Applications of the mathematical apparatus of quantum theory to cognition, decision making and finances

Andrei Khrennikov

in cooperation with Irina Basieva and Emmanuel Pothos

Center Math Modeling in Physics and Cognitive Sciences, Linnaeus University, Växjö , Sweden
Department of Psychology, City University London, UK

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Quantum Information Revolution: Impact to Foundations? (QIRIF?), 9-13 June, 2019

This is the jubilee 20th Vxj conference devoted to quantum foundations and applications of quantum theory, especially quantum information and probability. This conference is aimed to highlight and at the same time to question the foundational impact of the recent quantum information revolution and to enlighten recent novel contributions to quantum foundations, theory and experiment

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Quantum(-like) operational representation of the process of decision making by cognitive systems

This talk is not about **quantum brain** in the spirit of Umezawa and Vitiello or Penrose and Hameroff. We do not try to reduce information processing by cognitive system to quantum **physical effects**.

The brain is a **black box** which information processing cannot be described by classical probability theory. And there is a plenty of such "nonclassical statistical data" - in cognitive psychology, game theory, decision making, social sicne, economics, finances, and politics.

A. Khrennikov, *Ubiquitous quantum structure: from psychology to finances*, Springer, Berlin-Heidelberg-New York, 2010.

E. Haven and A. Khrennikov, *Quantum Social Science*, Cambridge Press, Cambridge, 2013.

In decision theory such data was coupled to **probability fallacies and irrational behavior** of agents. We propose to apply the most well developed non-classical theories of information and probability, namely, based on the mathematical formalism of QM.

One may think that the appeal to quantum probability (and information) to model decision making by humans is too exotic. However, we recall that as early as the 1970s, Tversky (one of the most cited psychologists of all time) and Kahneman (Nobel prize in economics in 2002, for prospect theory, which he co-developed with Tversky) have been demonstrating **cases where CP prescription and human behavior persistently diverge (Tversky and Kahneman 1973, 1983).**

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Today, we are at the theoretical cross-roads, with huge divisions across conflicting, entrenched theoretical positions.

Should we continue relying on CP as the basis for descriptive and normative predictions in decision making (and perhaps ascribe inconsistencies to methodological idiosyncrasies)?

Should we abandon probability theory completely and instead pursue explanations based on heuristics, as Tversky and Kahneman proposed?

However, the use of the probabilistic and statistical methods is really the cornerstone of the modern scientific methodology. Thus, although the heuristic approach to decision making cannot be discarded completely, it seems more natural to search novel probabilistic models for decision making.

Slogan: **QP instead of heuristics of Tversky and Kahneman!**

The use of quantum information and probability, instead of their classical counterparts, can resolve some paradoxes of classical theory of decision making, economics, and game theory; e.g., the Elsberg and Machina paradoxes:

E. Haven and A. Khrennikov, Quantum mechanics and violation of the sure-thing principle: the use of probability interference and other concepts. *J. Math. Psychology*, **53**, 378-388 (2009).

The number of paradoxes generated by the classical decision making theory is really amazing. The authors of the recent review (Erev and Ert 2015) counted 35 basic paradoxes. During many years DM-theory was developed through creation of paradoxes and resolving them through modifications of the theory, e.g., from expected utility theory to the prospect theory. But any modified theory suffered of new paradoxes. **The use of QP can resolve all such paradoxes**, at least this is claimed in the recent paper:

I. Basieva, P. Khrennikova, E. M. Pothos, M. Asano, A. Khrennikov, Quantum-like model of subjective expected utility. *J. Math. Economics*, (2018); https://doi.org/10.1016/j.jmateco.2018.02.001

M. Asano, I. Basieva, E. M. Pothos, and A. Khrennikov, State entropy and differentiation phenomenon. *Entropy* **20** (6), 394 (2018).

F. Bagarello, I. Basieva, E. Pothos, A. Khrennikov, Quantum like modeling of decision making: Quantifying uncertainty with the aid of HeisenbergRobertson inequality. *J. Math. Psychology* **84**, 49-56 (2018).

Quantum-like paradigm which was formulated in (Khrennikov 1999):

The mathematical formalism of quantum information and probability theories can be used to model behavior not only of genuine quantum physical systems, but **all context-sensitive systems, e.g., humans.** Contextual information processing cannot be based on complete resolution of ambiguity. It is meaningless to do this for the concrete context, if tomorrow context will be totally different. Therefore **such systems process ambiguities, process superpositions of alternatives.**

Towards opening the black-box

The brain as a quantum-like processor of noisy classical random fields:

A. Khrennikov, Quantum-like model of processing of information in the brain based on classical electromagnetic field. *Biosystems* 105(3), 250-262 (2011).

The brain is not "interested" in explicit structure of random fields inside it and coming from the environment. It operates only with covariance matrices of such fields. The classical→ quantum correspondence is given by the simple formula:

$$\boldsymbol{B} \to \boldsymbol{\rho} = \boldsymbol{B}/\mathrm{tr}\mathbf{B}.$$

For physics, see:

A. Khrennikov, *Beyond Quantum*, Pan Stanford Publ., Singapore, 2014.

Conditional probability in CP

Bayes' formula for conditional probabilities.

$$p(B|A) = p(B \cap A)/p(A), p(A) > 0. \tag{1}$$

Consider two random variables $a = \pm 1, b = \pm 1.$ The b-variable describes decisions. So, we can make the decision $b = +1,$ "yes", or $b = -1,$ "no". The a-variable describes possible conditions, **contexts**, preceding the decision making.

For example, $a = +1$: the climate will change towards warming, $a = -1$: not; $b = +1$: to buy a property near sea, $b = -1$: not.

The main constraint imposed by the Bayes formula is appealing to CONJUCNTION of events, or joint measurement of two observables. However, for some (so-called incompatible) observables this is impossible. QP-formalism was specially designed to proceed without conjuctions. The cornerstone of the QP-modeling is a new definition of conditional probability and the new way of probability update.

Law of total probability (LTP) .

LTP *The prior probability to obtain the result, e.g.,* $\boldsymbol{b = +1}$ *for the random variable* $\boldsymbol{b}$ *is equal to the prior expected value of the posterior probability of* $\boldsymbol{b = +1}$ *under conditions* $\boldsymbol{a = +1}$ *and* $\boldsymbol{a = -1}$*.*

$$\boldsymbol{p(b=j)=}$$

$$\boldsymbol{p(a=+1)p(b=j|a=+1)+p(a=-1)p(b=j|a=-1),}$$

where $\boldsymbol{j = +1}$ *or* $\boldsymbol{j = -1}$*.*

LTP gives a possibility to predict the probabilities for the $\boldsymbol{b}$-variable on the basis of conditional probabilities and the $\boldsymbol{a}$-probabilities.

The cornerstone of Kolmogorov's approach is the postulation of a possibility to embed all complexes of conditions (contexts) preceding the decision making into one probability space. This embedding provides a possibility to apply to contexts the set-theoretical algebra, *Boolean algebra*, operations of intersection, union and complement.

Feynman (The Concept of Probability in Quantum Mechanics, can be found on Internet):

From about the beginning of the twentieth century experimental physics amassed an impressive array of strange phenomena which demonstrated the inadequacy of classical physics. The attempts to discover a theoretical structure for the new phenomena led at first to a confusion in which it appeared that light,and electrons, sometimes behaved like waves and sometimes like particles. This apparent inconsistency was completely resolved in 1926 and 1927 in the theory called quantum mechanics. The new theory asserts that there are experiments for which the exact outcome is fundamentally unpredictable, and that in these cases one has to be satisfied with computing probabilities of various outcomes. **But far more fundamental was the discovery that in nature the laws of combining probabilities were not those of the classical probability theory of Laplace.**

Probability structure of two slit experiment

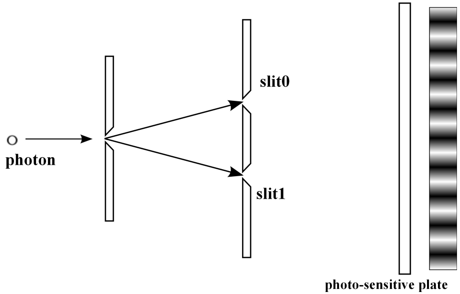

FIGURE 1. Context with both slits are open, C01

https://lnu.se/en/research/conferences/quantum-information-revolution-impact-to-foundations-qirif-9-13-juni-2019/

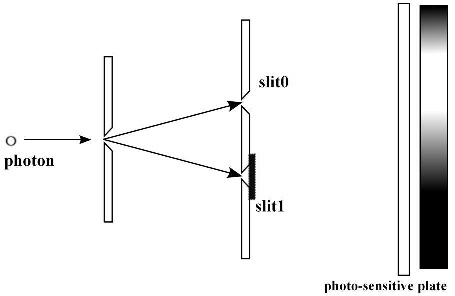

FIGURE 2. Context with one slit is open C0.

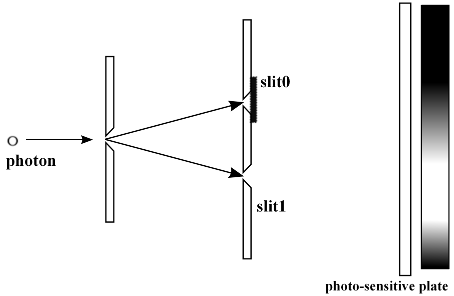

FIGURE 3. Context with one slit open, C1.

Observables: a is "which slit observable", i.e., $a = 0, 1$, b as the position on the photo-sensitive plate.

Set $P(i) = P(a = i), P(x) = P(b = x)$. Then

$$P(x) = \left|\frac{1}{\sqrt{2}}(\psi_0(x) + \psi_1(x))\right|^2$$

$$= \frac{1}{2}\left|\psi_0(x)\right|^2 + \frac{1}{2}\left|\psi_1(x)\right|^2 + \left|\psi_0(x)\right|\left|\psi_1(x)\right|\cos\theta, \tag{2}$$

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where ψ_0 and ψ_1 are two wave functions, whose squared absolute values $|\psi_i(x)|^2$ give the probability distributions $P(x|i)$ of photons passing through the slits. The last term represents the interference effect of two wave functions.

Then Eq. (2) is represented as

$$P(x) = P(0)P(x|0) + P(1)P(x|1) + 2\sqrt{P(0)P(x|0)P(1)P(x|1)}\cos\theta. \tag{3}$$

The violation of FTP is a consequence of the special **contextual structure of the two slit experiment** (in fact, a group of experiments). As Feynman pointed out, the interference formula (3) involves three contexts: $C_i, i = 0, 1$, only the ith split open and C_{01} both slits are open.

Interference effects in social science

Savage Sure Thing Principle: Savage, L.J. The foundations ofstatistics. New York: Wiley and Sons (1954).

STP *If you prefer prospect* $\boldsymbol{b}_+$ *to prospect* $\boldsymbol{b}_-$ *if a possible future event* $\boldsymbol{A}$ *happens* $(\boldsymbol{a} = \mathbf{+1})$, *and you prefer prospect* $\boldsymbol{b}_+$ *still if future event* $\boldsymbol{A}$ *does not happen* $(\boldsymbol{a} = \mathbf{-1})$, *then you should prefer prospect* $\boldsymbol{b}_+$ *despite having no knowledge of whether or not event* $\boldsymbol{A}$ *will happen.*

Savage's illustration refers to a person deciding whether or not to buy a certain property shortly before a presidential election, the outcome of which could radically affect the property market. "Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event will obtain".

Rationality

A decision maker has to be *rational.* Thus the STP was used as one of foundations of *rational decision making* and rationality in general. It plays an important role in economics in the framework of Savage's *utility theory.*

Savage's STP is a simple consequence of LTP.

LTP: Bayes conditioning + additivity of probability.

Violation of LTP implies violation of STP.

Prisoners' Dilemma, behavioral game theory.

Real players do not select Nash equilibrium.

A. Grib, A. Khrennikov, G. Parfionov, and K. Starkov, Quantum equilibria for macroscopic systems. *J. Phys. A.: Math. Gen.*, **39, 8461-8475 (2006).**

The mathematical formalism of quantum theory was successfully applied to model the basic effects of cognitive psychology: the order, conjunction, and disjunction effects (the works of Busemeyer, Pothos, Haven and Khrennikov).

No Aumann's theorem: Quantum agents can agree on disagree, see:

A. Khrennikov, Quantum version of Aumanns approach to common knowledge: Sufficient conditions of impossibility to agree on disagree. *J. Math. Economics* **60, 89104 (2015).**

Social Laser (Stimulated Amplification of Social Actions):

A. Khrennikov, "Social laser:" action amplification by stimulated emission of social energy. *Phil. Trans. Royal Soc.* **374, N 2054, 20150094 (2016).**

Order Effects

In a typical opinion-polling experiment, a group of participants is asked one question at a time, e.g., a =“Is Bill Clinton honest and trustworthy?” and then b =“Is Al Gore honest and trustworthy?”

The joint probability distribution is found $p(a = \alpha, b = \beta), \alpha, \beta = \pm 1$. Then these questions are asked in the opposite order, the joint probability distribution is found $p(b = \beta, a = \alpha), \alpha, \beta = \pm 1$. And these distributions do not coincide.

Such noncommutative effect cannot be represented in the Kolmogorov model, by representing questions by random variables. In the quantum formalism we can easily model this effect by using representation of observables by non-commutative Hermitian operators - projector valued measures. or more generally Positive operator valued measures, POVMs.

(a-a)-problem.

We remark that in fact we have to use projection valued measures, since if, e.g., the value $a = +1$ was received and the question a asked the second time, the answer $a = +1$ is obtained with probability 1. The same is happens for the b-question.

Repeatable measurement implies that, in fact, POVM is a projector valued measure! In the finite dimensional case!

Buscemi, F., D' Ariano, G. M. and Perinotti, P.: There exist nonorthogonal quantum measurements that are perfectly repeatable. Phys. Rev. Lett. 92, 070403-1 - 070403-4 (2004).

Thus, if we want to describe the Clinton-Gore experiment in the quantum-like manner we have to represent the questions A and B by projection-type observables.

(a-b-a)-problem.

However, the real situation is more complicated. Even in the sequence (a-b-a), if the first result was $a = +1$, then for any result $b = \beta$ the result of the second a-measurement is again $a = +1$ with probability 1. It is possible to show that this is possible only if a and b are projector valued measures and they commute. However, commutativity is incompatible with order effect.

A. Khrennikov, Basieva, I., Dzhafarov, E.N., Busemeyer, J.R. (2014). Quantum Models for Psychological Measurements : An Unsolved Problem. PLoS ONE. 9. Article ID: e110909.

For atomic instruments, this was proven in:

I. Basieva, A. Khrennikov, On a possibility to combine the order effect with sequential reproducibility for quantum measurements. *Found. Phys.* **45, N 10, 1379-1393 (2015).**

Quantum-like modeling of decision making

The mental state (belief state) of Alice is represented as a quantum state; questions or tasks as quantum observables (Hermitian operators or POVMs). Probabilities are determined by Born's rule. This model was presented in my old paper:

A. Khrennikov, *Found. of Physics*, **29, N. 7, 1065-1098 (1999).**

This model does not describe state dynamics in the process of decision making. This dynamics was accounted in the work:

E. M. Pothos, J. R. Busemeyer, A quantum probability explanation for violation of rational decision theory Proc. Royal. Soc. B, 276, 2171-2178 (2009).

Let observable A (Hermitian operator) represents a question (task) to Alice. There is introduced Hamiltonian H, considered dynamics of the initial belief state and then Alice's decision is represented as measurement of the observable A at some instant of time. The authors even present some cognitive arguments to determine the instance t_m of measurement.

The main problem is to construct operator-representation of questions and decision Hamiltonians.

In contrast to quantum physics, we do not classical Hamiltonian model of DM, i.e., we cannot use the quantization procedure to transform functions on the classical phase space into operators. However, one can use the algebra of operators of creation and annihilation. Let a question A is dichotomous, "no/ "yes". Creation operator $a^\star$ creates "yes" from "no, annihilation operator a transforms "yes" to"no. Then we can compound (similarly e.g. to quantum optics) Hamiltonian and observables of creation and annihilation operators.

This approach was actively explored by F. Bagarello:

Bagarello, F., 2012. Quantum dynamics for classical systems: with applications of the Number operator. J. Wiley, New York.

Type of canonical commutation relations? Bagarello used the Fermionic commutation relations to model "no/ "yes" questions.

However, the situation seems to be more complicated and other types of commutation relations algebras

Who are the decision makers? Bosons? Fermions? Qubits?

Polina Khrennikova, Modeling behavior of decision makers with the aid of algebra of qubit creation-annihilation operators. J. Math. Psychology 78, 76-85 (2017).

All quantum physical physical systems are either bosons or fermions, hence: (anti)-commutation relations for creation-annihilation operators.

Quantum information is done in n-qubit space. **Qubit is neither boson nor fermion!** It combines both fermionic and bosonic features. In quantum theory qubit representation is just a math model

In the quantum-like model of decision making qubit by itself is the basic entity of the quantum-like model.

Measurement problem in decision making

Majority of physicists are fine with collapse of the wave function. In cognitive community there is common opinion that the mental state dynamics is continuous and selection of different alternatives in the process of decision making cannot be modeled by the collapse type process.

One of attempts to solve the measurement problem is based on consideration of measurement process as the decoherence process, W. Zurek and recently G. Lindblad. In the limit $t \to \infty$ the state $\rho(t)$ approaches the state $\bar{\rho}$ which is diagonal in "pointer basis".

Decision making as decoherence

The model is pure informational. Both a "quantum-like system" and bath are represented by their states, ψ and ϕ. In the PD, the ψ represents Alice's possible decisions and ϕ the information bath having some degree of relevance to this concrete problem; in particular, Alice's recollections about Bob. Then we apply theory of open quantum systems and Gorini-Kossakowski-Sudarshan-Lindblad dynamics (often called simply Lindblad equation) to model experimental data.

Decision making dynamics should drive the belief state of Alice $\rho(t)$ to the diagonalized state $\bar{\rho} = \lim_{t\to\infty} \rho(t)$. Its diagonal elements give probabilities of possible decisions.

M. Asano, M. Ohya, Y. Tanaka, A. Khrennikov, and I. Basieva, On application of Gorini-Kossakowski-Sudarshan-Lindblad equation in cognitive psychology. *Open Systems and Information Dynamics* **17, 1-15 (2010).**

Part 2:

A. Khrennikov, Towards information lasers. *Entropy*, 17, N 10, 6969-6994 (2015).

Purpose: To present the basic assumptions for creation of social lasers and attract attention of other researchers (both from physics and socio-political science) to the problem of modeling of **Stimulated Amplification of Social Actions** (SASA).

Design/methodology/approach: The model of SASA and its analysis are based on the mathematical formalism of quantum thermodynamics and field theory (applied outside of physics).

Findings: The presented quantum-like model provides the consistent operational model of such complex socio-political phenomenon as SASA.

Research limitations/implications: The model of SASA is heavily based on the use of the notion of social energy. This notion has not yet been formalized.

Practical implications: Evidence of SASA ("functioning of social lasers") is rapidly accumulating, from color revolutions to such democratically structured protest actions as Brexit and the recent election of Donald Trump as the president of USA. The corresponding socio-political studies are characterized by diversity of opinions and conclusions. The presented social laser model can be used to clarify these complex socio-political events and even predict their possibility.

Social implications: SASA is the powerful source of social instability. Understanding its informational structure and origin may help to stabilize the modern society.

Originality/value: Application of the quantum-like model of laser technology in social and political sciences is really novel and promising approach.

Indistinguashability: physical versus social systems

One way to approach indistinguishability of information quanta is to create a massive flow of information such that, for people (receivers of information), it would be difficult to inspect its content. They would "absorb information only to absorb information." Here the amount of the social energy carried by information communication is the subject of interest and not its content. ' 'They eat without to feel food's test, they do not like or dislike food, they just eat, eat, and eat.. " People absorb information from TV, newspapers, Internet, mobiles. There is no neither time nor mental resources to analyze its content.

Information-processing by a "post-human" who is permanently connected to a variety of information channels pumping huge amount of information differs from information-processing by a "human" (who lived at the Earth say 50 years ago). It makes a big difference: to process information either from one letter coming with post every 2-3 weeks or from 20-100 emails per day.

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The main feature of all channels is massive (but short, "quantized") presentation of the same event - excitation of the information field. This generates a kind of shock-wave in which the real content of communication plays a subsidiary role.

For example, consider communications about wars and their victims throughout the world. We do not more go deeply in the moral and rational analysis of the continuous flow of brutal video.

At the same time such brutal video transfer to us "energy"; we become "excited". Do you remember the wave of video about the war in Ukraine? about the Malaysian Boeing(s)? about the war in Syria?

The everyday homogeneity of the brutal news makes them indistinguishable; humans (receivers of information) treat them merely as information quanta carrying portions of the social energy.

The problem of indistinguishability is complicated even in quantum physics. Quantum systems are indistinguishable from observational viewpoint.

And we remind that quantum mechanics is theory of observations. Therefore it is not important whether quantum systems can be distinguished at the subquantum level.

By the straightforward interpretation events are distinguishable even at the level of observations. To get a closer analogy with quantum theory, we have to consider a kind of "meta-observations" - information processing endowed with a kind of content-filter.

In such a meta-observation process, a person works similarly to a photo-detector. He just absorbs the portion of the social energy carried by a communication delivered by mass-media and practically ignores its content. Thus a social media working in such a regime of meta-observations can serve as gain-medium for social laser.

Part 3: Quantum-like model of subjective expected utility

M. Asano, I. Basieva, A. Khrennikov, M. Ohya, Y. Tanaka, A quantum-like model of selection behavior. *J. Math. Psychology* **78, 2-12 (2017).**

I. Basieva, P. Khrennikova, E. M. Pothos, M. Asano, A. Khrennikov, Quantum-like model of subjective expected utility. *J. Math. Economics*, **(2018); https://doi.org/10.1016/j**

Classical probability modeling

There are two lots, say $A = (x_i, P_i)$ and $B = (y_i, Q_i)$, where (x_i) and (y_i) are outcomes and (P_i) and (Q_i) are probabilities of these otcomes. All of the outcomes are different from each other. **Which lot do you select?**

An agent, say Alice, can simulate the experience that she draws the lot A (or B) and gets the outcome x_i (or y_i). Let us represent such an event by (A, x_i) (or (B, x_i)).

Alice assigns the utilities $u(x_i)$ and $y(x_i)$ of (A, x_i) and (B, y_i), respectively. Here, $u(x)$ is a utility function of outcome x. By using the utility function the agent evaluates various comparisons for making the preference $A \succeq B$ or $B \succeq A$.

Expected utility theory: an agent calculates the expectation values $E_A = \sum u(x_i) P_i$ and $E_B = \sum u(y_i) Q_i$, and uses their difference as the criterion for making the preference.

Representation of lotteries by orthonormal bases in belief-state space

Consider the space of **belief states of an agent.** Belief-states are represented by normalized vectors of a complex Hilbert space $\boldsymbol{H}$. These are so-called pure states.

Lotteries $\boldsymbol{A}$ and $\boldsymbol{B}$ are mathematically realized as two orthonormal bases in $\boldsymbol{H}$: $(|\boldsymbol{i}_a\rangle)$ and $(|\boldsymbol{j}_b\rangle)$.

Any vector $|\boldsymbol{i}_a\rangle$ represents the event $(\boldsymbol{A}, \boldsymbol{x}_i)$ - "selecting of the $\boldsymbol{A}$-lottery which generates the outcome $\boldsymbol{x}_i$." The same can be said about vectors of the $\boldsymbol{B}$-basis.

These **events are not real, but imaginable.** Alice plays with potential outcomes of the lotteries and compares them.

We can also represent lotteries by Hermitian operators, the **lotteries operators**:

$$\boldsymbol{A} = \sum_i \boldsymbol{x}_i |\boldsymbol{i}_a\rangle \ , \boldsymbol{B} = \sum_j \boldsymbol{y}_j |\boldsymbol{j}_b\rangle. \tag{4}$$

Representation of utilities by bases

As in the classical theory, each outcome x_i has some utility $u_i = u(x_i)$ (say amount of money). Starting with two lotteries A and B with outcomes (x_i) and (y_j) with corresponding utilities $u_i = u(x_i)$ and $v_j = u(y_j)$,

Alice couples these utilities with two orthonormal bases in the belief-state space H :

$$u_i \sim |i_a\rangle, \; v_j \rightarrow\sim j_b\rangle. \tag{5}$$

We emphasize coupling of utilities to lotteries. Utility (derived from some monetary amount) has not only the value, but also so to say the "color" determined by circumstances surrounding the corresponding lottery - *lottery's context.*

Complementary lotteries

The lotteries operators can be noncommuting, i.e., $[A, B] \neq 0$, *complementary lotteries* In the DM-process for complementary lotteries, Alice does not create the the joint image of outcomes of both of them.

In math terms: the impossibility to determine the joint probability distribution for the pairs of outcomes (x_i, y_j).

Instead of weighting probabilistically the pairs of outcomes, Alice analyzes the possibility of realization of an outcome say x_i of the A-lottery and she accounts its utility $u(x_i)$. Then under the assumption of such realization she imagines possible realizations (y_j) of the B-lottery and compares the utilities $u(y_j)$ and $u(x_i)$.

"Suppose I have selected the A-lottery and its outcome x_i was realized. What would be my earning (lost) if (instead) I **were selected** the B-lottery and its outcome y_j were realized?"

This kind of **counterfactual reflections** is mathematically described by the Hilbert space formalism and transition from the A-basis to the B-basis.

Outputs of these comparisons are weighted through accounting Hilbert space coordinates This accounting is described by the special comparison operator D.

Since Alice cannot handle both lotteries simultaneously, she starts with imaging one of them say A, as in the above consideration. Then she performs similar counterfactual reasoning starting with the B-lottery.

The comparison operator D has two counterparts representing the processes of reflections about preferences, $A \to B$ and $B \to A$: comparisons. In the operator terms transitions from one basis to another are represented by transition operators $E_{i_a \to j_b}, E_{j_b \to i_a}$. And the comparison operator D is compounded of these operators.

Belief-state

The state of Alice's beliefs about the lottery A can be represented as superposition

$$|\Psi_A\rangle = \sum_i \sqrt{P_i} e^{i\theta_{ai}} |i_a\rangle.$$

The probability of realization of the event (A, x_i) is given by the Born rule and equals to $P_i = |\langle i_a|\Psi_A\rangle|^2$. In the same way the state of beliefs about the lottery B can be represented as superposition

$$|\Psi_B\rangle = \sum_i \sqrt{Q_i} e^{i\theta_{bi}} |i_b\rangle.$$

Alice superposes her belief-states about the lotteries and her total belief-state is created via superposition of her beliefs about the A-lottery and the B-lottery. Thus the overall ψ is the superposition of the ψ's s for two individual lotteries.

$$\Psi = \Psi_A + \Psi_B. \tag{6}$$

Belief state: operator representation

In further calculations it is useful to use the operator representation of $|\boldsymbol{\Psi}\rangle$:

$$(7) \qquad \boldsymbol{\sigma} \equiv \boldsymbol{\sigma}_{\Psi} = |\boldsymbol{\Psi}\rangle\langle\boldsymbol{\Psi}| = \boldsymbol{\sigma}_A + \boldsymbol{\sigma}_B + \boldsymbol{\sigma}_{B\to A} + \boldsymbol{\sigma}_{A\to B},$$

where

$$\boldsymbol{\sigma}_A = |\boldsymbol{\Psi}_A\rangle\langle\boldsymbol{\Psi}_A| = \sum_{i,j}\sqrt{P_iP_j}e^{i(\theta_{ai}-\theta_{aj})}|i_a\rangle\langle j_a|$$

$$\boldsymbol{\sigma}_B = |\boldsymbol{\Psi}_B\rangle\langle\boldsymbol{\Psi}_B| = \sum_{i,j}\sqrt{Q_iQ_j}e^{i(\theta_{bi}-\theta_{bj})}|i_b\rangle\langle j_b|$$

$$\boldsymbol{\sigma}_{B\to A} = |\boldsymbol{\Psi}_A\rangle\langle\boldsymbol{\Psi}_B| = \sum_{i,j}\sqrt{P_iQ_j}e^{i(\theta_{ai}-\theta_{bj})}|i_a\rangle\langle j_b|$$

$$\boldsymbol{\sigma}_{A\to B} = |\boldsymbol{\Psi}_B\rangle\langle\boldsymbol{\Psi}_A| = \sum_{i,j}\sqrt{P_iQ_j}e^{-i(\theta_{ai}-\theta_{bj})}|j_b\rangle\langle i_a|.$$

Comparison operator

In the classical expected utility theory Alice calculates the averages of the utility function. In the quantum-like model the utility function determines the *comparison operator*. Invention of such an operator is based on coupling between the eigenstates of the "lottery-operators" and utilities (amounts of money).

We borrow the utility function from classical (objective or subjective) utility theory. Then we use QP to model subjective probabilities.

The crucial step is operational description of the process of comparison of lotteries with the aid of quantum states transitions which are encoded in the comparison operator.

This process can be structured as combination of comparison of a few SEUs and the interference type factors of the $\cos\theta$-form, where θ represents the combination of phases of a few processes of preferring of outcomes of the lotteries.

Transition operators

Let us introduce the *transition operators*

$$\boldsymbol{E}_{i_a \to j_b} = |\boldsymbol{j_b}\rangle\langle\boldsymbol{i_a}|, \ \boldsymbol{E}_{j_b \to i_a} = |\boldsymbol{i_a}\ \rangle\langle\boldsymbol{j_b}|. \tag{8}$$

We have, e.g., $\boldsymbol{E}_{i_a \to j_b}|\boldsymbol{i_a}\rangle = |\boldsymbol{j_b}\rangle$. This operator describes the process of transition from preferring the state $|\boldsymbol{i_a}\rangle$ to preferring the state $|\boldsymbol{j_b}\rangle$. The operator $\boldsymbol{E}_{j_b \to i_a} = |\boldsymbol{i_a}\rangle\langle\boldsymbol{j_b}|$ describes transition in the opposite direction. We stress that these are transitions between the belief-states of Alice. We remark that $\boldsymbol{E}_{j_b \to i_a} = \boldsymbol{E}^{\star}_{i_a \to j_b}$, i.e., elementary transitions in opposite directions are represented by adjoint operators.

Now we introduce the two comparison operators:

$$\boldsymbol{D}_{B\to A} = \sum_{n,m} (u(x_n) - u(y_m)) e^{i\gamma_{m_b\to n_a}} \boldsymbol{E}_{m_b\to n_a}$$

$$= \sum_{n,m} (u(x_n) - u(y_m)) e^{i\gamma_{m_b\to n_a}} |n_a\rangle\langle m_b|.$$

$$\boldsymbol{D}_{A\to B} = \sum_{n,m} (u(y_m) - u(x_n)) e^{i\gamma_{n_a\to m_b}} \boldsymbol{E}_{n_a\to m_b}$$

$$= \sum_{n,m} (u(y_m) - u(x_n)) e^{i\gamma_{n_a\to m_b}} |m_b\rangle\langle n_a|.$$

The operator $D_{B\to A}$ represents the utility of selection of the lottery A relatively to the utility of selection of the lottery B. We can say that by transition from the potential outcome (B, y_m) to the potential outcome (A, x_n) Alice earns utility $u(x_n)$ and at the same time she loses utility $u(y_m)$. (If $u(x) = x$ and x has the meaning of cash amounts (say USD), then by such a transition Alice (potentially) earns $x_n - y_m$ USD.)

In the same way we interpret the transition operator $D_{A\to B}$. This operator represents the utility of selection of the lottery B relatively to the utility of selection of the lottery A. These operators represent the process of Alice's reflections in the process of decision making. Her mind fluctuates between preferring outcomes of the A-lottery to outcomes of the B-lottery (formally represented by the operator $D_{B\to A}$) and inverse preferring (formally represented by the operator $D_{A\to B}$). Finally, she has to compare how much she can earn (in average) by preferring A to B comparing with preferring B to A.

This process is formally described by *the complete comparison operator:*

$$D = D_{B\to A} - D_{A\to B}. \tag{9}$$

This operator has the form:

$$D = \sum_{n,m} (u(x_n) - u(y_m)) e^{i\gamma_{m_b\to n_a}} |n_a\rangle\langle m_b| - \tag{10}$$

$$\sum_{n,m} (u(y_m) - u(x_n)) e^{i\gamma_{n_a\to m_b}} |m_b\rangle\langle n_a|$$

$$= \sum_{n,m} u_{nm} (e^{i\gamma_{m_b\to n_a}} |n_a\rangle\langle m_b| + e^{i\gamma_{n_a\to m_b}} |m_b\rangle\langle n_a|),$$

where

$$u_{nm} = u(x_n) - u(y_m).$$

Since all quantum observables are represented by Hermitian operators, the phases should be related as follows:

$$\gamma_{n_a \to m_b} = -\gamma_{m_b \to n_a}. \tag{11}$$

The comparison operator D gives us the integral judgment. Only heuristically can we treat the D-based judgment as the result of comparison of two relative utilities represented by the operators $D_{B\to A}$ and $D_{A\to B}$. We remark that the operators $D_{B\to A}$ and $D_{A\to B}$ are not Hermitian. Hence, they cannot be treated as observables. We have that $D^\star_{A\to B} = -D_{B\to A}$ and $D = D_{B\to A} + D^\star_{B\to A}$.

The quantum analog of (subjective) expected utility theory is based on the natural decision rule:

Decision rule. *If the average of the comparison operator* $\boldsymbol{D}$ *is non-negative, i.e.,* $\langle \boldsymbol{D} \rangle = \mathbf{tr} \boldsymbol{D\sigma} = \langle \boldsymbol{D}\boldsymbol{\Psi} | \boldsymbol{\Psi} \rangle \geq \mathbf{0}$, *then* $\boldsymbol{A} \succeq \boldsymbol{B}$.

Using Eqs. (7) and (9) the trace can be written as the sum of four components:

$$\mathbf{tr}\boldsymbol{D\sigma} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{tr}\boldsymbol{D\sigma}_{\boldsymbol{A}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{tr}\boldsymbol{D\sigma}_{\boldsymbol{B}} + \boldsymbol{\Delta}_{\mathbf{1}} + \boldsymbol{\Delta}_{\mathbf{2}},$$

where

$$\boldsymbol{\Delta}_{\mathbf{1}} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{tr}(\boldsymbol{D}_{\boldsymbol{B}\to\boldsymbol{A}}\boldsymbol{\sigma}_{\boldsymbol{A}\to\boldsymbol{B}} - \mathbf{tr}\boldsymbol{D}_{\boldsymbol{A}\to\boldsymbol{B}}\boldsymbol{\sigma}_{\boldsymbol{B}\to\boldsymbol{A}}),$$

$$\boldsymbol{\Delta}_{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{tr}(\boldsymbol{D}_{\boldsymbol{B}\to\boldsymbol{A}}\boldsymbol{\sigma}_{\boldsymbol{B}\to\boldsymbol{A}} - \mathbf{tr}\boldsymbol{D}_{\boldsymbol{A}\to\boldsymbol{B}}\boldsymbol{\sigma}_{\boldsymbol{A}\to\boldsymbol{B}}).$$

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Classical Aumann theorem

Mutual knowledge: everybody in a group of people is aware about some fact or event.

Common Knowledge: Alice and Bob knows about an event $\boldsymbol{E}$ and Alice knows that Bob knows about $\boldsymbol{E}$ and so on...

The celebrated Aumann theorem states that if two agents have common priors, and their posteriors for a given event $\boldsymbol{E}$ are common knowledge, then their posteriors must be equal;

Agents with the same priors and common knowledge about posteriors cannot agree to disagree.

Criticized assumptions:

a). **common priors**, but typically it is justified – as the result of information exchange.

b). common knowledge about posteriors, but again Aumann's statement can be violated even in situations, where this assumption is valid.

This situation is disturbing and the debate about possible sources of violation Aumann's theorem are continued.

We point to an implicit assumption of Aumann:

Agents are rational, where rationality is understood as the use of Bayes' rule to update probabilities.

Agents may update probabilities with schemes different from CP. QP update is a possible math formalism describing non-Bayesian updates. Such agents may agree to disagree; even with common priors and common knowledge.

Quantum state update: projection postulate

There are given a state $\boldsymbol{\rho}$ and an observable $\boldsymbol{A = \sum_i a_i P_i}$. Then

$$p_\rho(a_i) = \mathrm{Tr}\rho\mathrm{P_i}. \tag{12}$$

However, if after measurement of the $\boldsymbol{A}$-observable one plans to perform measurement of another observable $\boldsymbol{B = \sum_i b_i P_i')}$, then one needs to know even the output state:

$$\rho_{a_i} = \frac{P_i \rho P_i}{\mathrm{Tr}\mathrm{P_i}\rho\mathrm{P_i}}. \tag{13}$$

This nothing else than **the quantum version of the classical rule for probability update.** But here we update not the prior probability, but the *prior state*.

For the $\boldsymbol{B}$-measurement following the $\boldsymbol{A}$-measurement, this state plays the same role as the state $\boldsymbol{\rho}$ played for the $\boldsymbol{A}$-measurement. In particular, by applying the Born rule once again we obtain:

$$\boldsymbol{p_{\rho_{a_i}}(b_j) = \mathrm{Tr}\rho_{\mathrm{a_i}}\mathrm{P}'_{\mathrm{j}} = \frac{\mathrm{Tr}\mathrm{P_i}\rho\mathrm{P_i}\mathrm{P}'_{\mathrm{j}}}{\mathrm{Tr}\mathrm{P_i}\rho\mathrm{P_i}}.} \tag{14}$$

In quantum theory this probability is treated as the **conditional probability** $\boldsymbol{p_\rho(P'_j|P_i) \equiv p_\rho(B = b_j|A = a_i)}$.

Quantum(-like) viewpoint on the Aumann's theorem

Disagree from quantum(-like) interference

Theorem 1. *Let the assumption of common prior holds. Then:*

(15)

$$\boldsymbol{q_i - q_s = \frac{1}{\mathrm{Tr}\rho\kappa \mathrm{C}_{\mathrm{q_1\ldots q_N}}}\Big(\sum_{j\neq m}\mathrm{Tr}\mathrm{P}^{(i)}_{\mathrm{k_j}}\rho\mathrm{P}^{(i)}_{\mathrm{k_m}}\mathrm{E} - \sum_{\mathrm{j\neq m}}\mathrm{Tr}\mathrm{P}^{(s)}_{\mathrm{k_j}}\rho\mathrm{P}^{(s)}_{\mathrm{k_m}}\mathrm{E}\Big).}$$

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