Stochastic Modelling
Exercises on Stochastic Calculus*

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Elementary Problems

Q1. (i) Use Itô's Lemma with the function $f(t, x) = x^2$ to show that $\int_0^t B_\tau dB_\tau = \frac{1}{2}(B_t^2 - t)$.

(ii) Use Itô's Lemma with the function $f(t, x) = x^3$ to show that $\int_0^t B_\tau^2 dB_\tau = \frac{1}{3}B_t^3 - \int_0^t B_\tau d\tau$.

Q2. Let f_t be a non-random function and consider $I[f] = \int_0^t f_\tau dB_\tau$, where $\{B_t\}$ is standard Brownian motion.

(i) Briefly explain why $I[f]$ is normally distributed, stating its mean and variance.

(ii) Briefly explain whether $I[f]$ is a martingale with respect to the natural filtration of $\{B_t\}$.

Q3. The price of a stock is modelled by the stochastic differential equation $\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$ with initial condition $S_0 \in \mathbb{R}$.

(i) The stochastic differential equation above is the abbreviated form of a formal integral equation governing S_t. State this formal equation.

(ii) State Itô's Lemma.

(iii) Use Itô's Lemma to show that the logarithm of the stock price is a Brownian motion with drift coefficient $\mu - \sigma^2/2$ and diffusion coefficient σ.

(iv) State the probability distribution of S_t.

(v) State the expectation and variance of S_t.

Past Exam Questions

Q1. (i) State Itô's Lemma as it applies to a stochastic process $\{X_t : t \geq 0\}$ and a function $f(X_t)$ which is not explicitly dependent on t.

*Elementary problems should be attempted first. Past exam questions are included for exam practice and could be attempted later. They are adapted from papers set by the Exam Board of the Institute and Faculty of Actuaries. Papers set by the Exam Board, Faculty of Actuarial Science and Statistics, Cass Business School, City University, are separately available.

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(ii) Apply Itô's Lemma with $f(x) = x^4$ to calculate the stochastic differential $d(B_t^4)$, where B_t is standard Brownian motion.

(iii) Hence express the Itô integral $\int_0^t B_s^3 dB_s$ in terms of B_s and of an ordinary integral involving B_s.

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Q2. (i) (a) Define standard Brownian motion B_t, $t \geq 0$ and give its transition probability density.

(b) Write down the transition probability density of general Brownian motion $W_t = \sigma B_t + \mu t$.

Let S_t represent a share price at time t.

(ii) Solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

(iii) Calculate, given the parameters $\mu = 25\%$ p.a., $\sigma = 20\%$ on an annual basis, the probability that the share price will exceed 45 in four months' time given that its current price is 38.

(iv) Calculate the probability that the share price will exceed 45 *at any stage* during the next four months given that its current value is 38.

[You may use the formula

$$\mathbb{P}\left(\max_{0 \leq s \leq t} (B_s + \lambda s) > y\right) = G\left(\frac{\lambda t - y}{\sqrt{t}}\right) + e^{2\lambda y} G\left(\frac{-y - \lambda t}{\sqrt{t}}\right).$$

where $y \geq 0$ and G denotes the normalised Gaussian probability distribution function.]

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Q3. (i) State Itô's lemma.

(ii) Solve the following stochastic differential equation for the spot rate of interest:

$$dr_t = a(b - r_t)dt + \sigma dB_t.$$

(iii) What is the probability distribution of the solution r_t? What is the limit of this distribution as $t \to \infty$?

(iv) For the parameters $a = 150\%$ p.a., $b = 3\%$ p.a., $\sigma = 6\%$ on an annual basis, what is the probability that the rate of interest will be less than 4% p.a. in six months time given that its current value is 2.5%?

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