Engineering Mathematics: Complex Variables

Analytic functions

Polynomials and rational functions

It is clear from the definition of differentiability that both $f(z) = 1$ and $g(z) = z$ are analytic functions. From the rules of differentiation (sums of analytic functions are analytic, products of analytic functions are analytic, etc.) we can show that all *integer* powers of z and hence all **polynomials** are analytic.

[e.g. If z^n is analytic, the $z^n \times z = z^{n+1}$ is analytic, since the product of two analytic functions is analytic. Since we know that z^n is analytic for $n = 0$ and $n = 1$, by induction z^n is analytic for all integer $n > 0$.]

Also $f(z) = 1/z$ is analytic for all $z \neq 0$ [See Examples Sheet] and so, using similar arguments to the above, we can show that all negative integer powers of z are also analytic except at $z = 0$.

We can combine these results to show that **rational functions** of the form

$$f(z) = \frac{p(z)}{q(z)},$$

where $p(z)$ and $q(z)$ are polynomials, are also analytic except at the zeros of $q(z)$ (assuming that all the common factors of p and q have been canceled) [see Examples Sheet].

Exponential function e^z

It is possible to define the complex exponential function in terms of a power series, however we will take a different approach and define the **exponential function** as

$$e^z = e^x(\cos y + i \sin y).$$

This definition makes e^z a natural extension of the real function e^x since if $y = 0$, then $\cos 0 = 1$ and $\sin 0 = 0$, and we recover the real exponential function as we would hope.

This function satisfies the Cauchy-Riemann equations since

$$u(x, y) = e^x \cos y \qquad \text{and} \qquad v(x, y) = e^x \sin y$$

Giving

$$\frac{\partial u}{\partial x} = \boxed{} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \boxed{},$$

and

$$\frac{\partial v}{\partial x} = \boxed{} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \boxed{}.$$

Hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

It also satisfies the differential equation $f'(z) = f(z)$, since

$$f'(z) = \boxed{} = \boxed{} + i\boxed{} = f(z).$$

Further properties:

- The product of exponentials is given by

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1) \times e^{x_2}(\cos y_2 + i\sin y_2) \\ &= \boxed{} \\ &= \boxed{} \\ &= e^{z_1+z_2} \end{aligned}$$

  Hence also $e^z{\cdot}e^{-z} = e^0 = 1$, and so $e^{-z} = 1/e^z$.

  This last result tells us that since all values of e^z have multiplicative inverses then $e^z \neq 0$ for all values of z.

- For the case $z = iy$ we obtain the Euler formula

$$e^{iy} = \cos y + i\sin y.$$

  Hence the polar form of z can be written

$$z = r(\cos\theta + i\sin\theta) = \boxed{}.$$

- Note $|e^z| = e^x$. This also has the implication noted above that e^z is never zero.

- The function $f(z) = e^z$ is periodic with period $2\pi i$. I.e.

$$e^{z+2\pi i} = e^z \quad \forall z.$$

  This follows from the periodicity of cos and sin.

Trigonometric functions

Just as we extended the real function e^x to e^z we would like to extend $\cos x$ and $\sin x$, the real trigonometric functions, into the whole complex plane. From the Euler formula

$$e^{ix} = \cos x + i\sin x$$

and

$$e^{-ix} = \cos x - i\sin x.$$

Hence we obtain

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

and

$$\sin x = \boxed{\qquad\qquad\qquad}.$$

This would suggest the following definitions for complex z

$$\cos z = \boxed{\qquad\qquad\qquad}$$

$$\sin z = \boxed{\qquad\qquad\qquad}.$$

The analyticity of e^z ensures the analyticity of these two function. Also Euler's formula carries over without modification to complex values

$$e^{iz} = \cos z + i \sin z.$$

In addition we can define, just as in the real case, the trigonometric functions

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z},$$
$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}.$$

It is straight forward to show from $(e^z)' = e^z$ that

$$(\cos z)' = \boxed{\qquad\quad}, \qquad (\sin z)' = \boxed{\qquad\quad}, \qquad (\tan z)' = \boxed{\qquad\quad}.$$

We can rewrite

$$\begin{aligned} \cos z &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\ &= \boxed{\qquad\qquad\qquad\qquad\qquad\qquad} \\ &= \boxed{\qquad\qquad\qquad\qquad}. \end{aligned}$$

Similarly

$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$

From this it is clear that both $\sin z$ and $\cos z$ are periodic with period 2π. Hence the periodicity of $\tan z$, etc. follows.

<u>Note:</u> Although $\cos^2 z + \sin^2 z = 1$ just as for reals, it is no longer the case that $|\cos z|$ is bounded since

$$\begin{aligned} |\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \boxed{\qquad\qquad\qquad\qquad\qquad\qquad} \\ &= \boxed{\qquad\qquad\qquad\qquad\qquad\qquad} \\ &= \cos^2 x + \sinh^2 y, \end{aligned}$$

and so the absolute value of $\cos z$ tends to ∞ as $|y| \to \infty$.

Hyperbolic functions

We define the **hyperbolic cosine** and **sine** by

$$\cosh z = \frac{1}{2}\left(e^z + e^{-z}\right) \qquad \text{and} \qquad \sinh z = \frac{1}{2}\left(e^z - e^{-z}\right).$$

These definitions coincide with the real functions when z is real, and are analytic. In addition their derivatives are as you would expect;

$$(\cosh z)' = \sinh z \qquad \text{and} \qquad (\sinh z)' = \cosh z.$$

We can also define in the obvious way

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z},$$

We also have from the definitions of the trigonometric and hyperbolic functions the relationships

$$\cosh iz = \boxed{\qquad\qquad}, \qquad \text{and} \qquad \sinh iz = \boxed{\qquad\qquad},$$

$$\cos iz = \boxed{\qquad\qquad}, \qquad \text{and} \qquad \sin iz = \boxed{\qquad\qquad},$$

Note: In real analysis there is no clear connection between exponentials and the trigonometric functions. However their complex analogues are clearly interrelated.

Logarithm

The **natural logarithm** of $z = x + iy$ is denote by $\log z$ or $\ln z$, and is defined as the inverse of the exponential function. That is to say, the function $w = \log z$ is defined by the relation

$$e^w = z.$$

Note that since e^w is never zero this means that the logarithm is not defined for $z = 0$.

If we set $w = u + iv$, and $z = re^{i\theta}$ and substitute these into the above expression we see

$$e^{u+iv} = e^u e^{iv} = re^{i\theta}.$$

Comparing the modulus and the argument gives

$$e^u = r, \qquad v = \theta,$$

and so

$$\log z = \log r + i\theta, \quad (r = |z| > 0, \quad \theta = \arg z).$$

Note that just as $\arg z$ can take many values (all differing my integer multiples of 2π) so can $\log z$! If we restrict ourselves to the principal value of the argument only then the logarithm becomes single valued, and is called the **principal value** of the logarithm. However with this definition the logarithm has a discontinuity along the negative real axis (this discontinuity, called a **branch cut**, cannot be avoided, although its position can be altered by taking different definitions of the argument). It is, however, conventional to ensure that the definition used of the argument to make the logarithm single valued leaves the logarithms of positive real numbers as reals.

Logarithms satisfy the Cauchy-Riemann equations since

$$u = \log r = \frac{1}{2}\log(x^2 + y^2), \qquad v = \arg z = \tan^{-1}\frac{y}{x}.$$

Then

$$u_x = \boxed{\qquad\qquad\qquad} = v_y = \boxed{\qquad\qquad\qquad},$$

$$u_y = \boxed{\qquad\qquad\qquad} = -v_x = \boxed{\qquad\qquad\qquad}.$$

The derivative of $\log z$ is given by

$$(\log z)' = u_x + iv_x = u_x - iu_y = \boxed{\qquad\qquad\qquad}$$

$$= \boxed{\qquad\qquad} = \boxed{\qquad\quad}.$$

General powers of z

General powers of z are defined by

$$z^c = e^{c\log z},$$

where c is complex and $z \neq 0$.

Since $\log z$ is many-valued then so in general is z^c. If we take the principal value of $\log z$ the we will obtain the **principal value of** z^c. If c takes an integer value then z^n is single valued and the definition here matches the normal definition of z^n. If $c = 1/n$ where $n = 2, 3, \ldots$ then z^c is determined up to multiples of $2\pi i/n$ and so we obtain n distinct roots of z. If, however, c is real irrational or has an imaginary part then z^c has infinitely may values.

E.g., find i^i [see Example Sheet]