ADVANCED HEAT TRANSFER

1. MODES OF HEAT TRANSFER

Conduction

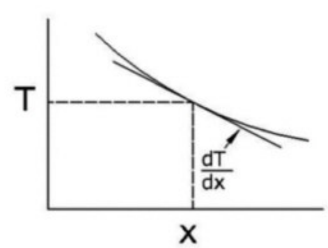
Conduction is the mode of heat transfer through solids, which may also occur in liquids and gases under circumstances where bulk motion of the fluid is not possible. It takes the form of a thermal wave propagated through the system by the vibration of molecules and free electrons. Thus, solids with a crystalline structure and rich in free electrons make the best conductors. It has been deduced that in crystalline type metals, the free electron motion is responsible for most of the conduction. It follows that good conductors of heat are also good conductors of electricity. Due to their larger intermolecular distances, liquids and gases have a correspondingly lower conducting ability than solids.

The ability of a material to conduct heat is called its thermal conductivity, k,
dimensions: Wm/m^2 deg C or Btu/hr deg F ft^2/ft.

The fundamental equation of conduction is the Fourier Rate Equation (1822)

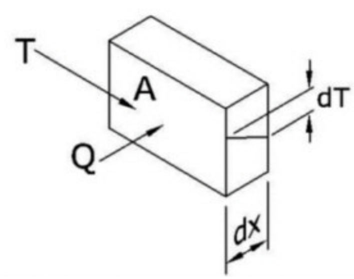
$$\dot{Q} = -k \cdot A \cdot \frac{\partial T}{\partial x} \qquad (1)$$

Where: Q = Rate of heat transfer (W or Btu/hr)
k = Thermal conductivity
A = Cross sectional area normal to heat flow
T = Temperature
x = Distance in direction of heat flow



or per unit area

$$\dot{q} = \frac{\dot{Q}}{A} = -k \frac{\partial T}{\partial x}$$

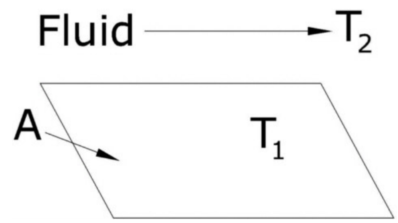


Convection

This occurs in fluids when the heat transfer by conduction across the fluid boundaries is augmented by motion of the fluid, which transports energy both by internal circulation of the fluid and its bulk motion. The study of convection is therefore associated with the study of fluid mechanics. It is most important in the study of heat flow between solid surfaces and the fluids in contact with them.

The fundamental equation of convection between a solid surface and a fluid is Newton's Rate Equation (1701).

$$\dot{Q} = h \cdot A \cdot (T_1 - T_{2l})$$



Where h = Convective heat transfer coefficient
(W/m^2/deg C or Btu/ft^2 deg F hr)

or per unit area

$$\dot{q} = \frac{\dot{Q}}{A} = h \cdot (T_1 - T_2) = h \cdot \theta \qquad (2)$$

Where θ is the temperature difference

The estimation of h is a very complex matter in all but the simplest cases since it is a function of fluid temperature, fluid motion and surface geometry. It is performed either by scaling the results of model tests by means of dimensional analysis or by the fundamental equations of fluid motion and heat conduction, usually with the aid of complex computer programs which are often compiled with the aid of existing software packages.

Radiation

This is significantly different in nature from conduction and convection in that energy transfer is by means of electromagnetic waves and these can travel through free space in the absence of a material medium. The rate of radiation from a substance is proportional to the fourth power of its absolute temperature. A substance can not only emit radiation but can also absorb or transmit it from another source depending on the wavelength of the radiation received and the physical properties of the substance. Thus glass is almost transparent to short wave high temperature emissions from the Sun but absorbs long wave low temperature emissions from the Earth. At low temperatures, gases are fairly transparent and emit little radiation. In such cases, heat transfer between a solid body and its surrounding environment can be accounted for by convective heat exchange and radiation between the body and its surroundings while ignoring the effect of the fluid on the radiant heat transfer. At high temperatures, as in the case of flames, gases become significant emitters.

The basic equation of radiation is the Stefan-Boltzmann Law for black body radiation (1879 Stefan by experiment, 1884 Boltzmann by theory).

$$\dot{Q}_b = \sigma \cdot A \cdot T^4$$

or $$\dot{q}_b = \frac{\dot{Q}_b}{A} = \sigma \cdot T^4 \qquad (3)$$

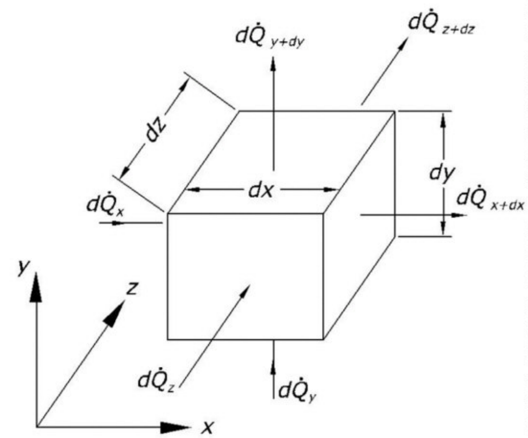
Where σ is the Stefan-Boltzmann Constant $= 5.67\text{x}10^{-8}$ W/m^2 $(K)^4$ and K = ^{o}C+273.16
$= 0.171\text{x}10^{-8}$Btu/ft^2 hr $(R)^4$ R = ^{o}F+459.7

2. CONDUCTION

The general differential equation of conduction

Consider a solid of infinitesimal volume with sides of length dx, dy and dz. In the x direction, the cross sectional area = dy.dz. For heat flow into the volume in this direction we may therefore write equation 1 as:

$$d\dot{Q}_x = -k \cdot dy \cdot dz \cdot \frac{\partial T}{\partial x}$$



Similarly for heat flow out in the x direction we may write equation 1 as:

$$d\dot{Q}_{x+dx} = -k \cdot dy \cdot dz \cdot \frac{\partial}{\partial x}(T + \frac{\partial T}{\partial x}dx)$$
$$= -k \cdot dy \cdot dz \frac{\partial T}{\partial x} - k \cdot dx \cdot dy \cdot dz \frac{\partial^2 T}{\partial x^2}$$

Hence the net heat flow into the element in the x direction is:

$$d\dot{Q}_x - d\dot{Q}_{x+dx} = k \cdot dx \cdot dy \cdot dz \cdot \frac{\partial^2 T}{\partial x^2}$$

It follows that summing up for all three directions, the total net heat flow into the element is:

$$k \cdot dx \cdot dy \cdot dz(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2})$$

In addition to the flow of heat through the element, heat may be generated within it though such processes as the flow of electricity (Joulean heating) or radioactivity. If q' is the rate of heat generation per unit volume, then the heat generation within the element is:

$$q' \cdot dx \cdot dy \cdot dz$$

These processes together create a storage of heat within the element. If the solid is of density ρ and specific heat capacity c_p, then the rate of increase of stored heat with time in the volume is:

$$dx \cdot dy \cdot dz \cdot \rho \cdot c_p \cdot \frac{\partial T}{\partial t}$$

Since the term dx.dy.dz is common to all terms, we may therefore express this as:

$$\rho \cdot c_p \cdot \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q'$$

$$\therefore \frac{\partial T}{\partial t} = \alpha \cdot \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q'}{\rho \cdot c_p} \qquad (4)$$

Where: $\alpha = \dfrac{k}{\rho \cdot c_p}$ is known as the thermal diffusivity of the material of the element and is the ratio of conduction to storage qualities of the material.

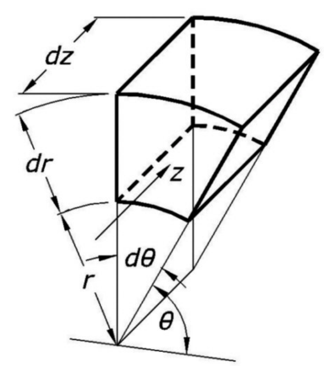
Equation 4 is known as the general differential equation of heat conduction and may be simplified for specific applications. Thus, if there is no internal heat generation and the flow of heat is steady, then the first term of the LHS and the last term of the RHS are both zero. The equation then becomes:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

This is, of course the well known Laplace equation which describes all processes in which a driving force produces a linear displacement, such as deformation of an elastic solid, viscous flow of fluids and flow of electric current. Also, terms in y and z may be removed if only one or two dimensional flow of heat is considered.

Equation 4 is written using Cartesian coordinates. When dealing with bodies of cylindrical shape such as pipes, shells, etc, it is often better to express it in cylindrical coordinates where $x = r \cdot \cos\theta$ and $y = r \cdot \sin\theta$. Either by substitution or deriving from first principles, using a cylindrical element of volume, we get:

$$\frac{\partial T}{\partial t} = \alpha \cdot \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q'}{\rho \cdot c_p} \qquad (5)$$



Equations 4 and 5 can generally be solved analytically for one dimensional flow problems but for heat flow in two or three dimensions, numerical methods are usually needed to obtain a solution. For many applications, existing software is available to obtain such solutions.

One Dimensional Conduction with Heat Generation

This is important for electrical conductors and nuclear reactors.

In electrical conductors: Heat generated/unit volume = $\rho^* i^2$ = R

Where: ρ^* = Resistivity and i = current density

In Cartesian coordinates

Equation 4 for one dimensional flow then can be written as:

$$\frac{\rho \cdot c_p}{k} \cdot \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\rho^* \cdot i^2}{k}$$

which in steady state becomes:

$$\frac{d^2 T}{dx^2} + \frac{\rho^* \cdot i^2}{k} = 0 \qquad \therefore \frac{dT}{dx} = -\frac{R}{k} \cdot x + A$$

and hence we get:

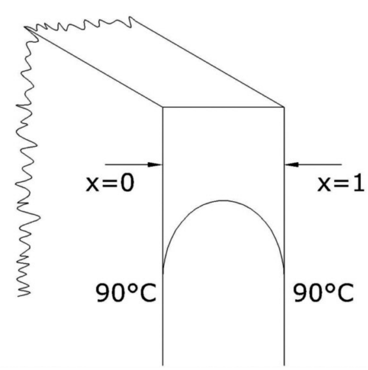
$$T = -\frac{R \cdot x^2}{2 \cdot k} + A \cdot x + B$$

A and B are constants to be determined from known boundary conditions i.e. T or $\frac{dT}{dx}$ at the boundaries. Thus: for a flat plate insulated on one side, $\frac{dT}{dx} = 0$ at that surface.

Ex *Consider a wide flat plate 1.0 cm thick. k = 380 W/m K, ρ^* = 1.8x10^{-8}ohm m, i = 2000 Amp/cm^2, and each surface is at 90^{o}C. Find the position across the plate at which the temperature is a maximum, the value of the maximum temperature and the rate of heat loss to the surroundings.*

$$R = 1.8x10^{-6} x2000^2 = 7.2W / cm3$$

$$\frac{R}{2k} = \frac{7.2}{2x3.8} = 0.948\ ^o\ C / cm^2$$



Boundary Conditions:

At x = 0: *90 = -0+0+B* $\therefore$ *B = 90*

At x = 1: *T = -0.948+A+90* $\therefore$ *A = 0.948*

$\therefore$ *T = 0.948x^2+0.948x+90*

$$\therefore \quad \frac{dT}{dx} = -1.896x + 0.948 = 0 \qquad \textit{at position of maximum temperature}$$

$$\therefore \quad x = \frac{0.948}{1.896} = 0.5\ cm \qquad \textit{i.e maximum is at the middle of the plate}$$

$$\therefore \quad At\ x = 0.5: \quad T = -0.237 + 0.474 + 90 = 90.237$$

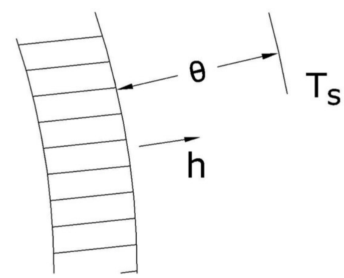
$$(\frac{dT}{dx})_{x=0} = \ 0.948\ {}^oC/cm \quad \therefore q_{x=0} = -3.8x0.948 = -3.6\ W/cm^2$$

$$(\frac{dT}{dx})_{x=1} = -0.948\ {}^oC/cm \quad \therefore q_{x=1} = -3.8x - 0.948 = +3.6\ W/cm^2$$

$$\therefore \quad q = q_{x=0} - q_{x=1} = -3.6 - 3.6 = -7.2\ W/cm^2$$

Note opposite signs of heat flow due to their opposite directions. This total, is equal to the rate of energy dissipated in the plate due to Joulean heating.

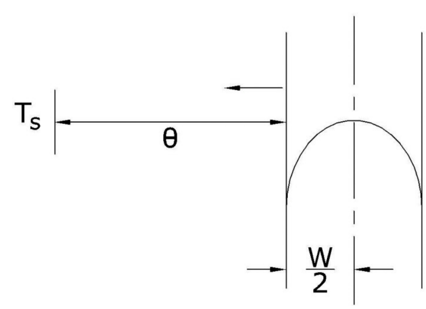
Procedure for estimating the heat transfer from a surface if $h = f(\theta)$ and the stream temperature T_s are known.



i) Guess θ. Hence obtain the surface temperature.
ii) Evaluate the constants A and B.
iii) Evaluate the temperature gradients at the surface.
iv) Evaluate the heat transfer by conduction at the surface from:
$q = -k\dfrac{dT}{dx}$ at the surface
v) This must be equal to the convection at the surface. i.e. $q = -k\dfrac{dT}{dx} = h \cdot \theta$
vi) If the conduction and convection terms are not equal, revise the estimate of θ i.e. use a trial and error method. (Can be programmed).

Flat plate with equal temperatures on both sides

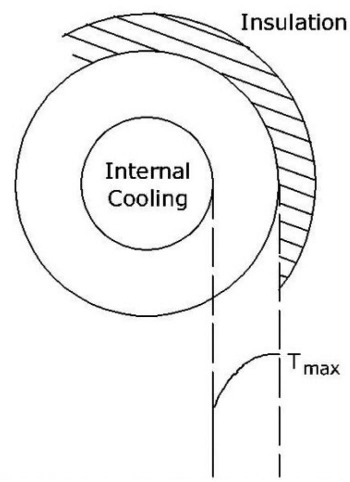
$$q = h \cdot \theta = \rho^* \cdot i^2 \frac{W}{2}$$



i.e. convection from one side/unit area = heat generated in half the bar width.

Hence θ is found.

In cylindrical coordinates



In cylindrical coordinates equation 5 in steady uniform radial flow can be written as:

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = -\frac{R}{k}$$

$$\therefore T = -\frac{R \cdot r^2}{4 \cdot k} + A \ln r + B$$

Circular solid conduction

Maximum temperature must be at the centre:

At the centre: $\frac{dT}{dr} = -\frac{2 \cdot R \cdot r}{4k} + \frac{A}{r} = 0$ $\quad \therefore A = 0$ $\quad \therefore T_{r=0} = B$

Find the value of B from T at the surface radius.

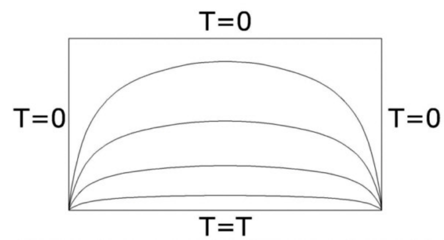
<u>Two-dimensional steady state conduction</u>

The two dimensional steady state form of equation 4 is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{R}{k} = 0 \qquad \text{without heat generation.}$$

The analytical form of the solution of this equation is of the form $T = f(x, y)$

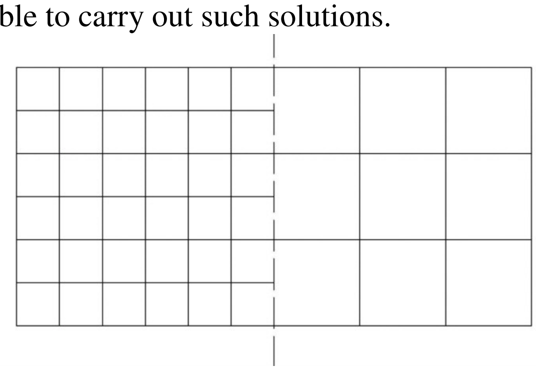
If this is known, then the heat flow can be estimated by solving $q_x = -k \cdot \frac{\partial T}{\partial x}$ and $q_y = -k \cdot \frac{\partial T}{\partial y}$ around the boundaries of the solid shape described.



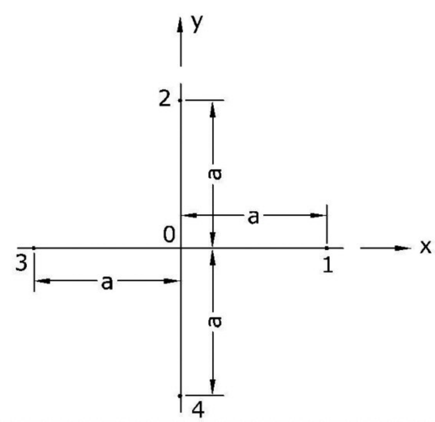
Analytical solutions are possible only for simple and generally non-representative shapes such as a rectangular plate heated along one edge, as shown.

In general, with complicated boundary conditions and shapes, numerical solutions are far more effective, especially now, when software is readily available to carry out such solutions.

A Numerical Method of Solution



A grid of mesh lines is placed over the system. The solution gives temperatures at the grid points only. For ease of computation, few points are required. For accuracy of solution, a fine grid is needed. Generally, a medium grid gives sufficient accuracy without excessive computation.



Consider a single mesh point surrounded by 4 others at a distance “a” from it. Since temperature functions are continuous, the temperatures at points 1,2, 3 and 4 may be expressed in terms of the temperature at 0 by Maclaurin’s theorem. Thus:

$$T_1 = T_0 + \left(\frac{\partial T}{\partial x}\right)_{x=0} \frac{a}{1!} + \left(\frac{\partial^2 T}{\partial x^2}\right)_{x=0} \frac{a^2}{2!} + \left(\frac{\partial^3 T}{\partial x^3}\right)_{x=0} \frac{a^3}{3!} + ...$$

$$T_3 = T_0 - \left(\frac{\partial T}{\partial x}\right)_{x=0} \frac{a}{1!} + \left(\frac{\partial^2 T}{\partial x^2}\right)_{x=0} \frac{a^2}{2!} - \left(\frac{\partial^3 T}{\partial x^3}\right)_{x=0} \frac{a^3}{3!} + ...$$

$$\therefore T_1 + T_3 = 2T_0 + \left(\frac{\partial^2 T}{\partial x^2}\right)_0 a^2 \quad similarly \quad T_2 + T_4 = 2T_0 + \left(\frac{\partial^2 T}{\partial y^2}\right)_0 \cdot a^2$$

$$\therefore \frac{T_1 + T_2 + T_3 + T_4 - 4T_0}{a^2} = \left(\frac{\partial^2 T}{\partial x^2}\right)_{x=0} + \left(\frac{\partial^2 T}{\partial y^2}\right)_{x=0} \qquad (6)$$

or $\dfrac{T_1 + T_3 - 2T_0}{a^2} = Finite \ \ difference \ \ form \ \ of \ \left(\dfrac{\partial^2 T}{\partial x^2}\right)$

and $\dfrac{T_2 + T_4 - 2T_0}{a^2} = Finite \ \ difference \ \ form \ \ of \ \left(\dfrac{\partial^2 T}{\partial y^2}\right)$

$\therefore$ The original equation may be written as :

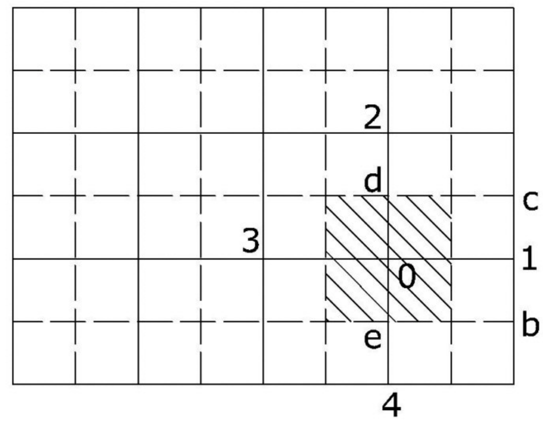
$$\frac{T_1 + T_2 + T_3 + T_4 - 4T_0}{a^2} = -\frac{R}{k} = 0 \ \ Without \ \ heat \ \ generation \qquad (7)$$

$$or \ \ T_1 + T_2 + T_3 + T_4 - 4T_0 = -\frac{a^2 R}{k} = 0 \ \ Without \ \ heat \ \ generation \qquad (8)$$

There will be an equation of this type for every point on the grid. Solution of these simultaneously is required and the accuracy of the result improves as the mesh is made finer. These calculations can be performed by hand (in very simple cases) or by a computer program.

Conducting Rod Analogy

Consider the grid shown:



——————— = Original Mesh Lines

– – – – – – – – – = Squares of Side "a" surrounding the original mesh points. These are squares of conducting material.

Assume heat flows only at right angles across the broken lines along the mesh lines joining the points. Also, assume bc is at temperature T_1 along its length and de is at temperature T_0 along its length and that this applies to all other selected points.

$$Q_{(1-0)} = -k \cdot (Area) \cdot (Temperature\ \ Gradient) = -k \cdot (a \times 1) \times \left(\frac{T_0 - T_1}{a} \right)$$

$$= k \cdot \left(T_1 - T_0 \right)$$

$\therefore$ Net heat flow to 0 along the mesh in the steady state

$$= k \cdot \left(T_1 + T_2 + T_3 + T_4 - 4T_0 \right) = -a^2 R\ \ With\ \ Heat\ \ Generation \tag{9}$$

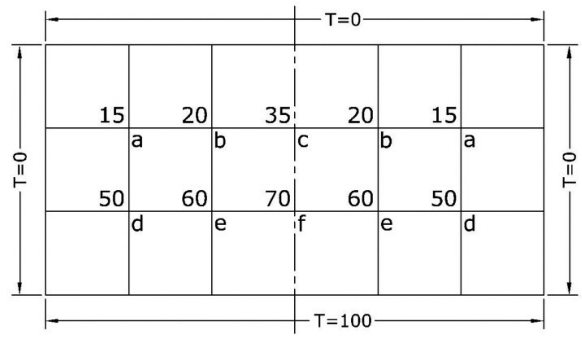
$$= 0\ \ Without\ \ Heat\ \ Generation \tag{10}$$

This is called the **Conducting Rod Analogy**. As may be seen it corresponds exactly with the finite difference relationships derived in equations (6), (7) and (8).

Solution of the Equations

Before the widespread use of computers, the temperatures at the grid points were derived by hand calculation. Initial guesstimated values were assigned to all the points and then gradually adjusted by a numerical procedure known as relaxation. Relaxation methods are still valid but modern computer programs use more complex procedures to produce a faster convergence. The method of solution is, however, best appreciated with an understanding of relaxation procedures.

Ex: *Determine the heat transfer to a rectangular plate heated along one edge when the heated edge is at 100^oC and all the other edges are at 0^oC.*



Trial: *Apply equation (10) to point "a" to see if it is valid:*

We get: $0 + 0 + 20 + 50 - 4\times 15 = 10$

There is a residual value of 10 on the RHS. If the temperatures were assigned correctly, then it would be 0.

Determine the residuals for all the remaining points:

$$
\begin{array}{lll}
b: & 15 + 35 + 0 + 60 - 80 & = 30 \\
c: & 70 + 20 + 20 + 0 - 140 & = -30 \\
d: & 0 + 100 + 60 + 15 - 200 & = -25 \\
e: & 100 + 50 + 70 + 20 - 240 & = 0 \\
f: & 60 + 60 + 100 + 35 - 280 & = -25
\end{array}
$$

Point e gives the correct residual but altering the surrounding points will probably upset it.

Starting with the largest residual, eliminate it by adding to the temperature at that point, an amount equal to ¼ of the residual. Then re-estimate each residual again, assuming the new relaxed value of temperature. Repeat for the next largest residual and continue till all residuals are close to zero, making up a table in the process.

	New	*Residuals at each point*					
	Value	*a*	*b*	*c*	*d*	*e*	*f*
Initial Values→		*10*	*30*	*-30*	*-25*	*0*	*-25*
Increment ↓							
Point c – 7^o	*28^o*		*23*	*-2*			*-32*
Point f – 8^o	*62^o*			*-10*		*-8*	*0*
Point b + 6^o	*26^o*	*16*	*-1*	*2**		*-2*	
Point d – 6^o	*44^o*	*10*			*-1*	*-8*	
Point a + 2^o	*17^o*	*2*	*1*		*1*		
Point e – 2^o	*58^o*		*-1*		*-1*	*0*	*-4**
Point f – 1^o	*61^o*	*+2*	*-1*	*1*	*-1*	*-1*	*0*

* *Increment comes from both halves due to axis of symmetry*

Check: *a:*   *26 +44 – 68*   =   *2*

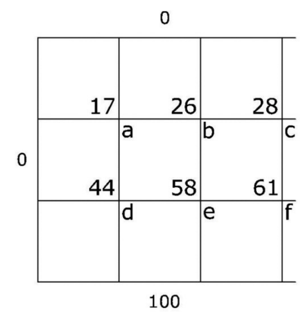
b: *17 + 27+ 58 – 104*   =   *-1*

c: *52 + 61 – 112*   =   *1*

d: *117 + 58 – 176*   =   *-1*

e: *126 + 105 –232*   =   *-1*

f: *116 + 128 -244*   =   *0*



Hence: *Energy flowing into the plate* $= k(100-61) + 2k(100-58) + 2k(100 - 44)$

$= 235k$ */unit thickness*

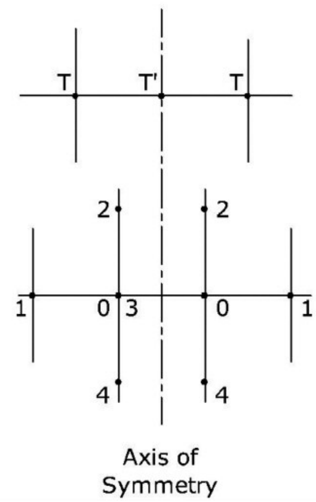
Check: *Energy flowing out of the plate*$= 2k(44 - 0)+ 4k(17 - 0) +2k(26 - 0) + k(28 - 0)$

$= 236k$ */unit thickness*

Units are in W/m (or Btu/ft hr)

Relaxation of points adjacent to lines of symmetry

1. As shown in the example, since points on either side of the line of symmetry have the same temperature, the residual of T' is altered by twice the value of the alteration to the other points surrounding T'.



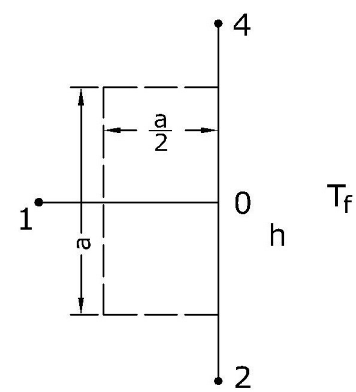
2. $T_1 + T_2 + T_3 + T_4 - 4T_0 \quad = \quad 0$

   or $\quad T_1 + T_2 + T_0 + T_4 - 4T_0 \quad = \quad 0$

   i.e. $\quad T_1 + T_2 + T_4 - 3T_0 \quad = \quad 0$

Hence if the residual has value +*r*, increase T_0 by $\frac{r}{3}$.

Application of the Conducting Rod Analogy to boundary points with convection at the surface



$$Q_{(1-0)} = \frac{-k \cdot a \cdot (T_0 - T_1)}{a} = -k \cdot (T_0 - T_1)$$

$$Q_{(2-0)} = \frac{-k \cdot a}{2} \cdot \frac{(T_0 - T_2)}{a} = \frac{-k}{2} \cdot (T_0 - T_2)$$

$$Q_{(4-0)} = \frac{-k \cdot a}{2} \cdot \frac{(T_0 - T_4)}{a} = \frac{-k}{2} \cdot (T_0 - T_4)$$

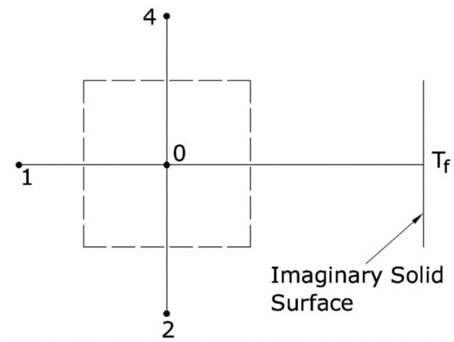
$$Q_{(f-0)} = -h \cdot a \cdot (T_0 - T_f)$$

$$\therefore \frac{\sum \dot{Q}}{k} = T_1 - T_0 + \frac{T_2 - T_0}{2} + \frac{T_4 - T_0}{2} + (T_f - T_0) \cdot \frac{h\,a}{k}$$

$$= T_1 + \frac{T_2 + T_4}{2} + \frac{h.aT_f}{k} - T_0\left(2 + \frac{h\,a}{k}\right) \quad = 0 \quad \textit{Without heat generation}$$

$$= \frac{-a^2 R}{2} \quad \textit{With heat generation}$$

This method is suitable for use in computer programs. However, when using relaxation methods, the following procedure is easier to implement.



$$Q_{1-0} = -k(T_0 - T_1)$$
$$Q_{2-0} = -k(T_0 - T_2)$$
$$Q_{4-0} = -k(T_0 - T_4)$$
$$Q_{f-0} = -h \cdot a \cdot (T_0 - T_f)$$

$$\therefore \frac{\sum \dot{Q}}{k} = T_1 + T_2 + T_4 + \frac{h.a.T_f}{k} - T_0 \cdot \left(3 + \frac{h.a}{k}\right) = 0 \ \textit{Without heat generation} \qquad (11)$$

Compare this result with equation 10, which we may write as:

$$T_1 + T_2 + T_3 + T_4 - 4T_0 = 0$$

If, in equation (11) we define:

$$T_3 \equiv \frac{h \cdot a \cdot T_f}{k} + T_0 \cdot \left(1 - \frac{h \cdot a}{k}\right)$$

Then choose the dimension of "a" such that $\frac{h \cdot a}{k} = 1$

Equation 11 becomes identical with equation 10. This is equivalent to extending the solid boundary 'a' into the fluid and replacing the temperature T_f by T_3 as given. The normal relaxation procedure can then be used as if the solid boundaries have a known temperature.

Transient Conduction

A knowledge of the rate at which thermal equipment heats up or cools down from the steady state at which heat generation and cooling rates exactly balance, is of the utmost importance in engineering. Such factors as how long an electric motor can run on overload, how quickly an I.C engine can develop full power from a cold start and whether or not an article will break due to a sudden cooling or heating are determined from estimates of heating and cooling rates.

Two approaches are employed.

i) For relatively small systems: The system may be assumed to be at a uniform temperature throughout, at all times.
ii) Where the uniform temperature assumption is not valid: The differential equations of conduction must be solved wrt time.

i) Uniform Temperature System

Criterion for the validity of the approach.

The thermal resistance of the body must be relatively small.

$$L = \frac{Volume}{Area \;\; of \;\; surface} \quad e.g. \;\; for \;\; a \;\; wire: L = \frac{\frac{\pi \cdot d^2}{4} \times Length}{\pi \cdot d \times Length} = \frac{d}{4}$$

Let $\frac{L}{k}$ = *Thermal resistance of the body and let* $\frac{1}{h}$ *is its surface convective resistance*

Then $\frac{h \cdot L}{k} = \frac{Thermal \;\; Resistance}{Surface \;\; Resistance} \equiv Biot \;\; Number$

If the value of the Biot Number is small, say < 0.1, then the assumption of uniform temperature is valid.

Uniform Temperature Equation

Change of energy in system in time dt = Heat flow to surroundings in time dt

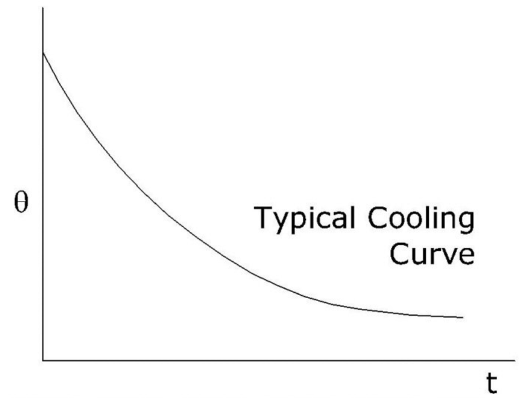
$$\therefore -\rho \cdot V \cdot c_p \cdot dT = h \cdot A(T - T_s) \cdot dt \qquad Where \; T_s = Surrounding \; Temperature$$

Let $T\text{-}T_s = \theta$, *then, if* T_s = *Constant,* $d\theta = dT$

$$\therefore -\rho \cdot V \cdot c_p \cdot d\theta = h \cdot A \cdot \theta \cdot dt$$

$$\therefore \frac{d\theta}{\theta} = \frac{-h \cdot A}{\rho \cdot V \cdot c_p} \cdot dt$$

$$\therefore \ln \frac{\theta_2}{\theta_1} = \frac{-h \cdot A}{\rho \cdot V \cdot c_p} \cdot t = e^{-\frac{t}{\mathrm{T}}}$$



If θ_1 *= initial temp difference,*
θ_2 *= final temp difference,*
t *= time elapsed.*

$$\boldsymbol{T} = \text{Time Constant} = \frac{\rho \cdot V \cdot c_p}{h \cdot A}$$

The value of the time constant is indicative of the rate of response of the system to a change in environmental temperature.

Uniform Temperature System with Heat Generation

Ex: to find the time required for an electric motor to reach its steady state temperature.

$$\text{Let } \dot{q} \cdot V = \text{Rate of heat generation}$$

The rate of increase in system energy can be expressed as:

$$\rho \cdot V \cdot c_p \cdot d\theta = (\dot{q} \cdot V - h \cdot A \cdot \theta) \cdot dt$$

$$\therefore \frac{dt}{\rho \cdot V \cdot c_p} = \frac{d\theta}{\dot{q} \cdot V - h \cdot A \cdot \theta}$$

$$\therefore \frac{t}{\rho \cdot V \cdot c_p} = \left[-\frac{1}{h \cdot A} \cdot \ln(\dot{q} \cdot V - h \cdot A \cdot \theta) \right]_0^\theta = -\frac{1}{h \cdot A} \cdot \ln \left(\frac{\dot{q} \cdot V - h \cdot A \cdot \theta}{\dot{q} \cdot V} \right)$$

$$\therefore -\frac{h \cdot A \cdot t}{\rho \cdot V \cdot c_p} = \ln \left(1 - \frac{h \cdot A \cdot \theta}{\dot{q} \cdot V} \right)$$

$$\text{or } e^{-\frac{t}{\mathrm{T}}} = 1 - \frac{h \cdot A}{\dot{q} \cdot V} \cdot \theta$$

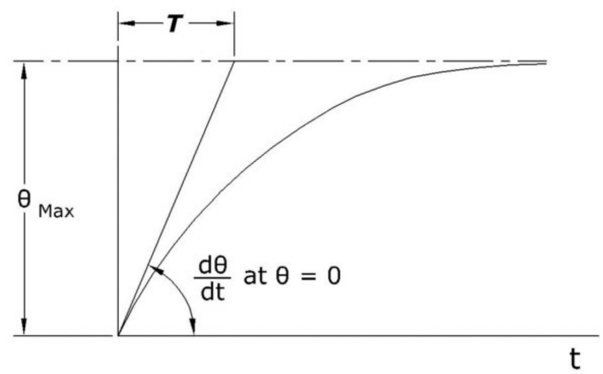
The maximum temperature is reached when the rate of heat generation is equal to the rate of convective heat loss from the system. Let the value of θ when this occurs = θ_{max}. Then:

$$\dot{q}.V = h.A.\theta_{max} \quad or \quad \theta_{max} = \frac{\dot{q}.V}{h.A}$$

$$\therefore \qquad e^{\frac{-t}{\mathrm{T}}} = 1 - \frac{\theta}{\theta_{max}}$$

$$\therefore \quad \theta_{max}.e^{\frac{-t}{\mathrm{T}}} = \theta_{max} - \theta$$

$$\therefore \qquad \theta = \theta_{max}.\left(1 - e^{\frac{-t}{\mathrm{T}}}\right)$$



Hence the rate of temperature rise when θ = 0 can be written as:

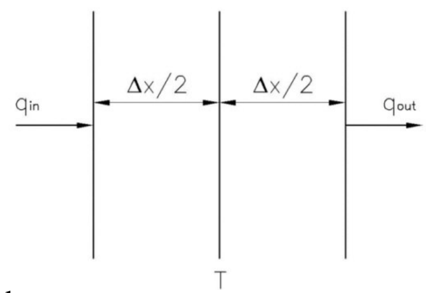
$$\frac{\dot{q} \cdot V}{\rho \cdot V \cdot c_p} = \frac{h \cdot A \cdot \theta_{max}}{\rho \cdot V \cdot c_p} = \frac{\theta_{max}}{\mathrm{T}}$$

Thus, the appearance of the time constant on a temperature - time graph.

ii) The Differential Equation of Transient Conduction in One Dimension (Rectangular Coordinates)

$$\dot{q}_{in} = -k \cdot \frac{\partial}{\partial x} \cdot \left(T - \frac{1}{2} \cdot \frac{\partial T}{\partial x} \cdot \frac{\Delta x}{1!} + \frac{1}{4} \cdot \frac{\partial^2 T}{\partial x^2} \cdot \frac{\Delta x^2}{2!} + \cdots\cdots \right)$$

$$\dot{q}_{out} = -k \cdot \frac{\partial}{\partial x} \cdot \left(T + \frac{1}{2} \cdot \frac{\partial T}{\partial x} \cdot \frac{\Delta x}{1!} + \frac{1}{4} \cdot \frac{\partial^2 T}{\partial x^2} \cdot \frac{\Delta x^2}{2!} + \cdots\cdots \right)$$



$\therefore \dot{q}_{in} - \dot{q}_{out} = -k \cdot \Delta x \cdot \frac{\partial^2 T}{\partial x^2}$ Neglecting Δx^2 and higher order terms.

= Rate of Energy Storage (Zero in the steady state)

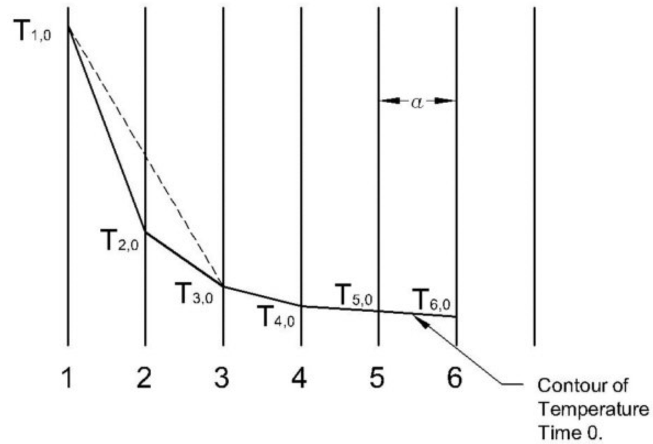
$$= \Delta x \cdot 1 \cdot \rho \cdot c_p \cdot \frac{\partial T}{\partial t}$$

$$\therefore \frac{\partial T}{\partial t} = \frac{k}{\rho \cdot c_p} \cdot \frac{\partial^2 T}{\partial x^2} = \alpha \cdot \frac{\partial^2 T}{\partial x^2} \quad where \quad \alpha = \frac{k}{\rho \cdot c_p} = Thermal \quad diffusivity$$

This equation may be solved by a finite difference procedure.

Method of Solution

Boundary conditions must be specified at positions 1 and 6.



a = Space intervals

Δt = Time interval

Consider points $T_{1,0}$ $T_{2,0}$ and $T_{3,0}$ on mesh lines "a" apart.

From the analysis of 2-dimensional steady state conduction, we obtained the finite difference relationship:

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{2,0} = \frac{T_{3,0} + T_{1,0} - 2 \cdot T_{2,0}}{a^2}$$

and a finite difference relationship for $\left(\frac{\partial T}{\partial t}\right)_{2,0}$ *is* $\frac{T_{2,1} - T_{2,0}}{\Delta t}$

Hence: $$\frac{T_{2,1} - T_{2,0}}{\Delta t} = \alpha \cdot \frac{(T_{3,0} + T_{1,0} - 2 \cdot T_{2,0})}{a^2}$$

$$\therefore T_{2,1} - T_{2,0} = \frac{2 \cdot \Delta t \cdot \alpha}{a^2} \cdot \left(\frac{T_{3,0} + T_{1,0}}{2} - T_{1,0}\right)$$

This is the basis of the Schmidt graphical solution, for if we choose "a" so that

$$\frac{2 \cdot \Delta t \cdot \alpha}{a^2} = 1 \text{ , } \textit{then it follows that } T_{2,1} = \frac{T_{3,0} + T_{1,0}}{2}$$

Hence joint $T_{1,0}$ and $T_{3,0}$ by a straight line. This cuts line 2 at $T_{2,1}$.

Ex: *Consider a sheet of glass 3.6 cm thick. Its surface temperature rises by 20^{o}C/min.*

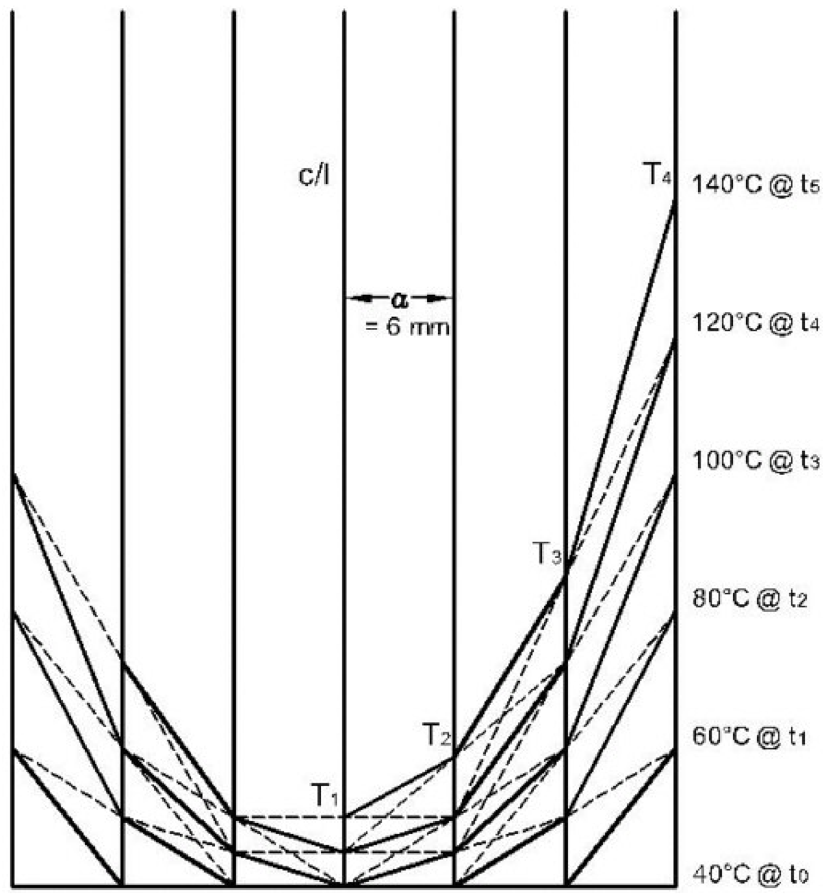
Plot the temperature - time history over the first 5 minutes and calculate the heat transferred to the glass in this time.

For glass, assume: α = *0.3x10^{-6} m^2/s,* ρ = *2700 kg/m^3,* c_p = *0,837 kJ/kg^{o}C*

Take a = 6 mm and assume the starting temperature = 40^{o}C.

$$\Delta t = \frac{a^2}{2\alpha} = \frac{0.006^2}{2\times 0.3\times 10^{-6}} = 60 \ \sec s$$

Position	*0 mm (Surface)*	*6 mm*	*12 mm*	*18 mm*	*24 mm*	*30 mm*	*36 mm (Surface)*
Time (secs)							
0	*40*	*40*	*40*	*40*	*40*	*40*	*40*
1	*60*	*40*	*40*	*40*	*40*	*40*	*60*
2	*80*	*50*	*40*	*40*	*40*	*50*	*80*
3	*100*	*60*	*45*	*40*	*45*	*60*	*100*
4	*120*	*72.5*	*50*	*45*	*50*	*72.5*	*120*
5	*140*	*85*	*58.75*	*50*	*58.75*	*85*	*140*



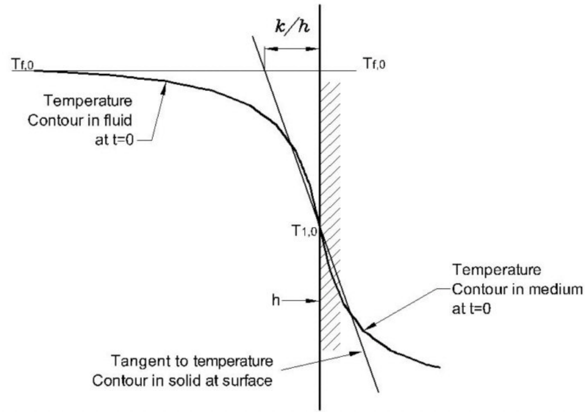
$$\text{Total heat transferred to glass} = 2\cdot\left(\frac{T_1+T_2}{2}\cdot\rho\cdot c_p\cdot a+\frac{T_2+T_3}{2}\cdot\rho\cdot c_p\cdot a+\frac{T_3+T_4}{2}\cdot\rho\cdot c_p\cdot a\right)$$

$$= 2\cdot\left(\frac{50+58.75}{2}+\frac{58.75+85}{2}+\frac{85+140}{2}\right)\times 2700\times 0.837\times 0.006$$

$$= 8364 \;\; kJ/m^2$$

The Effect of Surface Convection:

At the boundary between the solid surface and its surroundings the rate of heat conduction must be equal to the rate of heat convection.



$$\therefore\ \dot{q} = -k\cdot\left(\frac{\partial T}{\partial x}\right)_{1,0} = -h\cdot\left(T_{1,0}-T_{f,0}\right)$$

$$\therefore\left(\frac{\partial T}{\partial x}\right)_{1,0} = \frac{T_{1,0}-T_{f,0}}{k/h}$$

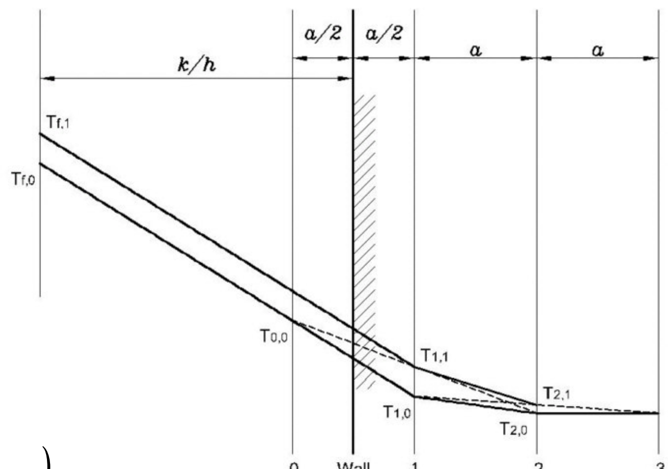
= Slope of tangent at x = 0

Imagine that the wall is extended by k/h outwards with the wall temperature contour from the imaginary boundary meeting that the temperature contour at the real wall tangentially. Conduction then occurs from k/h outside the wall to some point inside it.

If the fluid bulk temperature does not change;

$T_{f,1} = T_{f,0}$ but:

$T_{f,0}$ —— $T_{1,0}$	
$T_{f,1}$ —— $T_{1,1}$	must always be straight lines
etc	



Energy Stored $= \sum a\cdot\rho\cdot c_p\cdot\left(T_{Average}-T_{0Average}\right)$

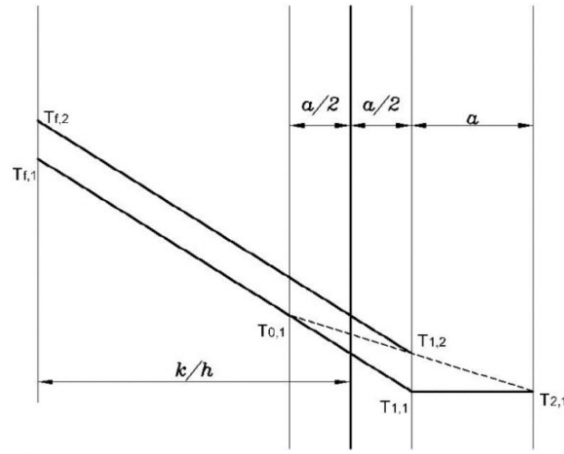
Energy Input by Convection at Surface $= \sum h\cdot\Delta t\cdot\left(\frac{T_{f,1}+T_{f,0}}{2}+\frac{T_{w,1}+T_{w,0}}{2}\right)$

Numerical Methods of Solution

i) Points within the "a" mesh are calculated by normal means.
ii) We know that $T_{f,2}$ is joined to $T_{1,2}$ by a straight line.
iii) To calculate $T_{1,2}$, $T_{0,1}$ must be known.. Hence, $T_{0,1}$ must be calculated from $T_{f,1}$ and $T_{1,1}$ as follows:

$$\frac{T_{f,1}-T_{1,1}}{\frac{k}{h}+\frac{a}{2}}=\frac{T_{0,1}-T_{1,1}}{a}$$

$$\therefore T_{0,1} = \frac{a\cdot T_{f,1}+T_{1,1}\cdot\left(\frac{k}{h}-\frac{a}{2}\right)}{\left(\frac{k}{h}+\frac{a}{2}\right)}$$



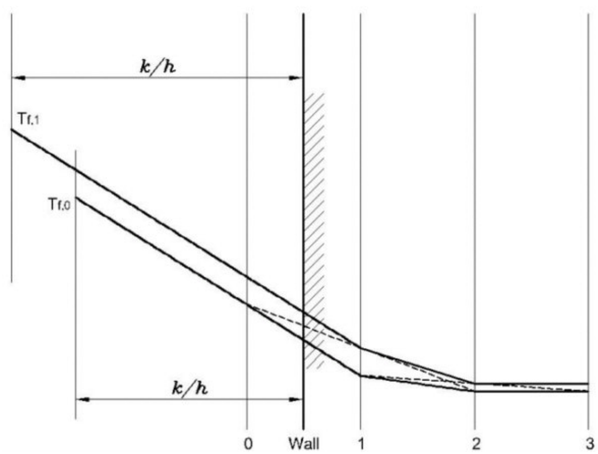
$T_{0,1}$ may be regarded as an equivalent surface temperature in a purely conducting situation and hence it replaces $T_{f,1}$.

Thus obtain values of temperatures at $\frac{a}{2}$, $\frac{3a}{2}$, $\frac{5a}{2}$ etc from the wall and the wall temperatures by interpolation.

Special Boundary Conditions

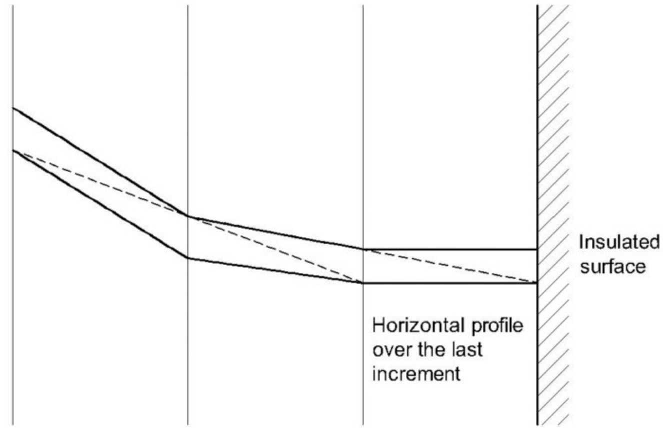
i) Varying value of h (which is a temperature function)

$\frac{k}{h}$ is no longer constant and a new value must be taken for each constructional step. This causes the point $T_{f,1}$ to move both horizontally and vertically relative to $T_{f,0}$, as shown.



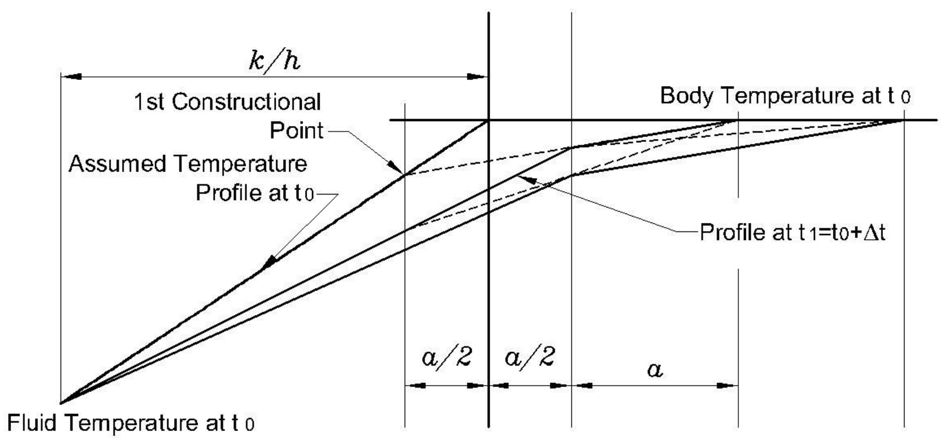
ii) Insulated surface

Since there is no heat transfer across an insulated surface, the temperature gradient at the surface must therefore be zero.



iii) Sudden surface temperature change

A temperature discontinuity must occur between the solid surface and its surrounding fluid when, for example, a body is suddenly plunged into a fluid of different temperature. A reasonable (but not the best) approximation is to assume the initial profile shown.

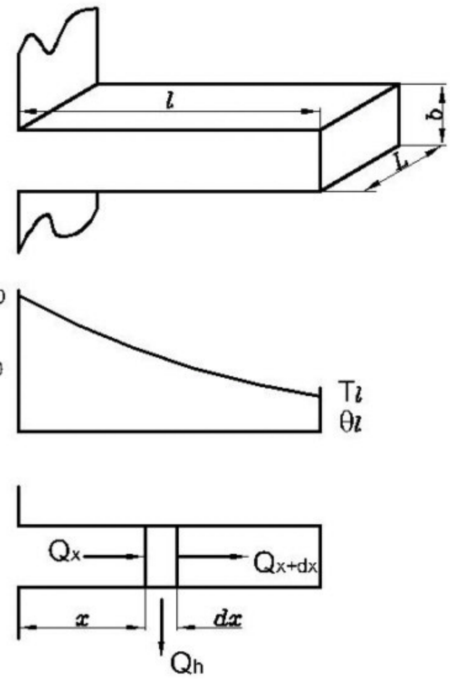


2 ONE DIMENSIONAL EXTENDED SURFACES (FINS)

These are used to increase the convective heat transfer at solid/fluid boundaries. They have many applications, particularly in heat exchangers and air cooled internal combustion engines.

Assume for a thin fin that the heat flow is one dimensional i.e. that there is no temperature gradient across any cross-section normal to "l".

Consider heat flow into the element at position x:



$$\dot{Q}_x = -k \cdot L \cdot b \cdot \frac{dT}{dx}$$

$$\dot{Q}_{x+dx} = -k \cdot L \cdot b \cdot \frac{d}{dx}\left(T + \frac{dT}{dx}dx\right)$$

$$\dot{Q}_h = 2 \cdot L \cdot dx \cdot h\left(T - T_s\right) \qquad (neglecting\ b)$$

Let $(T\text{-}T_s) = \theta$

If T_s is constant, then $\frac{dT}{dx} = \frac{d\theta}{dx}$

For steady state heat transfer conditions it must follow that:

$$Q_x = Q_{x+dx} + Q_h$$

$$\therefore \quad -kLb\frac{d\theta}{dx} = -kLb\left(\frac{d\theta}{dx} - \frac{d^2\theta}{dx^2}dx\right) + 2hL\theta \cdot dx$$

$$\therefore \quad kLb\frac{d^2\theta}{dx^2} = 2hL\theta$$

$$\therefore \quad \frac{d^2\theta}{dx^2} = \frac{2h}{kb} \cdot \theta$$

This is a second order linear differential equation of the type: $\frac{d^2\theta}{dx^2} = m^2\theta$ which has a solution of the form:

$$\theta = Me^{mx} + Ne^{-mx} \quad where \ \ m = \sqrt{\frac{2h}{kb}}$$

M and N are found from the boundary conditions of which there are usually two sets:

i) $\theta = \theta_0$ at x = 0 and $Q_\ell = 0$ at $x = \ell$ (long thin fins)

ii) $\theta = \theta_0$ at x = 0 and $-kA\left(\frac{d\theta}{dx}\right)_{x=\ell} = hA\theta_\ell$ (end heat flow significant)

Consider Case i)

At $x = 0$ $\theta_0 = M + N$ (1)

At $x = \ell$ $\left(\frac{d\theta}{dx}\right)_{x=\ell} = 0$ $\therefore m.M.e^{m\ell} - m.N.e^{-m\ell} = 0$ (2)

Solve for M and N. Hence:

$$M = \frac{\theta_0 e^{-m\ell}}{e^{m\ell} + e^{-m\ell}} \qquad N = \frac{\theta_0 e^{m\ell}}{e^{m\ell} + e^{-m\ell}}$$

Substituting in the original equation:

$$\theta = \theta_0 \cdot \left[\frac{e^{m(\ell - x)} + e^{-m(\ell - x)}}{e^{m\ell} + e^{-m\ell}}\right]$$

$$\therefore \theta = \theta_0 \cdot \left[\frac{\cosh \quad m(\ell - x)}{\cosh \quad m\ell}\right]$$ = Temperature distribution along the fin.

Now the total heat flow from the fin to the surroundings = Heat flow into the fin at x = 0

Hence: $$\dot{Q}_0 = -kLb\left(\frac{d\theta}{dx}\right)_{x=0} = +mkLb\theta_0 \cdot \left[\frac{\sinh m(\ell - x)}{\cosh m\ell}\right]_{x=0}$$

$$= mkLb\theta_0 \tanh m\ell \qquad (3)$$

In Case ii): Where the fins are relatively short, the assumption of zero heat transfer from the tip to the surroundings is not valid. Hence under these conditions equation (2) does not apply. The heat transfer at the tip is therefore given by:

$$-kLb\left(\frac{d\theta}{dx}\right)_{x=\ell} = +hLb\theta_\ell$$

$$\therefore -k \cdot \left(mMe^{m\ell} - mNe^{m\ell}\right) = h\theta_\ell \qquad (4)$$

By combining equations (1) and (4) it may be shown that in this case:

$$\therefore\ \theta = \theta_0 \cdot \left[\frac{\cosh m(\ell - x) + \dfrac{h}{km}\sinh m(\ell - x)}{\cosh m\ell + \dfrac{h}{km}\sinh m\ell} \right]$$

As in case i), the heat transfer from the fin is obtained by consideration of conduction into the fin at x = 0. From this we get:

$$\dot{Q}_0 = -kLb\left(\frac{d\theta}{dx}\right)_{x=0} = +mkLb\theta_0 \cdot \left[\frac{\tanh ml + \dfrac{h}{km}}{1 + \dfrac{h}{km}\tanh m\ell} \right] \tag{5}$$

Limit of Usefulness of a Fin

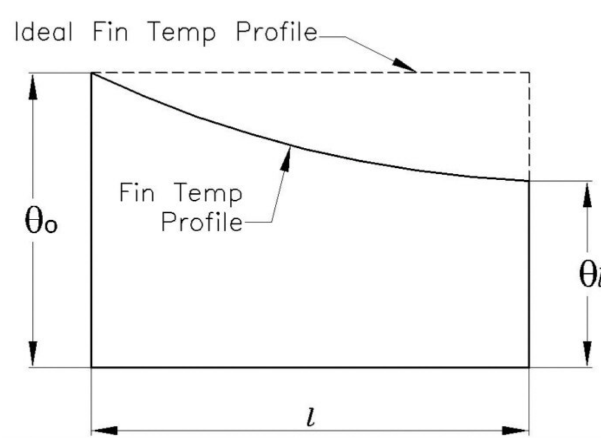
Under some circumstances, the addition of fins does not improve the heat transfer. This occurs when $\dfrac{d\dot{Q}_0}{dl} = 0$. Taking the more general case, where heat transfer at the tip is not negligible (case 2 above), then differentiating equation (5), it can be shown that the differential is zero when the numerator of the differential = 0 and that this occurs when*:* *mk* = *h*.

Substituting the original derivation of $m = \sqrt{\dfrac{2h}{kb}}$, it therefore follows that for a straight fin to lead to an improvement in heat transfer from the surface from which it extends: $\sqrt{\dfrac{2k}{hb}} > 1$.

Efficiency of Fins

An ideal fin of infinite thermal conductivity would maintain a constant temperature difference θ_0 with its surroundings throughout its length.

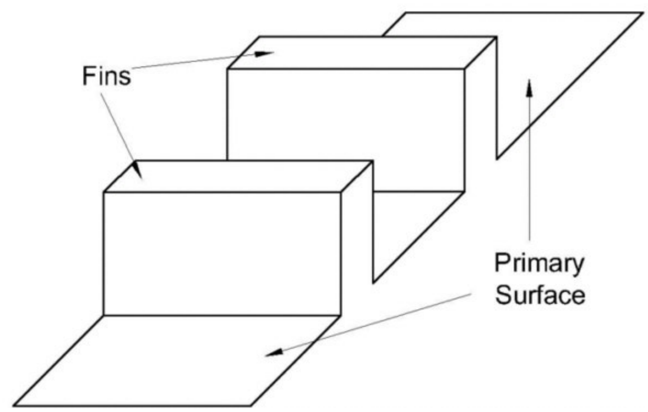
Assuming the fin to be thin (ignoring edge effects), this would permit a heat flow $= 2 \cdot \ell \cdot L \cdot h \cdot \theta_0$



If we define fin efficiency η_f as the ratio of heat flow/ideal heat flow through the fin, we get:

Neglecting End Heat Transfer	Including End Heat Transfer
$\eta_f = \dfrac{m \cdot k \cdot L \cdot b \cdot \theta_0 \cdot \tanh m\ell}{2 \cdot \ell \cdot L \cdot h \cdot \theta_0} = \dfrac{\tanh m\ell}{m\ell}$	$\eta_f = \dfrac{m \cdot k \cdot L \cdot b \cdot \theta_0}{2 \cdot \ell \cdot L \cdot h \cdot \theta_0} \cdot \dfrac{\tanh m\ell + \frac{h}{km}}{1 + \frac{h}{km} \cdot \tanh m\ell} = \dfrac{\tanh m\ell + \frac{h}{km}}{m\ell + \frac{h\ell}{k} \cdot \tanh m\ell}$

Effectiveness of a Finned Surface



Let A_s = Fin Surface Area/unit Area of Primary Surface

The primary surface at θ_0 is 100% efficient but the fin surface at θ is not.

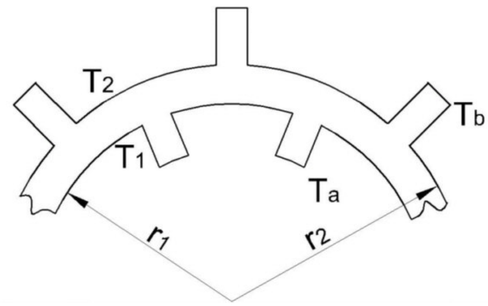
Let $\eta_f A_s$ be a surface that is 100% efficient.

$\therefore$ The equivalent 100% efficient surface $= 1 + \eta_f A_s$

But $1 + A_s$ = Total surface area

$$\therefore \quad \eta_{fe} = \textit{Effectiveness of a Finned Surface} = \frac{1 + \eta_f A_s}{1 + A_s}$$

__Overall Heat Transfer Coefficient of a Finned Surface__



Consider the tube surface in a heat exchanger.

Internal Convection:

$$Q = -2\pi r_1 \lfloor \eta_{fe}(1 + A_s) \rfloor_a \cdot h_a (T_1 - T_a) \quad \textit{per unit length of tube}$$

The term $\lfloor \eta_{fe}(1 + A_s) \rfloor$ is the total inside surface at temperature T_1/unit area of ***bare tube*** surface.

This could be replaced by $\left(1 + \eta_f \cdot A_s - n \cdot b\right)_a$ $\quad$ *where* $n = \textit{number of fins of thickness } b$.

Conduction:

$$Q = \frac{-2\pi k}{\ln \frac{r_2}{r_1}} \cdot (T_2 - T_1)$$

External Convection:

$$Q = -2\pi r_2 \lfloor \eta_{fe}(1 + A_s) \rfloor_b \cdot h_b (T_b - T_2) \quad \textit{per unit length of tube}$$

Since $\quad Q = U_L (T_b - T_a)$

$$\therefore U_L = \frac{1}{\dfrac{1}{2\pi \; r_1 \left[\eta_{fe}(1 + A_s)\right]_a \cdot h_a} + \dfrac{\ln \frac{r_2}{r_1}}{2\pi \; k} + \dfrac{1}{2\pi \; r_2 \left[\eta_{fe}(1 + A_s)\right]_b \cdot h_b}}$$

Numerical Solution of Steady State Conduction in Fins

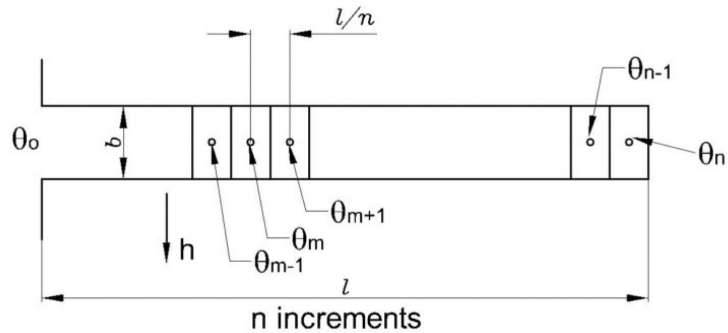
For increment m:

Conduction into the element = Conduction out + Convection from the edges

i.e. $$\frac{-k\cdot(b\cdot 1)(\theta_m-\theta_{m-1})}{\ell/n}=\frac{-k\cdot(b\cdot 1)(\theta_{m+1}-\theta_m)}{\ell/n}-2\frac{\ell}{n}\cdot 1\cdot h_m\cdot\theta_m$$

$$\therefore\quad \theta_{m-1}+\theta_{m+1}-2\theta_m+\frac{2\left(\ell/n\right)^2 h_m\theta_m}{kb}=0$$

$$\therefore\quad \theta_{m-1}+\theta_{m+1}+\theta_m\left[\frac{2\left(\ell/n\right)^2 h_m}{kb}-2\right]=0$$



For the end increment at θ_n:

With Convection out of the end face
(Short Fin Boundary Condition)

$$\frac{k(b\cdot 1)(\theta_n-\theta_{n-1})}{\ell/n}=-2\left(\frac{\ell}{n}\right)\cdot h_n\cdot\theta_n-(b\cdot 1)h_n\theta_n$$

$$\therefore\qquad \theta_n-\theta_{n-1}=-\frac{2\left(\ell/n\right)^2 h_n\theta_n}{kb}-\frac{b\,h_n\theta_n}{kb}$$

$$\therefore\qquad \theta_{n-1}+\theta_n\left[\frac{2(\ell/n)^2 h_n}{kb}+\left(\ell/n\right)\frac{hn}{k}-1\right]=0$$

No Convection out of the end face
(Long Fin Boundary Condition)

$$\frac{-k(b\cdot 1)(\theta_n-\theta_{n-1})}{\ell/n}=-2\left(\ell/n\right)h_n$$

$$\therefore\quad \theta_{n-1}+\theta_n\left(\frac{2\left(\ell/n\right)^2 h_n}{kb}-1\right)=0$$

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PART 3 HEAT TRANSFER

Steady State Conduction – 1

Starting from the Fourier rate equation, show that for a hollow sphere with uniform temperature distribution: the rate of heat transfer by conduction through it may be expressed as:

$$Q = \frac{4 \cdot \pi \cdot k \cdot r_o \cdot r_i}{r_o - r_i} \cdot (T_i - T_o)$$

Where: suffix i relates to the inner radius
suffix o relates to the outer radius

The thermal conductivity of a solid is given by the equation:
$k = k_o + 2 \cdot b \cdot T + 3 \cdot c \cdot T^2$ where k_o is the thermal conductivity at $T = 0$.

Show that the steady heat flow through a plane wall of this solid of thickness Δx with surface temperatures T_1 and T_2 may be expressed as:

$$Q = -\left[k_0 + b \cdot (T_2 + T_1) + c \cdot \left(T_2^2 + T_2 \cdot T_1 + T_1^2\right)\right] \cdot A \cdot \frac{T_2 - T_1}{\Delta x}$$

A pipe having an external diameter of 80 mm is covered with two layers of insulating material, each 30 mm thick. The thermal conductivity of one layer is four times that of the other. Show that the combined conductivity of the two layers is 30.65% more when the better insulating material is on the outside than when it is on the inside.

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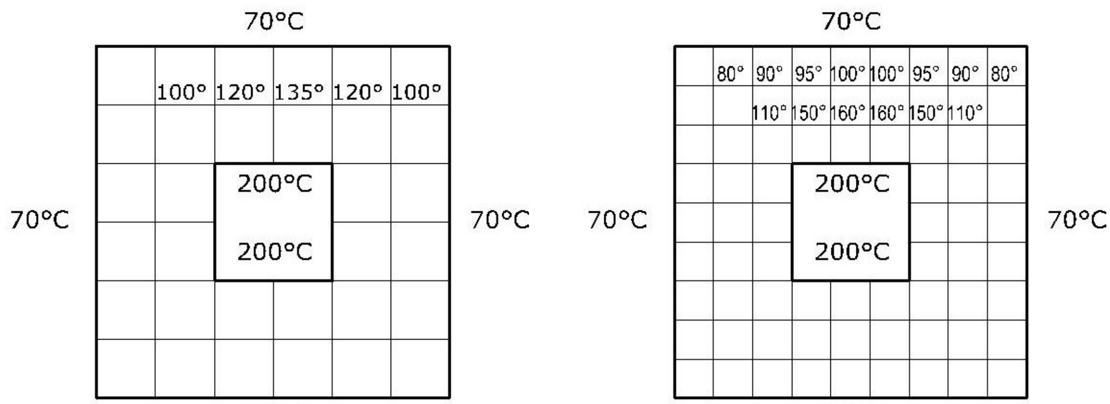
PART 3 HEAT TRANSFER

Steady State Conduction – 2

1. A square hollow duct with thick walls has surface temperatures as shown in the figures. Using (a) 9 mm mesh and (b) 6 mm mesh find by relaxation the temperatures within the cross-section using the initial guessed values given. The material has a thermal conductivity of 120 W/m K. Determine the heat transfer per metre length in both cases.



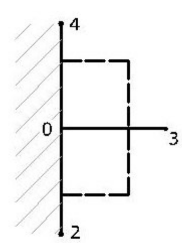
a) b)

(106,000 W/m)

2. By the "conducting rod analogy" it can be shown that in two-dimensional steady state conduction at a point is related to values at the surrounding points by the equation $t_1 + t_2 + t_3 - 4t_0 = 0$. By similar reasoning establish the equations given for the following boundary points.

   a. Point on an insulated surface.

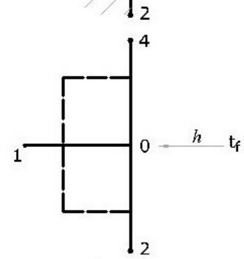
$$\frac{t_0 + t_4}{2} + t_3 - 2t_0 = 0$$



(a)

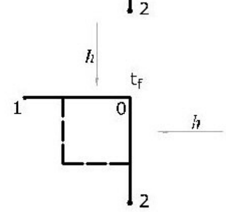
   b) Convection at the surface, with energy generation in the material.

$$t_1 + \frac{t_2 + t_4}{2} + \frac{h \cdot a \cdot t_f}{k} - t_0\left(2 + \frac{h \cdot a}{k}\right) = \frac{-a^2 R}{2k}$$



(b)

   b. Exterior corner, with convection.



(c)

$$\frac{t_1 + t_2}{2} + \frac{h \cdot a \cdot t_f}{k} - t_0\left(1 + \frac{h \cdot a}{k}\right) = 0$$

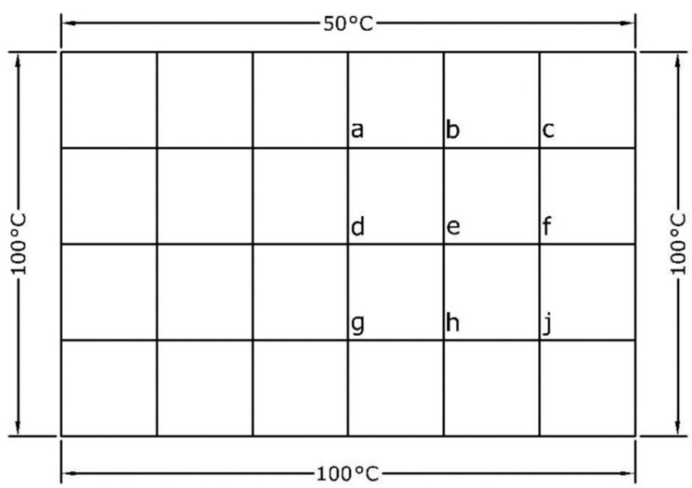
3. Establish the temperatures at the mesh points on the cross-section of the rectangular bar carrying a current of 4472 amps/cm^2.

For the material of the bar: k = 200 W/m K, $\rho = 4x10^{-8}$ ohm metre. Surface temperatures are as shown. Mesh size is 0.5 cm.

Calculate the heat transfer from the bar:

i) from the total heat generation

ii) from conduction along mesh lines at the boundary



(i)480 W/cm ii) 296 W/cm)

4. Show that in one-dimensional radial conduction with internal heat generation *q'* per unit volume, the temperature distribution is governed by the differential equation:

$$\frac{d^2 t}{dr^2} + \frac{1}{r}\frac{dt}{dr} = -\frac{q'}{k}$$

and hence: $t = -\frac{q'}{4k} r^2 + A \ln r + B$

where A and B are constants of integration to be determined from the boundary conditions.

An internally cooled copper conductor of 4 cm outer diameter and 1.5 cm inner diameter carries a current density of 5000 amp/cm^2. The temperature of the inner surface is 70^{o}C and the external surface is insulated. Determine the equation for the temperature distribution through the copper. Hence find the maximum temperature for the copper, the radius at which it occurs and the internal heat transfer rate. Check that this last answer is equal to the total energy generation in the bar.

For copper: $k = 380$ W/m K $\rho = 2\text{x}10^{-8}$ ohm metre.

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PART 3 HEAT TRANSFER

Transient Conduction

A bare copper cable 2.5 cm in diameter is situated in still air at 15°C. A current of 750 Amps is passed along the cable. Heat is transferred from the cable to its surroundings by natural convection for which the heat transfer coefficient may be expressed as $2.95\ \theta^{0.25}$ $W/m^2\ K$. Determine whether or not it may be assumed that the temperature across the copper cross section is uniform and what the equilibrium temperature of the cable surface will be.

If the current is stopped, estimate the time for the copper to cool to a temperature of 18°C.

Assume the following for copper: density $\rho = 8950\ kg/m^3$,

specific heat capacity $c_p = 380$ J/kg K, electrical resistivity $\rho^* = 17\times10^{-9}\ \Omega$ m,
thermal conductivity k = 400 W/m K.

(49.65°C, 2.783 hrs)

Given the differential equation $\left(\dfrac{\partial T}{\partial \tau}\right) = \alpha\left(\dfrac{\partial^2 T}{\partial x^2}\right)$ for unsteady conduction in a "one dimensional" wall show that the temperature $T_{n,p+1}$ at some section n and time instant p+1 can be calculated approximately from:

$$T_{n,p+1} = F\left[T_{n+1,p} + T_{n-1,p} + \left(\frac{1}{F} - 2\right)T_{n,p}\right]$$

The temperatures in the right-hand bracket are values at equidistant sections (n-1), n, (n+1), preceding (p+1) by a finite time interval $\Delta\tau$ and $F = \dfrac{\alpha\Delta\tau}{\Delta x^2}$ is the Fourier Number.

A steel slab 2.54 cm thick is initially at a uniform temperature of 650°C. It is cooled by quenching in water, which may be assumed to reduce the surface temperature suddenly to 93.5°C. Derive a numerical method to deal with this case by dividing the slide into eight slices. The heat flow may be assumed to be normal to the sides of the slab. Use the method to determine the time required to reduce the centre temperature to 450°C.

For steel take the thermal diffusivity = 1.16×10^{-5} m/s

(3.48 secs)

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PART 3 HEAT TRANSFER

Extended Surfaces

A vertical surface is at a temperature of 50°C in an atmosphere at 15°C. The surface convection coefficient is $h = 1.31\theta^{1/3}$ W/(m^2 K) where θ is the temperature difference in °C. Heat transfer from the surface is to be increased by 40% by adding vertical fins, 0.65 cm thick. Assume the surface temperature is unchanged by the addition of the fins , that there is a new convection coefficient of $h = 1.1\theta^{1/3}$ W/(m^2 K) over the whole surface*, and that there is no heat transfer from the tips of the fins. Determine the height of the fins (i.e. distance from root to tip), if they are to be spaced 15 cm apart. The thermal conductivity k for the fin and wall is 260 W/(m K).

* where θ is the root to atmosphere temperature difference. (5.35. cm)

A steel pipe of 3 cm bore and 0.5 cm wall thickness has eight axial fins on its exterior surface, which are 0.3 cm thick and 2 cm high. The pipe carries a fluid at 150°C and is in an exterior environment of 30°C. Interior and exterior convection coefficients are 400 W/(m^2 K) and 50 W/(m^2 K). Determine

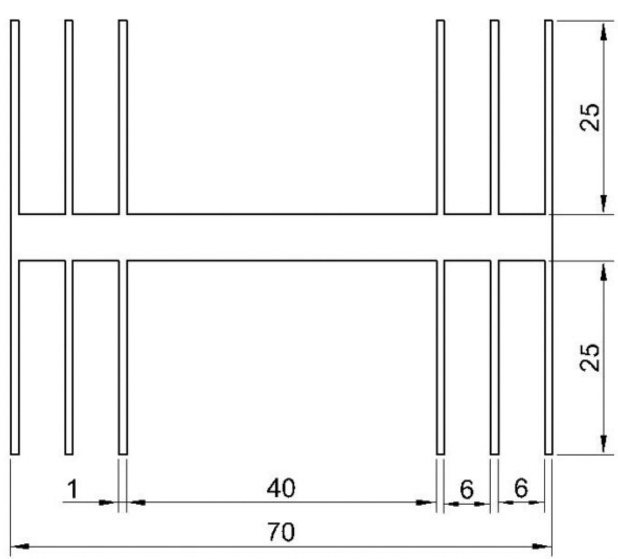
i) the efficiency of the fins. (90.2%)

ii) an overall heat transfer coefficient between the interior and exterior fluids per length of pipe. (13.17 W/m K)

iii) the rate of heat transfer between the fluids. (1572 W/m)

A transistor heat sink is a 10 cm length of an aluminium section, as shown.
It may be regarded as a horizontal plate 70 mm x 100 mm on which 12 fins, 1 mm thick by 25 mm high, are mounted. Assuming the horizontal plate is everywhere at 45°C above the temperature of the surroundings, determine the percentage of the total heat transfer from the sink that occurs from the surfaces of the fins. Neglect heat transfer from the edges of the plate and fins.

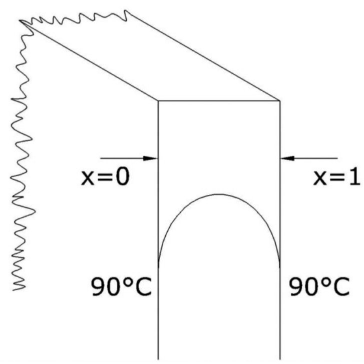
k = 150 W/m K and h = 30 W/(m^2 K) (81.3%)



Ex. 1 *Consider a wide flat plate 1.0 cm thick. k = 380 W/m K,* ρ^* *= 1.8x10*$^{-8}$*ohm m, i = 2000 Amp/cm*^{2}*, and each surface is at 90*^{o}*C. Find the position across the plate at which the temperature is a maximum, the value of the maximum temperature and the rate of heat loss to the surroundings.*

$$R = 1.8x10^{-6}x2000^2 = 7.2W/cm3$$

$$\frac{R}{2k} = \frac{7.2}{2x3.8} = 0.948\ ^oC/cm^2$$



Boundary Conditions:

At x = 0: $90 = -0+0+B$ $\therefore B = 90$

At x = 1: $T = -0.948+A+90$ $\therefore A = 0.948$

$\therefore T = 0.948x^2+0.948x+90$

$$\therefore \frac{dT}{dx} = -1.896x + 0.948 = 0 \qquad at\ position\ of\ \max imum\ temperature$$

$$\therefore \quad x = \frac{0.948}{1.896} = 0.5\ cm \qquad i.e\ \max imum\ is\ at\ the\ middle\ of\ the\ plate$$

$$\therefore \quad At\ x = 0.5: \quad T = -0.237 + 0.474 + 90 = 90.237$$

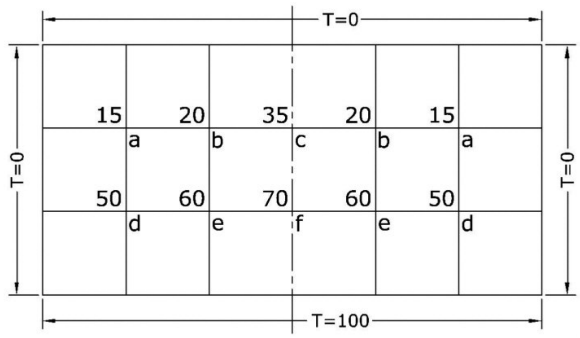
$$(\frac{dT}{dx})_{x=0} = \ 0.948\ ^oC/cm \quad \therefore q_{x=0} = -3.8x0.948 = -3.6\ W/cm^2$$

$$(\frac{dT}{dx})_{x=1} = -0.948\ ^oC/cm \quad \therefore q_{x=1} = -3.8x - 0.948 = +3.6\ W/cm^2$$

$$\therefore \quad q = q_{x=0} - q_{x=1} = -3.6 - 3.6 = -7.2\ W/cm^2$$

Note opposite signs of heat flow due to their opposite directions. This total, is equal to the rate of energy dissipated in the plate due to Joulean heating.

Ex. 2 *Determine the heat transfer to a rectangular plate heated along one edge when the heated edge is at 100^{o}C and all the other edges are at 0^{o}C.*



Trial: *Apply equation (10) to point "a" to see if it is valid:*

We get: $0 + 0 + 20 + 50 - 4\times 15 = 10$

There is a residual value of 10 on the RHS. If the temperatures were assigned correctly, then it would be 0.

Determine the residuals for all the remaining points:

$$
\begin{array}{lrl}
b: & 15 + 35 + 0 + 60 - 80 & = 30 \\
c: & 70 + 20 + 20 + 0 - 140 & = -30 \\
d: & 0 + 100 + 60 + 15 - 200 & = -25 \\
e: & 100 + 50 + 70 + 20 - 240 & = 0 \\
f: & 60 + 60 + 100 + 35 - 280 & = -25
\end{array}
$$

Point e gives the correct residual but altering the surrounding points will probably upset it.

Starting with the largest residual, eliminate it by adding to the temperature at that point, an amount equal to ¼ of the residual. Then re-estimate each residual again, assuming the new relaxed value of temperature. Repeat for the next largest residual and continue till all residuals are close to zero, making up a table in the process.

	New	*Residuals at each point*					
	Value	*a*	*b*	*c*	*d*	*e*	*f*
Initial Values→		10	30	-30	-25	0	-25
Increment ↓							
Point c – 7^o	28^o		23	-2			-32
Point f – 8^o	62^o			-10		-8	0
Point b + 6^o	26^o	16	-1	2*		-2	
Point d – 6^o	44^o	10			-1	-8	
Point a + 2^o	17^o	2	1		1		
Point e – 2^o	58^o		-1		-1	0	-4*
Point f – 1^o	61^o	+2	-1	1	-1	-1	0

* *Increment comes from both halves due to axis of symmetry*

Check:

a: $26 + 44 - 68 = 2$

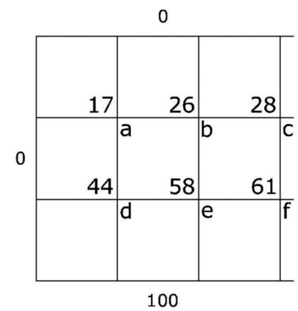
b: $17 + 27 + 58 - 104 = -1$

c: $52 + 61 - 112 = 1$

d: $117 + 58 - 176 = -1$

e: $126 + 105 - 232 = -1$

f: $116 + 128 - 244 = 0$



Hence: *Energy flowing into the plate* $= k(100-61) + 2k(100-58) + 2k(100-44)$

$= 235k$ */unit thickness*

Check: *Energy flowing out of the plate* $= 2k(44-0) + 4k(17-0) + 2k(26-0) + k(28-0)$

$= 236k$ */unit thickness*

Units are in W/m (or Btu/ft hr)

Ex. 3 *Consider a sheet of glass 3.6 cm thick. Its surface temperature rises by 20°C/min.*

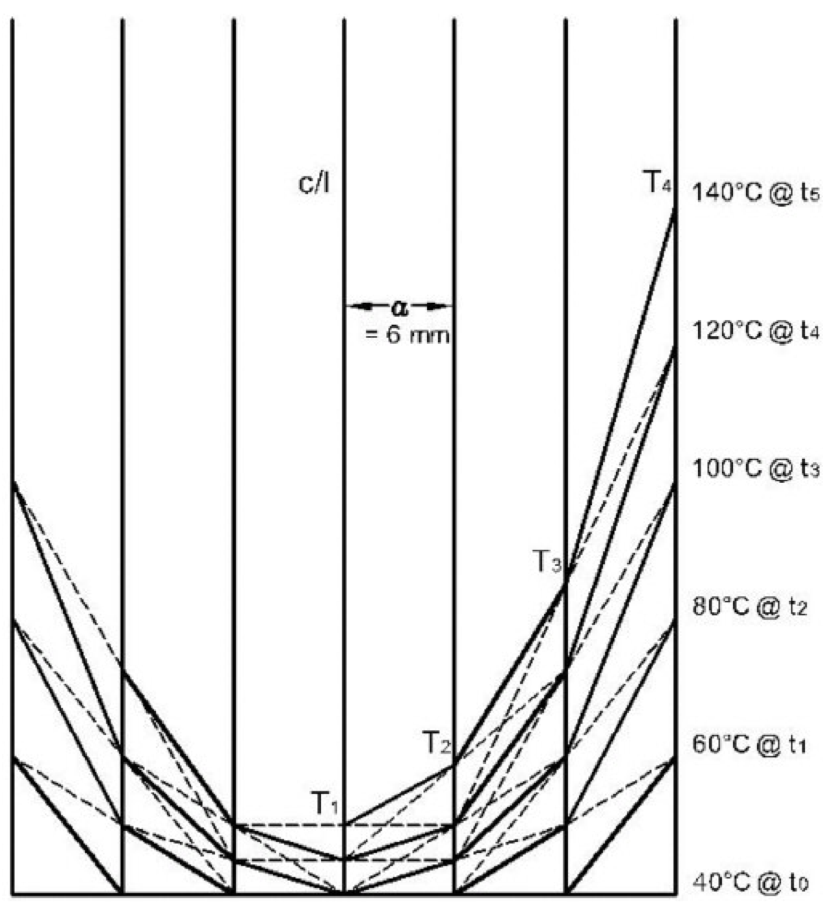
Plot the temperature - time history over the first 5 minutes and calculate the heat transferred to the glass in this time.

For glass, assume: $\alpha = 0.3x10^{-6}\ m^2/s$, $\rho = 2700\ kg/m^3$, $c_p = 0{,}837\ kJ/kg^oC$

Take a = 6 mm and assume the starting temperature = 40°C.

$$\Delta t = \frac{a^2}{2\alpha} = \frac{0.006^2}{2\times 0.3\times 10^{-6}} = 60 \ \sec s$$

Position	0 mm (Surface)	6 mm	12 mm	18 mm	24 mm	30 mm	36 mm (Surface)
Time (mins)							
0	40	40	40	40	40	40	40
1	60	40	40	40	40	40	60
2	80	50	40	40	40	50	80
3	100	60	45	40	45	60	100
4	120	72.5	50	45	50	72.5	120
5	140	85	58.75	50	58.75	85	140



$$\text{Total heat transferred to glass} = 2 \cdot \left(\frac{T_1 + T_2}{2} \cdot \rho \cdot c_p \cdot a + \frac{T_2 + T_3}{2} \cdot \rho \cdot c_p \cdot a + \frac{T_3 + T_4}{2} \cdot \rho \cdot c_p \cdot a \right)$$

$$= 2 \cdot \left(\frac{50 + 58.75}{2} + \frac{58.75 + 85}{2} + \frac{85 + 140}{2} \right) \times 2700 \times 0.837 \times 0.006$$

$$= 8364 \;\; kJ / m^2$$