Tutorial 4: Fourier Transform

1D Fourier Transform.

Fourier Transform is a mathematical operation that translates a signal from the spatial (or time) domain, into the frequency domain. The basis of transform is the analysis of exponential Fourier series, that is, represent a signal by the sum of the exponential signals that are orthogonal to each other. The exponential function e^{jx} can form a family of orthogonal functions within a certain interval (t_o, t_o +2p/w_o) with the functions { e^{jnwot} } n = 0, $\pm$ 1, $\pm$ 2, . . . where {e} is the complex Euler identity :

$$e^{j\,n\,Wo\,t} = \cos n\, w_0 t + j \sin n\, w_0 t$$

This function can be formed by a pair of sinusoidal signals, by which any signal can be expressed:

$$f(t) = F_0 + F_1 e^{j\,wo\,t} + F_2 e^{j\,2wo\,t} + F_3\, e^{j\,3wo\,t} + \ldots$$
$$+ F_{-1} e^{-j\,wo\,t} + F_{-2} e^{-j\,2wo\,t} + F_{-3} e^{-j\,3wo\,t} + \ldots$$

$$f(t) = \sum_{-\infty}^{+\infty} F_n e^{jn\omega_0 t} \qquad (t_o < t < t_o +T) \qquad T = 2p / w_o$$

The coefficients *Fn*, can be obtained from the similarity between the original signal and the exponential:

$$F_n = \frac{\int_{t_0}^{t_0+T} f(t)(e^{jn\omega_0 t})^*}{\int_{t_0}^{t_0+T} e^{jn\omega_0 t}(e^{jn\omega_0 t})^*} \qquad F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t}$$

It is important to notice that the previous analysis is limited to a certain interval in time (or space). For any interval (-∞,∞) is necessary thus to *transform* the signal and not just to express its series. To do this, the exponential Fourier series of a signal f(t) can be calculated by transforming the signal into a periodic signal where f(t) is the first period. The limit for the period T -> ∞ is evaluated and then the signal has only one cycle in the range of (-∞ < t <+

∞). By taking the limit (and some mathematical manipulations) we can reach the Fourier Transform pair:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

which sometimes is represented by:

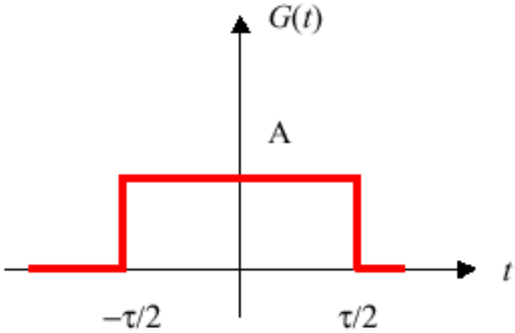
$$F(w) = \mathcal{F}\{ f(t) \} \qquad f(t) = \mathcal{F}^{-1}\{F(w)\}$$

The previous expressions can be complex and therefore two planes can be used to show it: (real/ imaginary) or (magnitude/phase).

In Matlab, the commands fft, and ifft perform the direct and inverse transformations (fft2, ifft2, fftn, ifftn for higher dimensions).

4.1 Generate the signal *G*(*t*) y obtain its Fourier transform.

$$G(t) = \begin{cases} 0 & -\infty < t < -\tau/2 \\ A & -\tau/2 < t < \tau/2 \\ 0 & \tau/2 < t < -\infty \end{cases}$$



Gate Function

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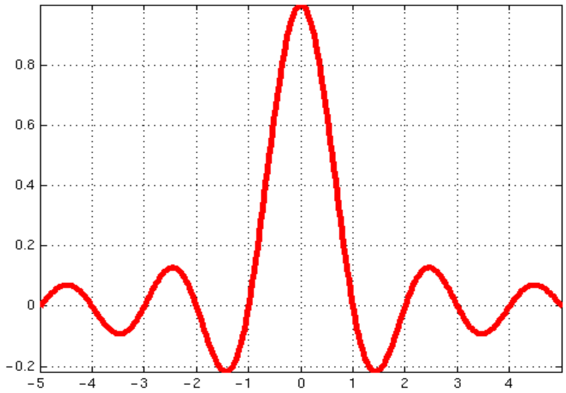
G_w= fft(G_t);
```

Remember that G_w is a complex function, and therefore to plot it you can use the functions: abs, angle, real and imag. Try

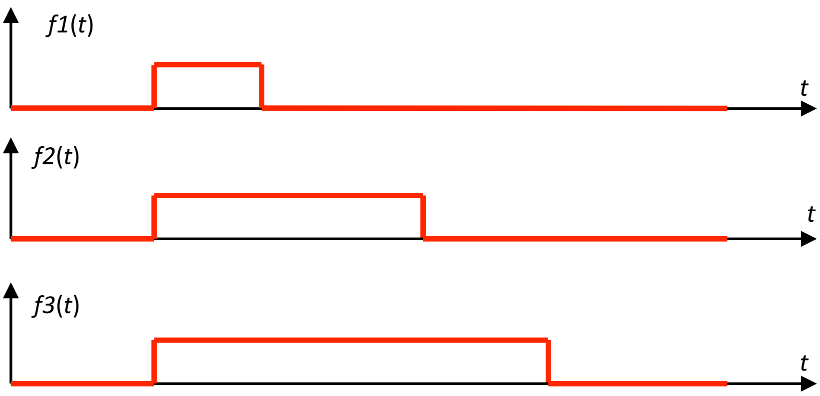
```
plot(abs(G_w)));
plot(angle(G_w));
plot(real(G_w));
plot(imag(G_w));
```

In most cases, only the magnitude is used, still in the following examples try to observe the angle, and the real and imaginary parts.

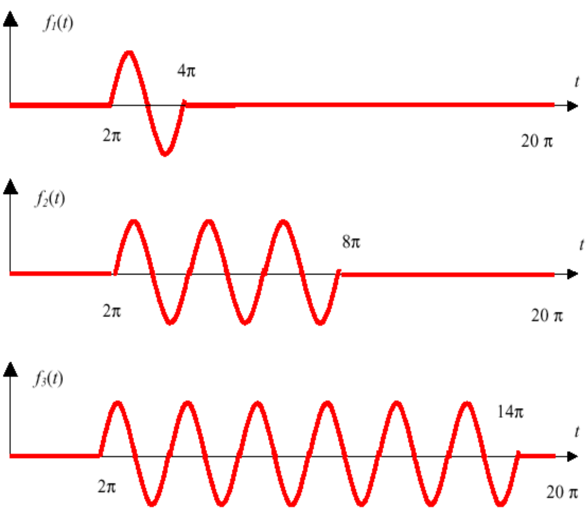


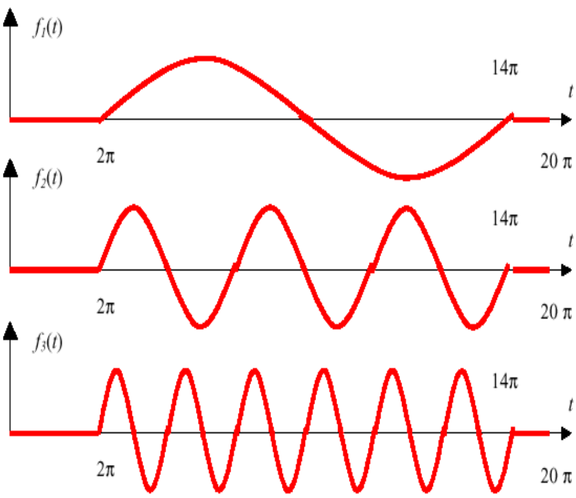
4.2 Generate a sampling function and transform it with *fft.* What can you observe? Are the results what you expected? Perhaps the command fftshift could help.

4.3 Generate the following signals in Matlab, transform them and study the results. Keep the length of the vectors constant.



4.4 Repeat step 1.3 for the following functions:





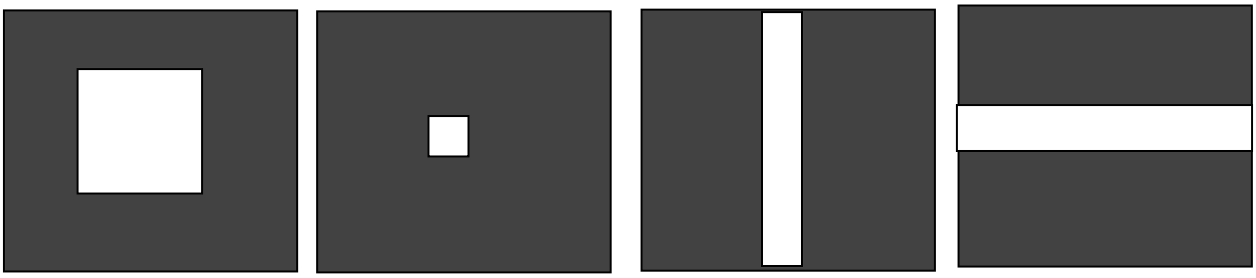
2D Fourier Transform.

Fourier Transform in two dimensions can be expressed as:

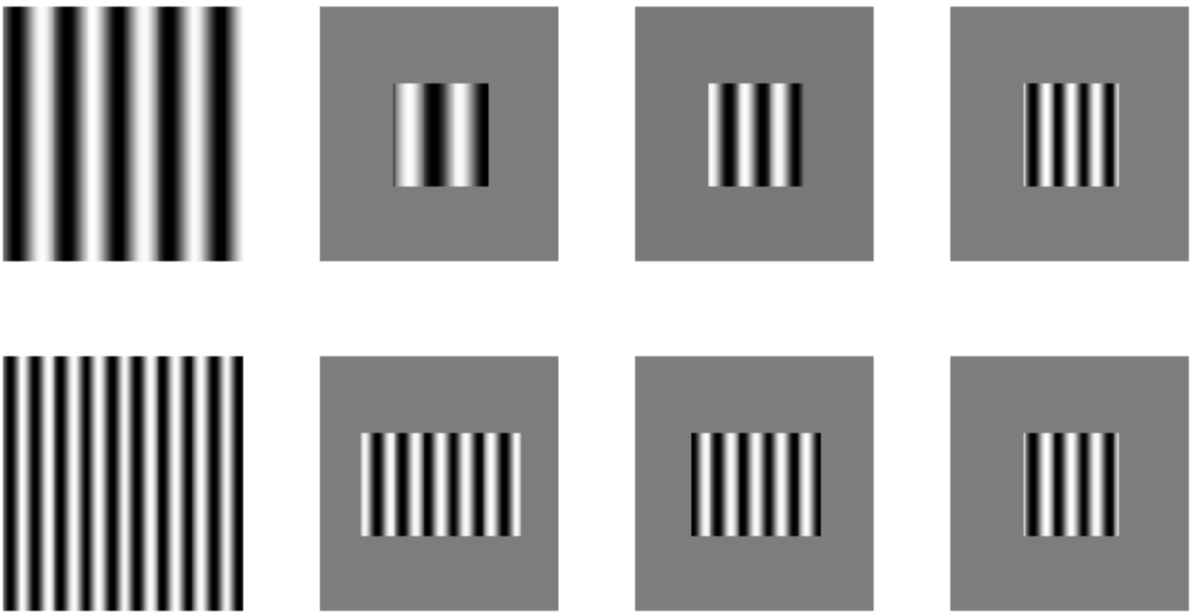
$$F(u,v) = \int_{-\infty}^{+\infty}\int f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy$$

$$f(x,y) = \int_{-\infty}^{+\infty}\int F(u,v)e^{j2\pi(ux+vy)}\,du\,dv$$

4.5 Generate the 2D signal analogous to exercises 1.1 and 1.3, transform and observe:



4.6 Generate the 2D signal analogous to exercise 1.4, transform and observe:



4.7 What happens if you rotate (for 90^{o} you can do this by transposing the matrix y=x';) the image before transforming?

4.8 Observe and compare the difference between the product of the two images and the sum of them (y1.*y2 vs. y1 +y2)