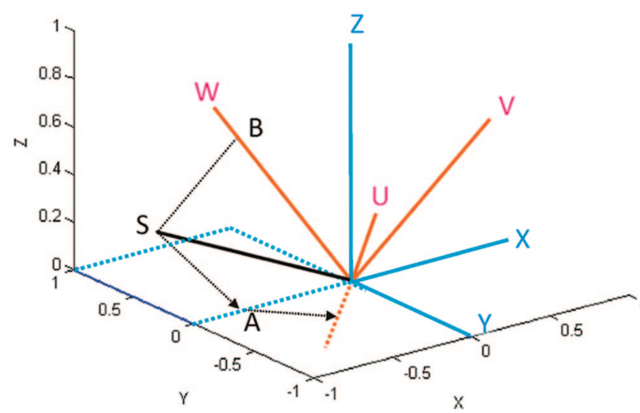

Figure 1. Order effect based on the vectors described in Appendix A. One basis is generated by the $X = [1, 0, 0]$, $Y = [0, 1, 0]$, and $Z = [0, 0, 1]$ basis vectors. The blue X, Y, Z basis could represent three mutually exclusive and exhaustive answers to an X or Y or Z question. A second basis is generated by the $U = [1/\sqrt{2}, 1/\sqrt{2}, 0]$, $V = [1/2, -1/2, 1/\sqrt{2}]$, and $W = [-1/2, 1/2, 1/\sqrt{2}]$ basis vectors. The orange U, V, W basis could represent three mutually exclusive and exhaustive answers to another incompatible U or V or W question. The initial state is represented by the black vector $S = [-.6963, .6963, .1741]$.

(*Appendices continue*)

$(-.6963) \cdot X$. In other words, the question is represented by a ray, and the ray spanned by the basis vector X does not have a positive or negative direction. Next consider the probability of an individual choosing W when asked question U or V or W from state S. The projection of S on the basis vector W is determined by the inner product $\langle W|S\rangle = (-1/2) \cdot (-.6963) + (1/2) \cdot (.6963) + (1/\sqrt{2}) \cdot (1741) = .8194$, the projection equals the point labeled $B = (.8194) \cdot W$, and the probability of choosing this answer equals $\|.8194 \cdot W\|^2 = |.8194|^2 \cdot \|W\|^2 = (.8194)^2 \cdot 1 = .6714$.

To examine the order effect, compare (a) asking U first and then X with (b) asking X first and then U. (Consider U the bank teller event, and consider X the feminist event.) Note that in the figure, the probability of X given U equals $|\langle X|U\rangle|^2 = .50 = |\langle U|X\rangle|^2$, which also equals the probability of U given X. In this example, the inner product between the initial state S and the basis vector U is 0, $\langle S|U\rangle = 0$, so these two vectors are orthogonal. (We made these two vectors orthogonal so that it is easy to visualize the relation in the figure. We could easily adjust all the vectors slightly so that the probabilities are small but not zero and make the same following point.) The fact that S and U are orthogonal implies that the probability of a person saying yes to question U directly from the initial state S is 0. But if we first ask whether X is true, then there is a probability (.4848) of answering yes; and if the answer is yes to X, then the projection of X on U equals $(1/\sqrt{2}) \cdot U$, and so now there is a probability $(1/\sqrt{2})^2 = .50$ of saying yes to U given yes to X. Thus, the probability from the direct path $S \rightarrow U$ equals 0, but the probability of the indirect path $S \rightarrow X \rightarrow U$ equals $.4848 \times .50 = .2424$. Therefore, this is an example in which the joint probability of first saying yes to X and then yes to U exceeds the single probability of saying yes to U when it is asked first.

The second part of this appendix explains why we can always choose a basis using basis vectors that produce amplitudes which are square roots of probabilities. The reason being that at the time of judgment, the phase of an amplitude is not meaningful, because it is not unique, and so it can be ignored, and we only need to consider the magnitude.

Consider a basis $\{|E_1\rangle, \ldots, |E_N\rangle\}$ for describing a state $|\psi\rangle$ in an N dimensional space. The state vector $|\psi\rangle$ can be represented in the $|E_j\rangle$ basis as a linear combination

$$|\psi\rangle = \Sigma|\mathrm{E_j}\rangle\langle\mathrm{E_j}|\psi\rangle.$$

The amplitude $\langle E_j|\psi\rangle$ assigned to the basis vector $|E_j\rangle$ equals the *inner product* between the state vector and the basis vector. In general, this inner product can be a complex number expressed as $\langle E_j|\psi\rangle = R_j \cdot e^{i\phi j}$, with $0 \leq R_j \leq 1$, and R_j^2 equals the probability for the ray spanned by the basis vector $|E_j\rangle$. Note that $e^{i\phi j} \cdot e^{-i\phi j} = 1$ so that

$$|\psi\rangle = \Sigma|E_j\rangle\langle E_j|\psi\rangle = \Sigma|E_j\rangle(e^{i\phi j} \cdot e^{-i\phi j})\langle E_j|\psi\rangle$$
$$= \Sigma e^{i\phi j} \cdot |E_j\rangle \cdot \left(e^{-i\phi j}\langle E_j|\psi\rangle\right) = \Sigma|F_j\rangle\langle F_j|\psi\rangle.$$

What we have done here is change from the $|E_j\rangle$ basis to the $|F_j\rangle$ basis for describing the state vector $|\psi\rangle$. But $|F_j\rangle = e^{i\phi j} \cdot |E_j\rangle$ spans the same ray as $|E_j\rangle$, and the squared magnitude of the amplitude $|\langle F_j|\psi\rangle|^2 = |e^{-i\phi j}\langle E_j|\psi\rangle|^2 = R_j^2$ produces the same probabilities as $|\langle E_j|\psi\rangle|^2 = R_j^2$. Suppose a question about an event corresponds to a subspace spanned by $\{|E_j\rangle, j \in X$, where X is the set of basis vectors that define the event in question$\}$. This subspace corresponds to the projector $\boldsymbol{P}_X = \Sigma|E_j\rangle\langle E_j|$ for $j \in X$. Then the matrix representation of $\boldsymbol{P}_X$ with respect to the $|E_j\rangle$ basis is the $N \times N$ matrix P_X with $\langle E_j|\boldsymbol{P}_X|E_j\rangle = 1$ in rows $j \in X$ and $\langle E_i|\boldsymbol{P}_X|E_j\rangle = 0$ otherwise; the matrix representation of $\boldsymbol{P}_X$ with respect to the $|F_j\rangle$ basis is exactly the same matrix P_X with the value $\langle F_j|\boldsymbol{P}_X|F_j\rangle = e^{-i\phi j} \cdot \langle E_j|\boldsymbol{P}_X|E_j\rangle \cdot e^{i\phi j} = \langle E_j|\boldsymbol{P}_X|E_j\rangle = 1$ in row $i \in X$ and zero otherwise. Finally, the probability of the event in question equals $\|\boldsymbol{P}_X|\psi\rangle\|^2 = \|P_X \cdot E\|^2 = \|P_X \cdot F\|^2$. Therefore, we can represent the state using either basis. To make the state more meaningful for cognition, we can choose to orientate the basis vectors so that they represent the state vector by using the square roots of probabilities. Then why do we need the phases?

The phases of the amplitudes are critical when a unitary transformation is used to change from one basis to another basis. Suppose A is an $N \times 1$ unit length column vector with complex coordinates $[a_1, \ldots, a_N] = [|a_1|e^{i\phi 1}, \ldots, |a_N|e^{i\phi N}]$; for this vector, we can define a unitary matrix $U_A = \mathrm{diag}[e^{-i\phi 1}, \ldots, e^{-i\phi N}]$ so that $(U_A \cdot A)$ is now a positive real unit length vector containing coordinates $[|a_1|, \ldots, |a_N|]$ in this new basis. Suppose B is another $N \times 1$ unit length column vector with coordinates $[b_1, \ldots, b_N] = [|b_1| \cdot e^{i\theta 1}, \ldots, |b_N| \cdot e^{i\theta N}]$; again for this vector we can define a unitary matrix $U_B = \mathrm{diag}[e^{-i\theta 1}, \ldots, e^{-i\theta N}]$ so that $(U_B \cdot B)$ is also a positive real unit length vector with coordinates $[|b_1|, \ldots, |b_N|]$. Finally, suppose the original complex vectors A and B are related by an $N \times N$ complex valued unitary transformation matrix U_{BA} so that $B = U_{BA} \cdot A$. Then we have the following relations:

$$B = U_{BA} \cdot A \rightarrow (U_B \cdot B) = (U_B \cdot U_{BA} \cdot U_A^{-1}) \cdot (U_A \cdot A).$$

The positive real vector $(U_A \cdot A)$ produces the same probabilities for events as the complex vector A, the positive real vector $(U_B \cdot B)$ produces the same probabilities for events as the complex vector B, and the matrix $(U_B \cdot U_{BA} \cdot U_A^{-1})$ is the unitary matrix that transforms $(U_A \cdot A)$ into $(U_B \cdot B)$. So we get the same exact answers using $\{A, B, U_{BA}\}$ or $\{(U_A \cdot A), (U_B \cdot B), (U_B \cdot U_{BA} \cdot U_B^{-1})\}$, and the latter only uses the square roots of probabilities. However, the phases remain important for the unitary transformation because $|b_j| = |\Sigma u_{ij} a_j| \neq \Sigma|u_{ij}| \cdot |a_j|$, and this is exactly where the interference enters the theory.

The unitary transformation can be interpreted as a fully interconnected hidden unit neural network: input $(U_A \cdot A) \rightarrow$ associative network $(U_B \cdot U_{BA} \cdot U_A{}^{-1}) \rightarrow (U_B \cdot B)$ output. Instead of using logistic hidden units as in a standard connectionist model, the unitary transformation uses sine-cosine units. We only require that the output amplitude $(U_B \cdot B)$ be explicitly available for awareness or reporting, and the phase captures implicit memory interference effects produced by the wave mechanical oscillations of the underlying neural based retrieval system represented by the unitary operator (Acacio de Barros & Suppes, 2009).

(*Appendices continue*)

Appendix B

In this appendix, we aimed to prove the following two propositions:

1. The conjunction and disjunction fallacies occur only if the events are incompatible.

2. The simultaneous explanation of both the conjunction and disjunction fallacies requires the following order constraint:

$$||\boldsymbol{P}_F\,\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2 < ||\boldsymbol{P}_B\,\boldsymbol{P}_{\mathrm{F}}|\psi\rangle||^2.$$

But before we begin, recall that $|\psi\rangle$ is a vector within a finite dimensional vector space defined on a field of complex numbers (technically, a finite dimensional Hilbert space). $\boldsymbol{P}_A$ denotes a projector on the subspace A which is a Hermitian matrix that satisfies $\boldsymbol{P}_A \cdot \boldsymbol{P}_A = \boldsymbol{P}_A$.

Proposition 1: The conjunction and disjunction fallacies occur only if the events are incompatible.

Proof:

If the events are compatible, then the projectors commute, $\boldsymbol{P}_B\,\boldsymbol{P}_F = \boldsymbol{P}_F\,\boldsymbol{P}_B$, and the interference term equals

$$\langle\psi_{B,\,\sim F}|\psi_{B,\,F}\rangle = \langle\psi|\boldsymbol{P}_B\boldsymbol{P}_{\sim \mathrm{F}}\boldsymbol{P}_B\,\boldsymbol{P}_F|\psi\rangle = \langle\psi|\boldsymbol{P}_B\boldsymbol{P}_{\sim F}\boldsymbol{P}_F\,\boldsymbol{P}_B|\psi\rangle$$
$$= 0 \text{ because } \boldsymbol{P}_{\sim F}\boldsymbol{P}_F = 0.$$

If the interference term is zero, then the probability of the single event, shown on the left-hand side of Equation 1, is simply the sum of the two conjunction probabilities, and so the left-hand side must be greater than or equal to each individual conjunction probability on the right-hand side. QED.

We need prove two lemmas before proving the second proposition.

Lemma 1: The interference term for event ~F is the negative of the interference term for event F.

Proof:

$$1 = ||\boldsymbol{P}_F|\psi\rangle||^2 + ||\boldsymbol{P}_{\sim F}|\psi\rangle||^2$$
$$= [||\boldsymbol{P}_F\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2 + ||\boldsymbol{P}_F\boldsymbol{P}_{\sim\mathrm{B}}|\psi\rangle||^2 + \delta_{\mathrm{F}}] + [||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2 + ||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\sim\mathrm{B}}|\psi\rangle||^2 + \delta_{\sim\mathrm{F}}]$$
$$= [||\boldsymbol{P}_F\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2 + ||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2] + [||\boldsymbol{P}_F\boldsymbol{P}_{\sim\mathrm{B}}|\psi\rangle||^2 + ||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\sim\mathrm{B}}|\psi\rangle||^2] + [\delta_{\mathrm{F}} + \delta_{\sim\mathrm{F}}]$$
$$= [||\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2 \cdot ||\boldsymbol{P}_{\mathrm{F}}|\psi_{\mathrm{B}}\rangle||^2 + ||\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2 \cdot ||\boldsymbol{P}_{\sim\mathrm{F}}|\psi_{\mathrm{B}}\rangle||^2] + [||\boldsymbol{P}_{\sim\mathrm{B}}|\psi\rangle||^2 \cdot ||\boldsymbol{P}_{\mathrm{F}}|\psi_{\sim\mathrm{B}}\rangle||^2 + ||\boldsymbol{P}_{\sim\mathrm{B}}|\psi\rangle||^2 \cdot ||\boldsymbol{P}_{\sim\mathrm{F}}|\psi_{\sim\mathrm{B}}\rangle||^2] + [\boldsymbol{\delta}_{\mathrm{F}} + \boldsymbol{\delta}_{\sim\mathrm{F}}] =$$
$$||\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2 \cdot [||\boldsymbol{P}_{\mathrm{F}}|\psi_{\mathrm{B}}\rangle||^2 + ||\boldsymbol{P}_{\sim\mathrm{F}}|\psi_{\mathrm{B}}\rangle||^2] + ||\boldsymbol{P}_{\sim\mathrm{B}}|\psi\rangle||^2 \cdot [||\boldsymbol{P}_{\mathrm{F}}|\psi_{\sim\mathrm{B}}\rangle||^2 + ||\boldsymbol{P}_{\sim\mathrm{F}}|\psi_{\sim\mathrm{B}}\rangle||^2] + [\boldsymbol{\delta}_{\mathrm{F}} + \boldsymbol{\delta}_{\sim\mathrm{F}}] =$$
$$||\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2 \cdot 1 + ||\boldsymbol{P}_{\sim\mathrm{B}}|\psi\rangle||^2 \cdot 1 + [\delta_{\mathrm{F}} + \delta_{\sim\mathrm{F}}] =$$
$$1 + [\delta_{\mathrm{F}} + \delta_{\sim\mathrm{F}}] \rightarrow [\delta_{\mathrm{F}} + \delta_{\sim\mathrm{F}}] = 0. \text{ QED.}$$

Lemma 2: The following two expressions for the interference terms are equivalent:

$$\delta_{\mathrm{B}} = \langle\psi_{\mathrm{B},\,\sim\mathrm{F}}|\psi_{\mathrm{B},\,\mathrm{F}}\rangle + \langle\psi_{\mathrm{B},\,\mathrm{F}}|\psi_{\mathrm{B},\,\sim\mathrm{F}}\rangle = 2 \cdot \{\mathrm{Re}[\langle\psi|\boldsymbol{P}_B\,\boldsymbol{P}_F|\psi\rangle] - ||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2\}$$

$$\delta_{\mathrm{F}} = \langle\psi_{\mathrm{F},\,\sim\mathrm{B}}|\psi_{\mathrm{F},\,\mathrm{B}}\rangle + \langle\psi_{\mathrm{F},\,\mathrm{B}}|\psi_{\mathrm{F},\,\sim\mathrm{B}}\rangle = 2 \cdot \{\mathrm{Re}[\langle\psi|\boldsymbol{P}_F\,\boldsymbol{P}_B|\psi\rangle] - ||\boldsymbol{P}_F\boldsymbol{P}_B|\psi\rangle||^2\}$$

Proof:

Note that $\langle\psi_{\mathrm{B},\,\mathrm{F}}|\psi_{\mathrm{B},\,\sim\mathrm{F}}\rangle = \langle\psi_{\mathrm{B},\,\sim\mathrm{F}}|\psi_{\mathrm{B},\,\mathrm{F}}\rangle^*$ (where * indicates the conjugate) so that

$$\delta_{\mathrm{B}} = \langle\psi_{\mathrm{B},\,\sim\mathrm{F}}|\psi_{\mathrm{B},\,\mathrm{F}}\rangle + \langle\psi_{\mathrm{B},\,\mathrm{F}}|\psi_{\mathrm{B},\,\sim\mathrm{F}}\rangle = 2 \cdot \mathrm{Re}[\langle\psi_{\mathrm{B},\,\sim\mathrm{F}}|\psi_{\mathrm{B},\,\mathrm{F}}\rangle]$$
$$\delta_{\mathrm{F}} = \langle\psi_{\mathrm{F},\,\sim\mathrm{B}}|\psi_{\mathrm{F},\,\mathrm{B}}\rangle + \langle\psi_{\mathrm{F},\,\mathrm{B}}|\psi_{\mathrm{F},\sim\mathrm{B}}\rangle = 2 \cdot \mathrm{Re}[\langle\psi_{\mathrm{F},\,\sim\mathrm{B}}|\psi_{\mathrm{F},\,\mathrm{B}}\rangle],$$

where $\mathrm{Re}(x)$ is the real part of the complex number x. It then follows that

$$\delta_{\mathrm{B}} = 2 \cdot \mathrm{Re}[\langle\psi_{\mathrm{B},\sim\mathrm{F}}|\psi_{\mathrm{B},\,\mathrm{F}}\rangle] = 2 \cdot \mathrm{Re}[\langle\psi|\boldsymbol{P}_{\sim F}\,\boldsymbol{P}_B\boldsymbol{P}_B\,\boldsymbol{P}_F|\psi\rangle] =$$
$$2 \cdot \mathrm{Re}[\langle\psi|\boldsymbol{P}_{\sim F}\,\boldsymbol{P}_B\,\boldsymbol{P}_F|\psi\rangle] =$$
$$2 \cdot \mathrm{Re}[\langle\psi|(\boldsymbol{I} - \boldsymbol{P}_F)\boldsymbol{P}_B\,\boldsymbol{P}_F|\psi\rangle] =$$
$$2 \cdot \{\mathrm{Re}[\langle\psi|\boldsymbol{P}_B\,\boldsymbol{P}_F|\psi\rangle] - ||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2\}.$$

A similar argument applies to produce the alternative expression for δ_{F}. QED.

Proposition 2. The simultaneously explanation of both the conjunction and disjunction fallacies requires the following order constraint: $||\boldsymbol{P}_F\,\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2 < ||\boldsymbol{P}_B\,\boldsymbol{P}_{\mathrm{F}}|\psi\rangle||^2$.

Proof:

Recall from Equation 1 the interference term from bank teller event equals $\delta_{\mathrm{B}} = \langle\psi_{\mathrm{B},\,\sim\mathrm{F}}|\psi_{\mathrm{B},\,\mathrm{F}}\rangle + \langle\psi_{\mathrm{B},\,\mathrm{F}}|\psi_{\mathrm{B},\,\sim\mathrm{F}}\rangle$, and the conjunction error requires $\delta_{\mathrm{B}} < -\,||\boldsymbol{P}_B\boldsymbol{P}_{\sim\mathrm{F}}|\psi\rangle||^2$. Recall from Equation 2 that the interference term from the not-feminist event equals $\delta_{\sim\mathrm{F}} = \langle\psi_{\sim\mathrm{F},\,\mathrm{B}}|\psi_{\sim\mathrm{F},\,\sim\mathrm{B}}\rangle + \langle\psi_{\sim\mathrm{F},\,\sim\mathrm{B}}|\psi_{\sim\mathrm{F},\,\mathrm{B}}\rangle$, and the disjunction error requires $\delta_{\sim\mathrm{F}} < -||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2$. Also note from Lemma 1 that $\delta_{\sim\mathrm{F}} = -\delta_{\mathrm{F}}$. From this last expression, it follows that $-\delta_{\mathrm{F}} < -||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2$ which then implies that $\delta_{\mathrm{F}} > ||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2$. Using the new expression for the interference based on Lemma 2, we see that the two inequalities require that

$$\delta_{\mathrm{F}} = 2 \cdot \{\mathrm{Re}[\langle\psi|\boldsymbol{P}_F\,\boldsymbol{P}_B|\psi\rangle] - ||\boldsymbol{P}_F\boldsymbol{P}_B|\psi\rangle||^2\} >$$
$$||\boldsymbol{P}_{\sim F}\boldsymbol{P}_{\mathrm{B}}|\psi\rangle||^2 - ||\boldsymbol{P}_B\boldsymbol{P}_{\sim\mathrm{F}}|\psi\rangle||^2 >$$
$$2 \cdot \{\mathrm{Re}[\langle\psi|\boldsymbol{P}_B\,\boldsymbol{P}_F|\psi\rangle] - ||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2\} = \delta_{\mathrm{B}}.$$

But $\mathrm{Re}[\langle\psi|\boldsymbol{P}_F\,\boldsymbol{P}_B|\psi\rangle] = \mathrm{Re}[\langle\psi|\boldsymbol{P}_B\,\boldsymbol{P}_F|\psi\rangle]$, which implies that $-||\boldsymbol{P}_F\boldsymbol{P}_B|\psi\rangle||^2 > -\,||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2$, and therefore we require $||\boldsymbol{P}_F\boldsymbol{P}_B|\psi\rangle||^2 < ||\boldsymbol{P}_B\boldsymbol{P}_F|\psi\rangle||^2$. QED.

(Appendices continue)

Appendix C

According to the vignette version of the memory model, information about the story is stored in a memory trace (column) vector denoted T. A single question A is represented by a probe (column) vector of the same length, P_A, with values assigned to features related to both the question and the story, and zeros otherwise. Retrieval strength (echo intensity) to a question is determined by the inner product between the memory trace vector and the question probe vector, $I_A = [\langle P_A|T\rangle/N_A]^3$. Note that the inner product is normalized by dividing it by a number, N_A, that depends on the number of nonzero elements in the question probe vector. Frequency or relative frequency judgments are assumed to be proportional to echo intensity (which requires the intensity to be nonnegative). A conjunctive question L and U is represented by a single conjunctive probe, which is the direct sum (concatenation) of two minivectors (this is the same as summing two nonoverlapping vectors). If P_L is a row minivector for L with length N_L, and P_U is a row minivector for U with length N_U, and 0_N is a row vector of N zeros, then $P_{L\&U} = [P_L|P_U] = [P_L|0_{NL}]+[0_{NU}|P_U] = P_L + P_U$. The echo intensity of this conjunction probe produces something akin to an average,

$$(I_{L\&U})^{1/3} = \langle P_{L\&U}|T\rangle/N_{L\&U} = \langle P_L + P_U|T\rangle/N_{L\&U} =$$

$$\langle P_L|T\rangle/N_{L\&U} + \langle P_U|T\rangle/N_{L\&U} =$$

$$(N_L/(N_L + N_U)) \cdot (I_L)^{1/3} + (N_U/(N_L + N_U)) \cdot (I_U)^{1/3}$$

which is a weighted average $=$

$$w_L \cdot (I_L)^{1/3} + w_U \cdot (I_U)^{1/3},$$

with weights $w_L = N_L/(N_L+N_U)$ and $w_U = N_U/(N_L+N_U)$. The intensity is the cube $[(I_{L\&U})^{1/3}]^3 = I_{L\&U}$, and the cubic function is monotonically increasing, so the intensity is ordered the same as $(I_{L\&U})^{1/3}$.

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