Nonlocal gauge equivalent integrable systems

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J. Cen, A. Fring, J. Phys A49 (2016) 365202; Physica D: Nonlin Phen. 397 (2019) 17; J. of Nonlin Math. Phys. 27 (2020) 17
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- Why study complex PT-symmetric solitons?

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- Conclusions and Outlook

The complex Korteweg-de Vries equation

The complex KdV equation equals two coupled real equations

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Leftrightarrow \quad \begin{cases} p_t + 6pp_x + p_{xxx} - 6qq_x = 0 \\ q_t + 6\,(pq)_x + q_{xxx} = 0 \end{cases}$$

with $u(x,t) = p(x,t) + iq(x,t)$, $p(x,t)$, $q(x,t) \in \mathbb{R}$

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  - for $q_{xxx} \to 0$ complex KdV $\Rightarrow$ Ito equations

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  $x \to -x$, $t \to -t$, $i \to -i$, $u \to u$, $p \to p$, $q \to -q$

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  $x \rightarrow -x$, $t \rightarrow -t$, $i \rightarrow -i$, $u \rightarrow u$, $p \rightarrow p$, $q \rightarrow -q$
- Integrability:
  Lax pair:

$$\begin{aligned} L_t &= [M, L] \\ L &= \partial_x^2 + \frac{1}{6}u \qquad \text{and} \qquad M = 4\partial_x^3 + u\partial_x + \frac{1}{2}u_x \end{aligned}$$

Solutions from Hirota's direct method

Convert KdV equation into Hirota's bilinear form

$$\left(D_x^4 + D_x D_t\right) \tau \cdot \tau = 0$$

with $u = 2(\ln \tau)_{xx}$. (D_x, D_t are Hirota derivatives)

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Expanding $\tau = \sum_{k=0}^{\infty} \lambda^k \tau^k$ gives multi-soliton solutions

$$\begin{aligned}
\tau_{\mu;\alpha}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} \\
\tau_{\mu,\nu;\alpha,\beta}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} + e^{\eta_{\nu;\beta}} + \varkappa(\alpha,\beta) e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}} \\
\tau_{\mu,\nu,\rho;\alpha,\beta,\gamma}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} + e^{\eta_{\nu;\beta}} + e^{\eta_{\rho;\gamma}} + \varkappa(\alpha,\beta) e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}} \\
&\quad + \varkappa(\alpha,\gamma) e^{\eta_{\mu;\alpha}+\eta_{\rho;\gamma}} + \varkappa(\beta,\gamma) e^{\eta_{\nu;\beta}+\eta_{\rho;\gamma}} \\
&\quad + \varkappa(\alpha,\beta)\varkappa(\alpha,\gamma)\varkappa(\beta,\gamma) e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}+\eta_{\rho;\gamma}}
\end{aligned}$$

with $\eta_{\mu;\alpha} := \alpha x - \alpha^3 t + \mu$, $\varkappa(\alpha,\beta) := (\alpha-\beta)^2/(\alpha+\beta)^2$
$\mu, \nu, \rho \in \mathbb{C}$, $\alpha, \beta, \gamma \in \mathbb{R}$

One-soliton solution

We find

$$u_{i\theta;\alpha}(x,t) = \frac{\alpha^2 + \alpha^2 \cos\theta \cosh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2} - i \frac{\alpha^2 \sin\theta \sinh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2}$$

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The solution found by Khare and Saxena is the special case

$$u_{\pm i\frac{\pi}{2};\alpha}(x,t) = \alpha^2 \operatorname{sech}^2(\alpha x - \alpha^3 t) \mp i\alpha^2 \tanh(\alpha x - \alpha^3 t) \operatorname{sech}(\alpha x - \alpha^3 t)$$

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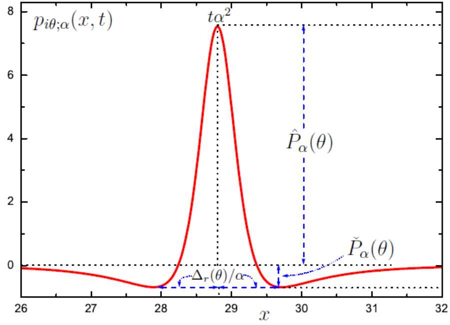

$$\hat{P}_\alpha(\theta) = \frac{\alpha^2}{2} \sec^2\left(\frac{\theta}{2}\right)$$

$$\check{P}_\alpha(\theta) = \frac{\alpha^2}{4} \cot^2(\theta)$$

$$\Delta_r(\theta) = \operatorname{arccosh}(\cos\theta - 2\sec\theta)$$

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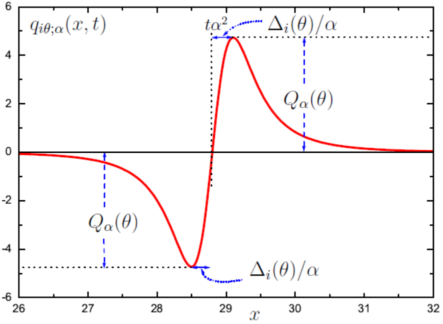

$$Q_\alpha(\theta) = \frac{8\alpha^2\sqrt{5+\cos(2\theta)+\cos\theta A}}{[6\cos\theta + A]^2 / \sin\theta}$$

$$\Delta_i(\theta) = \operatorname{arccosh}\left[\frac{1}{2}\cos\theta + \frac{1}{4}A\right]$$

$$A = \sqrt{2}\sqrt{17+\cos(2\theta)}$$

Real charges from one-soliton solution

$$
\begin{aligned}
\text{Mass}: m_\alpha &= \int_{-\infty}^{\infty} u_{i\theta;\alpha}(x,t)dx = 2\alpha \\
\text{Momentum}: p_\alpha &= \int_{-\infty}^{\infty} u_{i\theta;\alpha}^2 dx = \frac{2}{3}\alpha^3 \\
\text{Energy}: E_\alpha &= \int_{-\infty}^{\infty} \left[2u_{i\theta;\alpha}^3 - (u_{i\theta;\alpha})_x^2\right] dx = \frac{2}{5}\alpha^5
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Generic: $I_n = \int_{-\infty}^{\infty} w_{2n-2}(x,t)dx = \frac{2}{2n-1}\alpha^{2n-1}$

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This is not possible for N-soliton solutions with $N > 2$.

Nondegenerate two-soliton solutions

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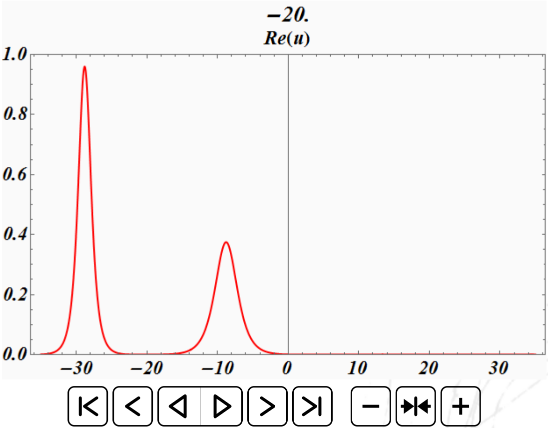

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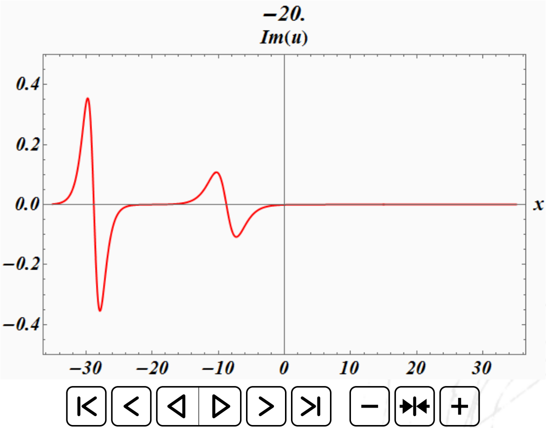

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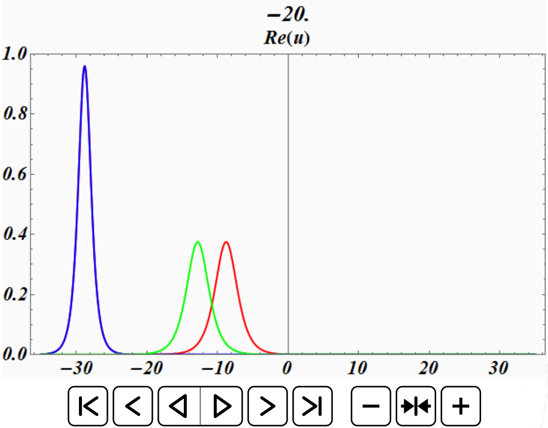

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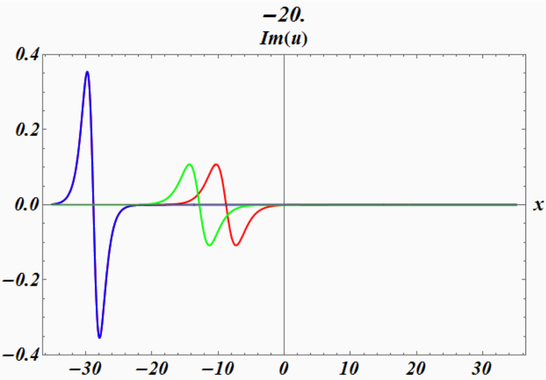

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Time-delays and lateral displacements

Comparing trajectories in the asymptotic past and future

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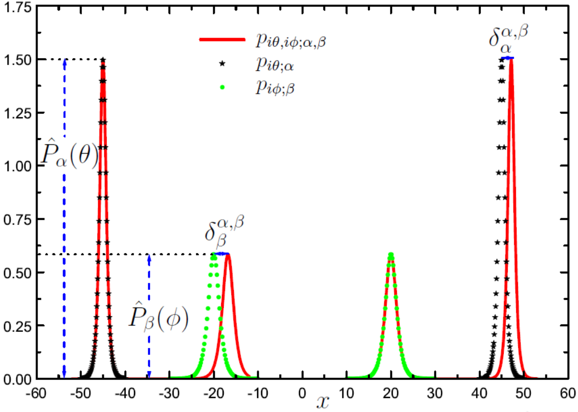
Comparing trajectories in the asymptotic past and future



$$\delta_x^{y,z} := \frac{2}{x} \ln\left(\frac{y+z}{y-z}\right)$$

Displacements: $(\Delta_x)_\alpha = \delta_\alpha^{\alpha,\beta}$ and $(\Delta_x)_\beta = -\delta_\beta^{\alpha,\beta}$

Time-delays: $(\Delta_t)_\alpha = -\frac{1}{\alpha^2}\delta_\alpha^{\alpha,\beta}$ and $(\Delta_t)_\beta = \frac{1}{\beta^2}\delta_\beta^{\alpha,\beta}$

Consistency relations: $\sum_k m_k(\Delta_x)_k = 0$ and $\sum_k p_k(\Delta_t)_k = 0$

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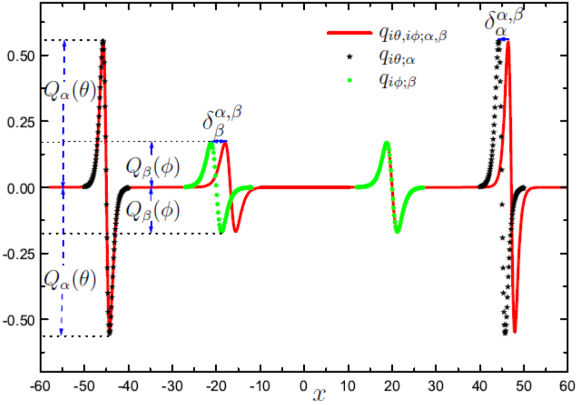

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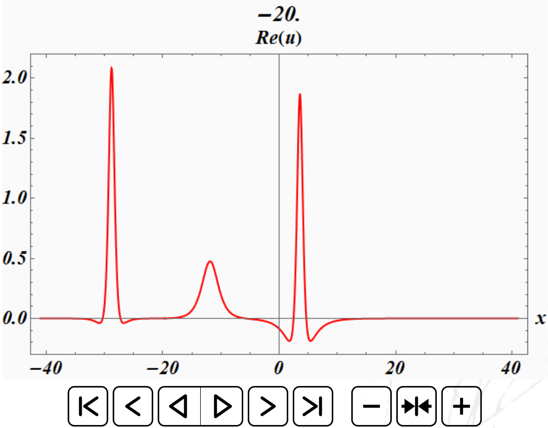

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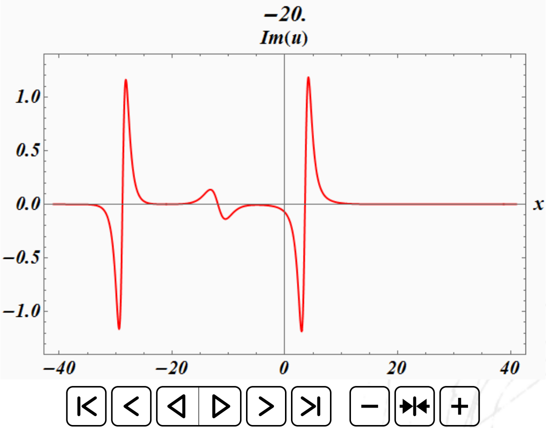

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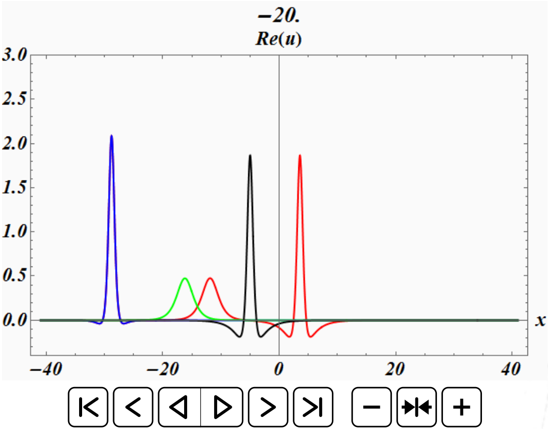

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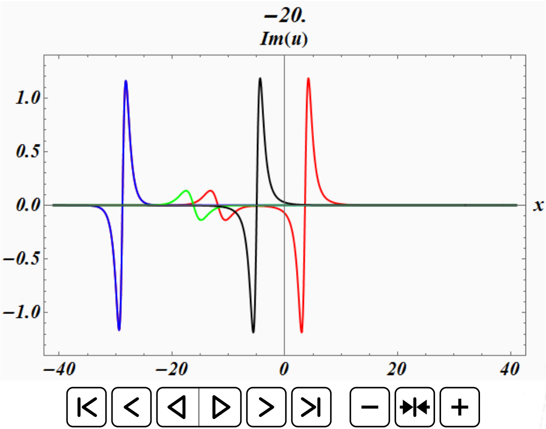

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Factorized Scattering

Displacements:

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(\Delta_x)_\beta &= \delta_\beta^{\beta,\gamma} - \delta_\beta^{\alpha,\beta} \\
(\Delta_x)_\gamma &= -\delta_\gamma^{\alpha,\gamma} - \delta_\gamma^{\beta,\gamma}
\end{aligned}
$$

Time-delays:

$$
\begin{aligned}
(\Delta_t)_\alpha &= -\frac{1}{\alpha^2}\left(\delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma}\right) \\
(\Delta_t)_\beta &= \frac{1}{\beta^2}\left(\delta_\beta^{\alpha,\beta} - \delta_\beta^{\beta,\gamma}\right) \\
(\Delta_t)_\gamma &= \frac{1}{\gamma^2}\left(\delta_\gamma^{\alpha,\gamma} + \delta_\gamma^{\beta,\gamma}\right)
\end{aligned}
$$

Consistency relations: $\sum_k m_k(\Delta_x)_k = 0$ and $\sum_k p_k(\Delta_t)_k = 0$

N-soliton time-delays $\equiv$ sum of two-soliton time-delays

Factorized Scattering

Displacements:

$$
\begin{aligned}
(\Delta_x)_\alpha &= \delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma} \\
(\Delta_x)_\beta &= \delta_\beta^{\beta,\gamma} - \delta_\beta^{\alpha,\beta} \\
(\Delta_x)_\gamma &= -\delta_\gamma^{\alpha,\gamma} - \delta_\gamma^{\beta,\gamma}
\end{aligned}
$$

Time-delays:

$$
\begin{aligned}
(\Delta_t)_\alpha &= -\frac{1}{\alpha^2}\left(\delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma}\right) \\
(\Delta_t)_\beta &= \frac{1}{\beta^2}\left(\delta_\beta^{\alpha,\beta} - \delta_\beta^{\beta,\gamma}\right) \\
(\Delta_t)_\gamma &= \frac{1}{\gamma^2}\left(\delta_\gamma^{\alpha,\gamma} + \delta_\gamma^{\beta,\gamma}\right)
\end{aligned}
$$

Consistency relations: $\sum_k m_k(\Delta_x)_k = 0$ and $\sum_k p_k(\Delta_t)_k = 0$

N-soliton time-delays $\equiv$ sum of two-soliton time-delays

Classical factorization

This corresponds to the factorization of the quantum S-matrix described by the Yang-Baxter and bootstrap equation.

Reality of complex N-soliton charges

Asymptotically complex N-solitons factor into N one-solitons

Charges based on one-solitons solutions are real by $\mathcal{PT}$-symmetry

Reality of complex N-soliton charges

Asymptotically complex N-solitons factor into N one-solitons

Charges based on one-solitons solutions are real by $\mathcal{PT}$-symmetry

Therefore

Reality condition

$\mathcal{PT}$-symmetry and integrability ensure the reality of all charges.

Regularized degenerate multi-solitons

- In general for real solutions:

  The limit $E_\alpha \rightarrow E_\beta$ gives $\lim_{\alpha\rightarrow\beta} u_{\alpha,\beta,\gamma,\ldots}(x,t) \rightarrow \infty$

  The best scenario still has cusps.

Regularized degenerate multi-solitons

- In general for real solutions:

  The limit $E_\alpha \to E_\beta$ gives $\lim_{\alpha\to\beta} u_{\alpha,\beta,\gamma,\ldots}(x,t) \to \infty$

  The best scenario still has cusps.

- In the complex case the limits become finite.

  Technically we use Wronskians as τ-functions involving solutions of the Schrödinger equation and Jordan states obtained from Darboux-Crum transformations.

  A link to Hirota's direct method and solutions obtained from a superpositon principle based on Bäcklund transformations is also established.

Degenerate two-soliton solutions

$$u_{i\theta,i\phi;\alpha,\alpha}(x,t) = \frac{2\alpha^2\left[\left(\alpha x - 3\alpha^3 t + i\phi\right)\sinh\left(\eta_{i\theta;\alpha}\right) - 2\cosh\left(\eta_{i\theta;\alpha}\right) - 2\right]}{\left[\alpha x - 3\alpha^3 t + i\phi + \sinh\left(\eta_{i\theta;\alpha}\right)\right]^2}$$

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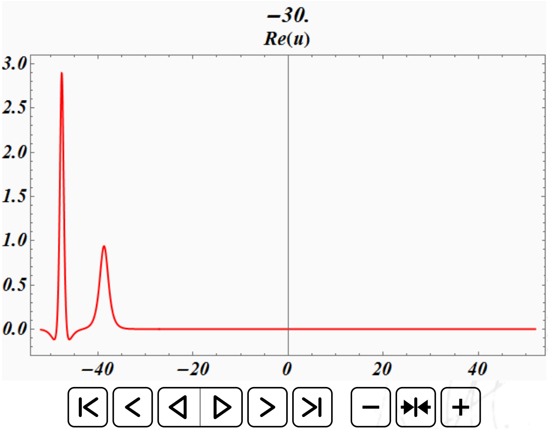

$\alpha = 6/5$, $\theta = \pi/3$, $\phi = \pi/4$

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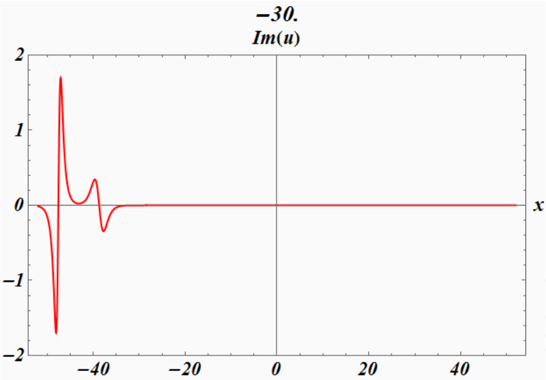

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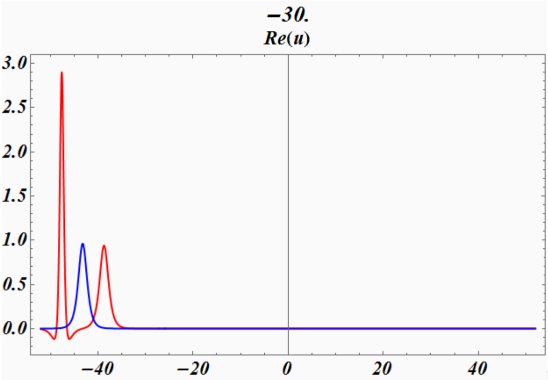

$\alpha = 6/5,\ \theta = \pi/3,\ \phi = \pi/4$

$p_{i\theta,i\phi;\alpha,\alpha}(x,t)$

$p_{i\theta;\alpha}(x,t)$

Relative displacement:
$\Delta(t) = \frac{1}{\alpha}\ln\left(4\alpha^3|t|\right)$

Total displacement:
$\pm 2\Delta(t)$

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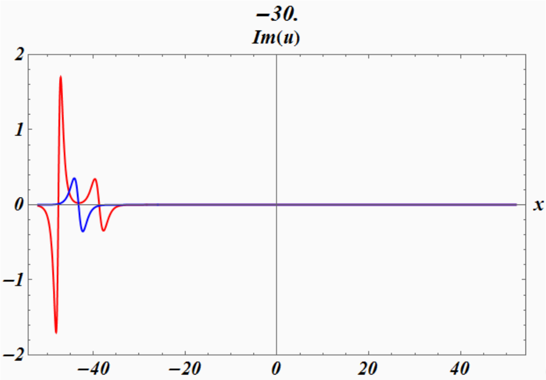

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Degenerate N-soliton solutions $(\alpha_1 = \alpha_2 = \ldots \alpha_N)$

Notation:

$\lim_{\alpha_2,\ldots,\alpha_N \to \alpha_1 = \alpha} u_{i\theta_1 = i\theta,\ldots,i\theta_N;\alpha_1,\ldots,\alpha_N} = p_{i\theta,\ldots,i\theta_N;N\alpha} + iq_{i\theta,\ldots,i\theta_N;N\alpha}$

Asymptotic limits:

$$\lim_{t\to\sigma\infty} p_{i\theta,\ldots,i\theta_{2n};2n\alpha}\left[t\alpha^2 + \sigma\Delta_{n,\ell,1}(t), t\right] = \hat{P}_\alpha\left(\theta + \frac{1-(-1)^{n+\ell+1}}{2}\pi\right)$$

$$\lim_{t\to\sigma\infty} p_{i\theta,\ldots,i\theta_{2n};2n\alpha}\left[t\alpha^2 - \sigma\Delta_{n,\ell,1}(t), t\right] = \hat{P}_\alpha\left(\theta + \frac{1-(-1)^{n+\ell}}{2}\pi\right)$$

for $n = 1, 2, \ldots,\ \ell = 1, 2, \ldots, n,\ \sigma = \pm 1$

$$\lim_{t\to\sigma\infty} p_{i\theta,\ldots,i\theta_{2n+1};(2n+1)\alpha}\left[t\alpha^2 \pm \Delta_{n,\ell,0}(t), t\right] = \hat{P}_\alpha\left(\theta + \frac{1-(-1)^{n+\ell}}{2}\pi\right)$$

for $n = 0, 1, 2, \ldots,\ \ell = 0, 1, 2, \ldots, n$

Time-dependent displacements:

$$\Delta_{n,\ell,\kappa}(t) = \frac{1}{\alpha}\ln\left[\frac{(n-\ell)!}{(n+\ell-\kappa)!}\left(4\,|t|\,\alpha^3\right)^{2\ell-\kappa}\right]$$

Higher order nonlinear Schrödinger equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q = 0$$

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$$iq_t + \frac{1}{2} q_{xx} + |q|^2 q + i\varepsilon \left[\alpha q_{xxx} + \beta |q|^2 q_x + \gamma q \left| q \right|_x^2 \right] = 0$$

$\mathcal{PT}$-symmetry: $\mathcal{PT} : x \to -x,\ t \to -t,\ i \to -i,\ q \to q$

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Integrable cases:

$\varepsilon = 0$ $\equiv$ nonlinear Schrödinger equation (NLSE)

$\alpha : \beta : \gamma = 0 : 1 : 1 \equiv$ derivative NLSE of type I

$\alpha : \beta : \gamma = 0 : 1 : 0 \equiv$ derivative NLSE of type II

$\alpha : \beta : \gamma = 1 : 6 : 3 \equiv$ Sasa-Satsuma equation

Higher order nonlinear Schrödinger equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon\left[\alpha q_{xxx} + \beta |q|^2 q_x + \gamma q |q|^2_x\right] = 0$$

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$\alpha : \beta : \gamma = 1 : 6 : 3 \equiv$ Sasa-Satsuma equation

$\alpha : \beta : \gamma = 1 : 6 : 0 \equiv$ Hirota equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon\left[q_{xxx} + 6|q|^2 q_x\right] = 0$$

Zero curvature condition

$$\partial_t U - \partial_x V + [U, V] = 0$$

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$$U = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}, \qquad V = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix}$$

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$$\begin{aligned} A_x(x,t) &= q(x,t)C(x,t) - r(x,t)B(x,t) \\ B_x(x,t) &= q_t(x,t) - 2q(x,t)A(x,t) - 2i\lambda B(x,t) \\ C_x(x,t) &= r_t(x,t) + 2r(x,t)A(x,t) + 2i\lambda C(x,t) \end{aligned}$$

$$\begin{aligned} A(x,t) &= -i\alpha qr - 2i\alpha\lambda^2 + \beta\left(rq_x - qr_x - 4i\lambda^3 - 2i\lambda qr\right) \\ B(x,t) &= i\alpha q_x + 2\alpha\lambda q + \beta\left(2q^2 r - q_{xx} + 2i\lambda q_x + 4\lambda^2 q\right) \\ C(x,t) &= -i\alpha r_x + 2\alpha\lambda r + \beta\left(2qr^2 - r_{xx} - 2i\lambda r_x + 4\lambda^2 r\right) \end{aligned}$$

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$$\begin{aligned} q_t - i\alpha q_{xx} + 2i\alpha q^2 r + \beta\left[q_{xxx} - 6qrq_x\right] &= 0 \\ r_t + i\alpha r_{xx} - 2i\alpha qr^2 + \beta\left(r_{xxx} - 6qrr_x\right) &= 0 \end{aligned}$$

Nonlocality (Zero curvature condition)

Complex conjugate pair: $r(x,t) = \kappa q^*(x,t)$ (Hirota equation)

$$
\begin{aligned}
iq_t &= -\alpha\left(q_{xx} - 2\kappa\left|q\right|^2 q\right) - i\beta\left(q_{xxx} - 6\kappa\left|q\right|^2 q_x\right) \\
-iq_t^* &= -\alpha\left(q_{xx}^* - 2\kappa\left|q\right|^2 q^*\right) + i\beta\left(q_{xxx}^* - 6\kappa\left|q\right|^2 q_x^*\right)
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\end{aligned}
$$

$\mathcal{P}$ conjugate pair: $r(x,t) = \kappa q^*(-x,t)$ (Nonlocal Hirota equ^n)

$$
\begin{aligned}
iq_t &= -\alpha\left[q_{xx} - 2\kappa\tilde{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q\tilde{q}^* q_x] \\
-i\tilde{q}_t^* &= -\alpha\left[\tilde{q}_{xx}^* - 2\kappa q(\tilde{q}^*)^2\right] - \delta(\tilde{q}_{xxx}^* - 6\kappa\tilde{q}^* q\tilde{q}_x^*)
\end{aligned}
$$

$$\beta = i\delta,\ \alpha, \delta \in \mathbb{R},\ q := q(x,t);\ \tilde{q} := q(-x,t)$$

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$$
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\end{aligned}
$$

$$\beta = i\delta,\ \alpha,\delta \in \mathbb{R},\ q := q(x,t);\ \tilde{q} := q(-x,t)$$

$\mathcal{T}$ conjugate pair: $r(x,t) = \kappa q^*(x,-t)$

$$
\begin{aligned}
iq_t &= -i\hat{\delta}\left[q_{xx} - 2\kappa\hat{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q\hat{q}^* q_x] \\
i\hat{q}_t^* &= i\hat{\delta}\left[\hat{q}_{xx}^* - 2\kappa q(\hat{q}^*)^2\right] + \delta(\hat{q}_{xxx}^* - 6\kappa\hat{q}^* q\hat{q}_x^*)
\end{aligned}
$$

$$\alpha = i\hat{\delta},\ \beta = i\delta,\ \hat{\delta},\delta \in \mathbb{R};\ \hat{q} := q(x,-t)$$

$\mathcal{PT}$-conjugate pair: $r(x,t) = \kappa q^*(-x,-t)$

$$
\begin{aligned}
q_t &= -\check{\delta}\left[q_{xx} - 2\kappa \check{q}^* q^2\right] - \beta[q_{xxx} - 6\kappa q \check{q}^* q_x] \\
-\check{q}_t^* &= -\check{\delta}\left[\check{q}_{xx}^* - 2\kappa q (\check{q}^*)^2\right] + \beta(\check{q}_{xxx}^* - 6\kappa \check{q}^* q \check{q}_x^*)
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$\mathcal{P}$ transformed pair: $r(x,t) = \kappa q(-x,t)$:

$$
\begin{aligned}
iq_t &= -\alpha\left[q_{xx} - 2\kappa \tilde{q} q^2\right] + \delta[q_{xxx} - 6\kappa q \tilde{q} q_x] \\
-i\tilde{q}_t &= -\alpha\left[\tilde{q}_{xx} - 2\kappa q \tilde{q}^2\right] - \delta(\tilde{q}_{xxx} - 6\kappa \tilde{q} q \tilde{q}_x)
\end{aligned}
$$

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$\mathcal{PT}$-conjugate pair: $r(x,t) = \kappa q^*(-x,-t)$

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$$\beta = i\delta;\ \alpha, \delta \in \mathbb{R}$$

$\mathcal{T}$ transformed pair: $r(x,t) = \kappa q(x,-t)$

$$\begin{aligned} iq_t &= -i\hat{\delta}\left[q_{xx} - 2\kappa \hat{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q \hat{q}^* q_x] \\ i\hat{q}_t^* &= i\hat{\delta}\left[\hat{q}_{xx}^* - 2\kappa q (\hat{q}^*)^2\right] + \delta(\hat{q}_{xxx}^* - 6\kappa \hat{q}^* q \hat{q}_x^*) \end{aligned}$$

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Nonlocality (Alice and Bob systems)

Korteweg-de Vries equation:

$$u_t + 6uu_x + u_{xxx} = 0$$

Take

$u(x,t) = \frac{1}{2}\left[a(x,t) + b(x,t)\right], \mathcal{PT} a(x,t) = a(-x,-t) = b(x,t)$

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$\mathcal{PT}$ pair: (Alice and Bob KdV equation)

$$\begin{aligned} a_t + 3/4(a+b)(3a_x + b_x) + a_{xxx} &= 0 \\ b_t + 3/4(a+b)(a_x + 3b_x) + b_{xxx} &= 0 \end{aligned}$$

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One -soliton solution

$$\begin{aligned} a_{\mu,\nu;\alpha}(x,t) &= u_{\mu,\alpha}(x,t) + \nu \tanh\left[\frac{1}{2}(\alpha x - \alpha^3 t + \mu)\right] \\ b_{\mu,\nu;\alpha}(x,t) &= u_{\mu,\alpha}(x,t) - \nu \tanh\left[\frac{1}{2}(\alpha x - \alpha^3 t + \mu)\right] \end{aligned}$$

$$u_{\mu,\alpha}(x,t) = \frac{\alpha^2}{2}\operatorname{sech}^2\left[\frac{1}{2}(\alpha x - \alpha^3 t + \mu)\right]$$

Nonlocality in Hirota's direct method

Bilinearisation of the local Hirota equation

$$f^3\left[iq_t+\alpha q_{xx}-2\kappa\alpha\left|q\right|^2q+i\beta\Big(q_{xxx}-6\kappa\left|q\right|^2q_x\Big)\right]=$$

$$f\left[iD_tg\cdot f+\alpha D_x^2g\cdot f+i\beta D_x^3g\cdot f\right]+\left[3i\beta\left(\frac{g}{f}f_x-g_x\right)-\alpha g\right]$$

$$\times\left[D_x^2f\cdot f+2\kappa\left|g\right|^2\right]$$

$$D_x^nf\cdot g=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)(-1)^k\frac{\partial^{n-k}}{\partial x^{n-k}}f(x)\frac{\partial^k}{\partial x^k}g(x)$$

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$$f\left[iD_t g\cdot f+\alpha D_x^2 g\cdot f+i\beta D_x^3 g\cdot f\right]+\left[3i\beta\left(\frac{g}{f}f_x-g_x\right)-\alpha g\right]$$
$$\times\left[D_x^2 f\cdot f+2\kappa\left|g\right|^2\right]$$

$$D_x^n f\cdot g=\sum\nolimits_{k=0}^{n}\binom{n}{k}(-1)^k\frac{\partial^{n-k}}{\partial x^{n-k}}f(x)\frac{\partial^k}{\partial x^k}g(x)$$

$$iD_t g\cdot f+\alpha D_x^2 g\cdot f+i\beta D_x^3 g\cdot f=0,\qquad D_x^2 f\cdot f=-2\kappa\left|g\right|^2$$

Nonlocality in Hirota's direct method

Bilinearisation of the local Hirota equation

$$f^3\left[iq_t+\alpha q_{xx}-2\kappa\alpha\left|q\right|^2 q+i\beta\left(q_{xxx}-6\kappa\left|q\right|^2 q_x\right)\right]=$$
$$f\left[iD_t g\cdot f+\alpha D_x^2 g\cdot f+i\beta D_x^3 g\cdot f\right]+\left[3i\beta\left(\frac{g}{f}f_x-g_x\right)-\alpha g\right]$$
$$\times\left[D_x^2 f\cdot f+2\kappa\left|g\right|^2\right]$$

$$D_x^n f\cdot g=\sum\nolimits_{k=0}^{n}\begin{pmatrix} n\\ k\end{pmatrix}(-1)^k\frac{\partial^{n-k}}{\partial x^{n-k}}f(x)\frac{\partial^k}{\partial x^k}g(x)$$

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Solve by formal power series that becomes **exact**

$$f(x,t)=\sum_{k=0}^{\infty}\varepsilon^{2k}f_{2k}(x,t),\quad \text{and}\quad g(x,t)=\sum_{k=1}^{\infty}\varepsilon^{2k-1}g_{2k-1}(x,t)$$

Bilinearisation of the nonlocal Hirota equation

$$f^3\tilde{f}^*\left[iq_t + \alpha q_{xx} + 2\alpha\tilde{q}^* q^2 - \delta(q_{xxx} + 6q\tilde{q}^* q_x)\right] =$$
$$f\tilde{f}^*\left[iD_t g\cdot f + \alpha D_x^2 g\cdot f - \delta D_x^3 g\cdot f\right] + \left(\frac{3\delta}{f}D_x g\cdot f - \alpha g\right)$$
$$\times\left(\tilde{f}^* D_x^2 f\cdot f - 2fg\tilde{g}^*\right)$$

not bilinear yet

$$iD_t g\cdot f + \alpha D_x^2 g\cdot f - \delta D_x^3 g\cdot f = 0, \qquad \tilde{f}^* D_x^2 f\cdot f = 2fg\tilde{g}^*$$

Bilinearisation of the nonlocal Hirota equation

$$
\begin{aligned}
& f^3 \tilde{f}^* \left[i q_t + \alpha q_{xx} + 2\alpha \tilde{q}^* q^2 - \delta (q_{xxx} + 6 q \tilde{q}^* q_x) \right] = \\
& f \tilde{f}^* \left[i D_t g \cdot f + \alpha D_x^2 g \cdot f - \delta D_x^3 g \cdot f \right] + \left(\frac{3\delta}{f} D_x g \cdot f - \alpha g \right) \\
& \times \left(\tilde{f}^* D_x^2 f \cdot f - 2 f g \tilde{g}^* \right)
\end{aligned}
$$

not bilinear yet

$$
i D_t g \cdot f + \alpha D_x^2 g \cdot f - \delta D_x^3 g \cdot f = 0, \qquad \tilde{f}^* D_x^2 f \cdot f = 2 f g \tilde{g}^*
$$

introduce auxiliary function

$$
D_x^2 f \cdot f = h g, \qquad \text{and} \qquad 2 f \tilde{g}^* = h \tilde{f}^*
$$

Solve again formal power series that becomes **exact**

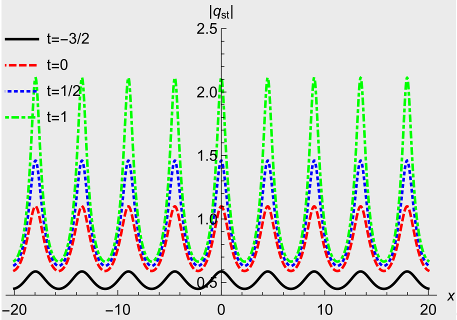
$$
h(x,t) = \sum\nolimits_k \varepsilon^k h_k(x,t).
$$

Two-types of nonlocal solutions (one-soliton)

Truncated expansions: $f = 1 + \varepsilon^2 f_2, \quad g = \varepsilon g_1, \quad h = \varepsilon h_1$

$$
\begin{aligned}
0 &= \varepsilon[i\,(g_1)_t + \alpha\,(g_1)_{xx} - \delta(g_1)_{xxx}] \\
&\quad + \varepsilon^3[2\,(f_2)_x\,(g_1)_x - g_1\,[(f_2)_{xx} + i\,(f_2)_t] + if_2\,[(g_1)_t + i\,(g_1)_{xx}]] \\
0 &= \varepsilon^2[2(f_2)_{xx} - g_1 h_1] + \varepsilon^4\left[2f_2(f_2)_{xx} - 2(f_2)_x^2\right] \\
0 &= \varepsilon[2\tilde{g}_1^* - h_1] + \varepsilon^3\left[2f_2\tilde{g}_1^* - \tilde{f}_2^* h_1\right]
\end{aligned}
$$

Standard solution, solve six equations independently, then $\varepsilon \to 1$



$$
q_{st}^{(1)} = \frac{\lambda(\mu - \mu^*)^2 \tau_{\mu,\gamma}}{(\mu - \mu^*)^2 + |\lambda|^2 \tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}}
$$

$$
\tau_{\mu,\gamma}(x,t) := e^{\mu x + \mu^2(i\alpha - \beta\mu)t + \gamma}
$$

Two-types of nonlocal solutions (one-soliton)

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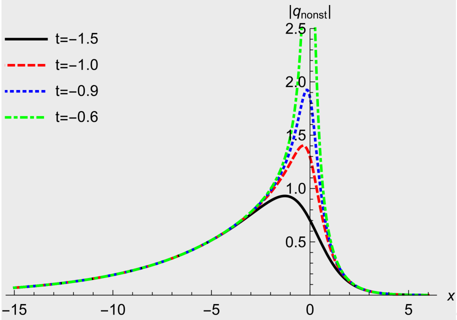
$$0 = \varepsilon[i\,(g_1)_t + \alpha\,(g_1)_{xx} - \delta(g_1)_{xxx}]$$
$$+\varepsilon^3[2\,(f_2)_x\,(g_1)_x - g_1\,[(f_2)_{xx} + i\,(f_2)_t] + if_2\,[(g_1)_t + i\,(g_1)_{xx}]]$$
$$0 = \varepsilon^2[2(f_2)_{xx} - g_1 h_1] + \varepsilon^4\left[2f_2(f_2)_{xx} - 2(f_2)_x^2\right]$$
$$0 = [2\tilde{g}_1^* - h_1] + \left[2f_2\tilde{g}_1^* - \tilde{f}_2^* h_1\right]$$

Nonstandard solution, solve five equations, last one for $\varepsilon = 1$



$$q_{\text{nonst}}^{(1)} = \frac{(\mu+\nu)\tau_{\mu,i\gamma}}{1 + \tau_{\mu,i\gamma}\tilde{\tau}^*_{-\nu,-i\theta}}.$$

$$\tau_{\mu,\gamma}(x,t) := e^{\mu x + \mu^2(i\alpha - \beta\mu)t + \gamma}$$

Two-soliton solution

Truncated expansions:

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4, \quad g = \varepsilon g_1 + \varepsilon^3 g_3, \quad h = \varepsilon h_1 + \varepsilon^3 h_3$$

$$q_{\mathrm{nl}}^{(2)}(x,t) = \frac{g_1(x,t) + g_3(x,t)}{1 + f_2(x,t) + f_4(x,t)}$$

$$\begin{aligned}
g_1 &= \tau_{\mu,\gamma} + \tau_{\nu,\delta} \\
g_3 &= \frac{(\mu-\nu)^2}{(\mu-\mu^*)^2(\nu-\mu^*)^2}\tau_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\mu,\gamma} + \frac{(\mu-\nu)^2}{(\mu-\nu^*)^2(\nu-\nu^*)^2}\tau_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta} \\
f_2 &= \frac{\tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}}{(\mu-\mu^*)^2} + \frac{\tau_{\nu,\delta}\tilde{\tau}^*_{\mu,\gamma}}{(\nu-\mu^*)^2} + \frac{\tau_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}}{(\mu-\nu^*)^2} + \frac{\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta}}{(\nu-\nu^*)^2} \\
f_4 &= \frac{(\mu-\nu)^2(\mu^*-\nu^*)^2}{(\mu-\mu^*)^2(\nu-\mu^*)^2(\mu-\nu^*)^2(\nu-\nu^*)^2}\tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta} \\
h_1 &= 2\tilde{\tau}^*_{\mu,\gamma} + 2\tilde{\tau}^*_{\nu,\delta} \\
h_3 &= \frac{2(\mu^*-\nu^*)^2}{(\mu-\mu^*)^2(\nu^*-\mu)^2}\tilde{\tau}^*_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}\tau_{\mu,\gamma} + \frac{2(\mu^*-\nu^*)^2}{(\mu^*-\nu)^2(\nu-\nu^*)^2}\tilde{\tau}^*_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}\tau_{\nu,\delta}
\end{aligned}$$

Two-soliton solution

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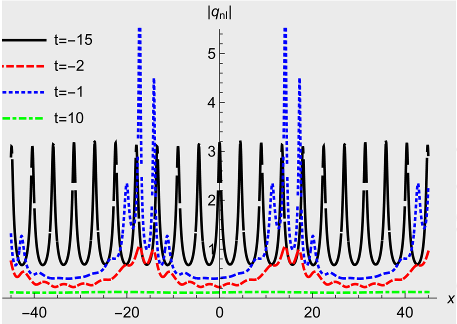
Nonlocal regular two-soliton solution



Two-soliton solution

Truncated expansions:

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4, \quad g = \varepsilon g_1 + \varepsilon^3 g_3, \quad h = \varepsilon h_1 + \varepsilon^3 h_3$$

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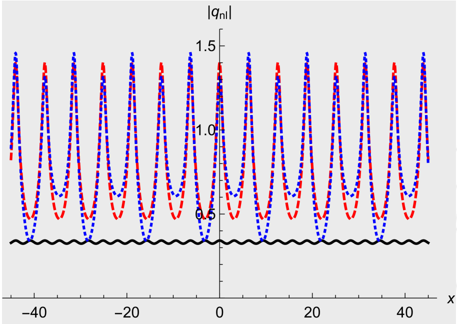
Nonlocal regular two one-soliton solution vs two-soliton solution



Nonlocality in Darboux-Crum transformations

Quantum mechanical analogue to supersymmetry, intertwining

$$L_n H_{n-1} = H_n L_n$$

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iteration $\mathcal{L}_n H_0 = H_n \mathcal{L}_n$, $\mathcal{L}_n := L_n L_{n-1} \ldots L_1$, $\Psi_n(\lambda) = \mathcal{L}_n \Psi(\lambda)$

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In Hirota case, Hamiltonian of Dirac type :

$$\Psi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \; \Psi_x = U\Psi \;\Leftrightarrow\; \begin{array}{c} -i\varphi_x + iq\phi = -\lambda\varphi \\ i\phi_x - ir\varphi = -\lambda\phi \end{array} \;\Leftrightarrow H\Psi(\lambda) = -\lambda\Psi(\lambda)$$

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with

$$H = \begin{pmatrix} -i\partial_x & iq \\ -ir & i\partial_x \end{pmatrix} = -i\sigma_3\partial_x + V,$$

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$$H_n = \begin{pmatrix} -i\partial_x & iq_n \\ -ir_n & i\partial_x \end{pmatrix} = -i\sigma_3\partial_x + V_n,$$

Solve the "seed" equations for $q = r = 0$:

$$\tilde{\Psi}_1(x,t;\lambda) = \begin{pmatrix} \varphi_1(x,t;\lambda) \\ \phi_1(x,t;\lambda) \end{pmatrix} = \begin{pmatrix} e^{\lambda x + 2i\lambda^2(\alpha - 2\delta\lambda)t + \gamma_1} \\ e^{-\lambda x - 2i\lambda^2(\alpha - 2\delta\lambda)t + \gamma_2} \end{pmatrix}$$

Implement nonlocality in the construction Ψ_2:
Two choices to achieve $r(x,t) = \pm q^*(-x,t)$

$$1:\ \varphi_2 = \pm\tilde{\phi}_1^*,\ \phi_2 = \tilde{\varphi}_1^* \qquad 2:\ \phi_1 = \tilde{\varphi}_1^*,\ \phi_2 = \pm\tilde{\varphi}_2^*$$

The second choice is not available in the local case.

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Choice 1:

$$\tilde{\Psi}_2(x,t;\lambda) = \begin{pmatrix} \varphi_2(x,t;\lambda) \\ \phi_2(x,t;\lambda) \end{pmatrix} = \begin{pmatrix} \mp e^{\lambda^* x + 2i(\lambda^*)^2(\alpha - 2\delta\lambda)t + \gamma_2^*} \\ e^{-\lambda^* x - 2i(\lambda^*)^2(\alpha - 2\delta\lambda^*)t + \gamma_1^*} \end{pmatrix}$$

with $\lambda, \gamma_1, \gamma_2 \in \mathbb{C}$

$$q_{\text{st}}^{(1)}(x,t) = \frac{2(\lambda^* - \lambda)e^{2\lambda^* x + 2i(\lambda^*)^2(\alpha - 2\delta\lambda^*)t - \gamma_1^* + \gamma_2^*}}{1 + e^{2(\lambda^* - \lambda)x + 4i\left[\alpha(\lambda^*)^2 - \alpha\lambda^2 + 2\delta\lambda^3 - 2\delta(\lambda^*)^3\right]t - \gamma_1 + \gamma_2 - \gamma_1^* + \gamma_2^*}}.$$

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The second choice is not available in the local case.
Choice 2:

$$\tilde{\Psi}_2(x,t;\nu) = \begin{pmatrix} \varphi_2(x,t;\nu) \\ \phi_2(x,t;\nu) \end{pmatrix} = \begin{pmatrix} e^{\nu x + 2i\nu^2(\alpha - 2\delta\nu)t + \gamma_3} \\ -e^{-\nu x - 2i\nu^2(\alpha - 2\delta\nu)t + \gamma_3^*} \end{pmatrix}$$

$$q^{(1)}_{\text{nonst}}(x,t) = \frac{2(\nu - \mu)e^{\gamma_1 - \gamma_1^* + 2\mu x + 4i\mu^2(\alpha - 2\delta\mu)t}}{1 + e^{2(\mu-\nu)x + 4i(\alpha\mu^2 - \alpha\nu^2 - 2\delta\mu^3 + 2\delta\nu^3)t + \gamma_1 - \gamma_1^* - \gamma_3 + \gamma_3^*}}.$$

Nonlocal n-soliton solutions: $q_n = q + 2\frac{\det D_n^q}{\det W_n}, \;\; r_n = r - 2\frac{\det D_n^r}{\det W_n}$

$$W_n = \begin{pmatrix} \varphi_1^{(n-1)} & \varphi_1^{(n-2)} & \cdots & \varphi_1 & \phi_1^{(n-1)} & \cdots & \phi_1' & \phi_1 \\ \varphi_2^{(n-1)} & \varphi_2^{(n-2)} & \cdots & \varphi_2 & \phi_2^{(n-1)} & \cdots & \phi_2' & \phi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{2n}^{(n-1)} & \varphi_{2n}^{(n-2)} & \cdots & \varphi_{2n} & \phi_{2n}^{(n-1)} & \cdots & \phi_{2n}' & \phi_{2n} \end{pmatrix}$$

$$D_n^q = \begin{pmatrix} \phi_1^{(n-2)} & \phi_1^{(n-3)} & \cdots & \phi_1 & \varphi_1^{(n)} & \cdots & \varphi_1' & \varphi_1 \\ \phi_2^{(n-2)} & \phi_2^{(n-3)} & \cdots & \phi_2 & \varphi_2^{(n)} & \cdots & \varphi_2' & \varphi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{2n}^{(n-2)} & \phi_{2n}^{(n-3)} & \cdots & \phi_{2n} & \varphi_{2n}^{(n)} & \cdots & \varphi_{2n}' & \varphi_{2n} \end{pmatrix}$$

$$D_n^r = \begin{pmatrix} \phi_1^{(n)} & \phi_1^{(n-1)} & \cdots & \phi_1 & \varphi_1^{(n-2)} & \cdots & \varphi_1' & \varphi_1 \\ \phi_2^{(n)} & \phi_2^{(n-1)} & \cdots & \phi_2 & \varphi_2^{(n-2)} & \cdots & \varphi_2' & \varphi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{2n}^{(n)} & \phi_{2n}^{(n-1)} & \cdots & \phi_{2n} & \varphi_{2n}^{(n-2)} & \cdots & \varphi_{2n}' & \varphi_{2n} \end{pmatrix}$$

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In the nonlocal case use

$$\tilde{S}_{2n}^{\text{st}} = \left\{ \tilde{\Psi}_1(x,t;\lambda_1), \tilde{\Psi}_2(x,t;\lambda_1), \tilde{\Psi}_1(x,t;\lambda_2), \tilde{\Psi}_2(x,t;\lambda_2), \ldots \right\}$$

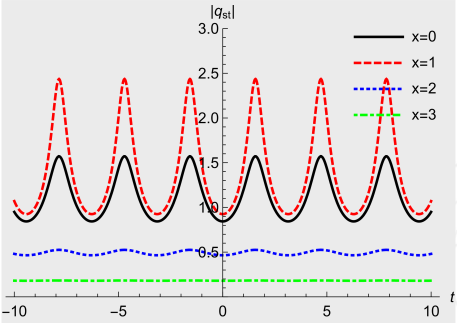
or

$$\tilde{S}_{2n}^{\text{nonst}} = \left\{ \tilde{\Psi}_1(x,t;\mu_1), \tilde{\Psi}_2(x,t;\nu_1), \tilde{\Psi}_1(x,t;\mu_2), \tilde{\Psi}_2(x,t;\nu_2), \ldots \right\}$$

Time-crystals: $[r(x,t) = \pm\hat{q}^*(x,-t)]$

$$q_{\text{st}}^{(1)}(x,t) = \frac{\pm 2(\lambda-\lambda^*)e^{\gamma_1+\gamma_2^*+2(\lambda+\lambda^*)x}}{e^{\gamma_1+\gamma_1^*+4(\lambda^*)^2(2\beta\lambda^*+\check{\delta})t+2\mu x} \pm e^{\gamma_2+\gamma_2^*+4\lambda\mu^2(2\beta\lambda+\check{\delta})t+2\lambda^* x}}$$

$$q_{\text{nonst}}^{(1)}(x,t) = \frac{2(\nu-\mu)e^{\gamma_1-\gamma_1^*+2(\mu+\nu)x}}{e^{4\mu^2(2\beta\mu+\check{\delta})t+2\nu x} + e^{4\nu^2(2\beta\nu+\check{\delta})t+2\mu x}}.$$



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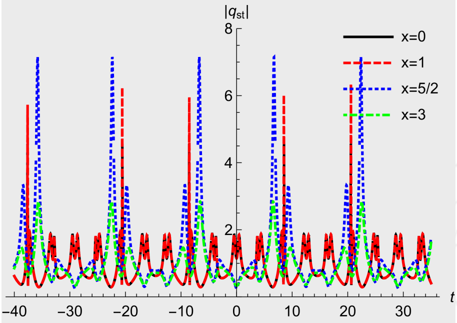

Nonlocal gauge equivalence

Two zero curvature conditions are related as

$$\partial_t U_i - \partial_x V_i + [U_i, V_i] = 0 \quad \Leftrightarrow \quad \Psi_{i,t} = V_i \Psi_i,\ \Psi_{i,x} = U_i \Psi_i \quad i = 1, 2$$

$$U_1 = G U_2 G^{-1} + G_x G^{-1} \qquad V_1 = G V_2 G^{-1} + G_t G^{-1}$$

when the auxiliary fields are related by a gauge field $\Psi_1 = G \Psi_2$

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System 1 $\equiv$ (nonlocal) Hirota equation $\Rightarrow$ system 2:

$$U_2 = -i\lambda G^{-1}\sigma_3 G, \qquad V_2 = \lambda G^{-1} B_1 G + \lambda^2 G^{-1} B_2 G + \lambda^3 G^{-1} B_3 G$$

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Extended continuous limit of the Heisenberg spin chain

For $\quad S := G^{-1} \sigma_3 G \Rightarrow S_t = i\alpha \left(S_x^2 + S S_{xx} \right) - \beta \left[\frac{3}{2} \left(S S_x^2 \right)_x + S_{xxx} \right]$

Nonlocal gauge equivalence

Two zero curvature conditions are related as

$$\partial_t U_i - \partial_x V_i + [U_i, V_i] = 0 \quad \Leftrightarrow \quad \Psi_{i,t} = V_i \Psi_i, \; \Psi_{i,x} = U_i \Psi_i \quad i = 1, 2$$

$$U_1 = G U_2 G^{-1} + G_x G^{-1} \qquad V_1 = G V_2 G^{-1} + G_t G^{-1}$$

when the auxiliary fields are related by a gauge field $\Psi_1 = G \Psi_2$

System 1 $\equiv$ (nonlocal) Hirota equation $\Rightarrow$ system 2:

$$U_2 = -i\lambda G^{-1} \sigma_3 G, \qquad V_2 = \lambda G^{-1} B_1 G + \lambda^2 G^{-1} B_2 G + \lambda^3 G^{-1} B_3 G$$

Extended continuous limit of the Heisenberg spin chain

For $\quad S := G^{-1} \sigma_3 G \Rightarrow S_t = i\alpha \left(S_x^2 + S S_{xx} \right) - \beta \left[\frac{3}{2} \left(S S_x^2 \right)_x + S_{xxx} \right]$

Extended version of the Landau-Lifschitz equation

For $\quad S = \mathbf{s} \cdot \boldsymbol{\sigma} \Rightarrow \mathbf{s}_t = -\alpha \mathbf{s} \times \mathbf{s}_{xx} - \frac{3}{2} \beta \left(\mathbf{s}_x \cdot \mathbf{s}_x \right) \mathbf{s}_x + \beta \mathbf{s} \times \left(\mathbf{s} \times \mathbf{s}_{xxx} \right)$

Parameterise

$$S = \begin{pmatrix} -\omega & u \\ v & \omega \end{pmatrix} \qquad \omega^2 + uv = 1$$

components of the matrix equation give

$$\begin{aligned} u_t &= i\alpha(u\omega_x - \omega u_x)_x - \beta\left[u_{xx} + 3/2u(u_x v_x + \omega_x^2)\right]_x \\ v_t &= -i\alpha(v\omega_x - \omega v_x)_x - \beta\left[v_{xx} + 3/2v(v_x u_x + \omega_x^2)\right]_x \end{aligned}$$

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Nonlocality via $u(x,t) = \kappa v^*(-x,t), \omega(x,t) = \omega^*(-x,t)$

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Solutions from auto-gauge transformation $\equiv$ DC transformation

$$\begin{aligned} u_1 &= \frac{2\varphi_1\varphi_2(\lambda_2\varphi_1\phi_2 - \lambda_1\varphi_2\phi_1)(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2(\varphi_2\phi_1 - \varphi_1\phi_2)^2} \\ v_1 &= \frac{2\phi_1\phi_2(\lambda_1\varphi_1\phi_2 - \lambda_2\varphi_2\phi_1)(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2(\varphi_2\phi_1 - \varphi_1\phi_2)^2} \\ \omega_1 &= 1 - \frac{2\varphi_1\varphi_2\phi_1\phi_2(\lambda_1 - \lambda_2)^2}{\lambda_1\lambda_2(\varphi_2\phi_1 - \varphi_1\phi_2)^2} \end{aligned}$$

Back to Hirota equation:
solve $S = G^{-1}\sigma_3 G$ for G and use $G_x = A_0 G, \; G_t = B_0 G$

$$A_0 = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix}$$

$$B_0 = i\alpha \left[\sigma_3 (A_0)_x - \sigma_3 A_0^2\right] + \beta \left[2A_0^3 + (A_0)_x A_0 - A_0 (A_0)_x - (A_0)_{xx}\right]$$

$$q(x,t) = \frac{\mu(t)}{2}\left(\frac{v_x}{v} + \frac{\omega v_x - \omega_x v}{v}\right)\exp\int^x \frac{\omega(s,t)v_s(s,t) - \omega_s(s,t)v(s,t)}{v(s,t)}ds$$

$$r(x,t) = \frac{1}{2\mu(t)}\left(\frac{v_x}{v} - \frac{\omega v_x - \omega_x v}{v}\right)\exp\int^x \frac{\omega_s(s,t)v(s,t) - \omega(s,t)v_s(s,t)}{v(s,t)}ds$$

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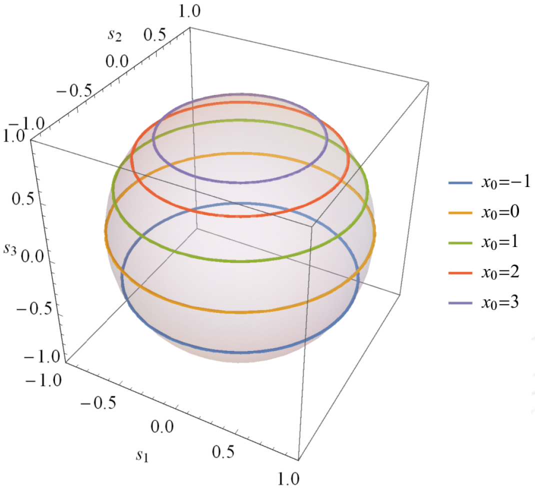
$$\begin{aligned} q_n(x,t) &= \frac{\mu_n}{\prod_{k=1}^{2n}\lambda_k}\left(\frac{(\det\mathcal{V}_n)_x}{\det\Upsilon_n} - \frac{(\det\mathcal{W}_n)_x \det\mathcal{V}_n}{\det\mathcal{W}_n \det\Upsilon_n}\right) \\ r_n(x,t) &= \frac{\prod_{k=1}^{2n}\lambda_k}{\mu_n}\left(\frac{(\det\Upsilon_n)_x}{\det\mathcal{V}_n} - \frac{(\det\mathcal{W}_n)_x \det\Upsilon_n}{\det\mathcal{W}_n \det\mathcal{V}_n}\right) \end{aligned}$$

Nonlocality: $r_n(x,t) = \kappa \prod_{i=1}^{n} \left|\lambda_{2i-1}\right|^4 / \mu_n^2 q_n^*(-x,t)$

Trajectories of local extended version of the LL equation

Gauge equivalent version of nonlinear Schrödinger equation $\beta = 0$



$|\mathbf{s}|^2 = 1$, $\mathbf{s}$ is real, real shift parameter

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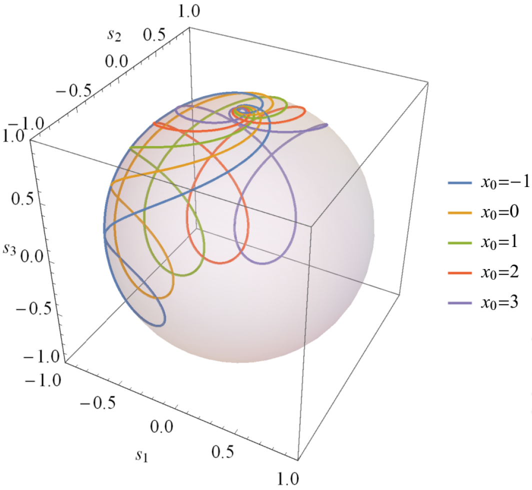

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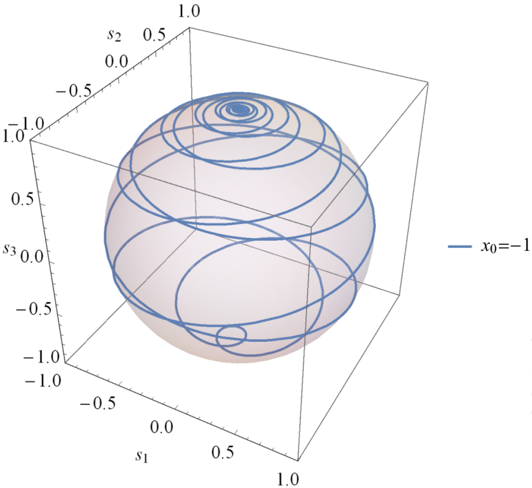

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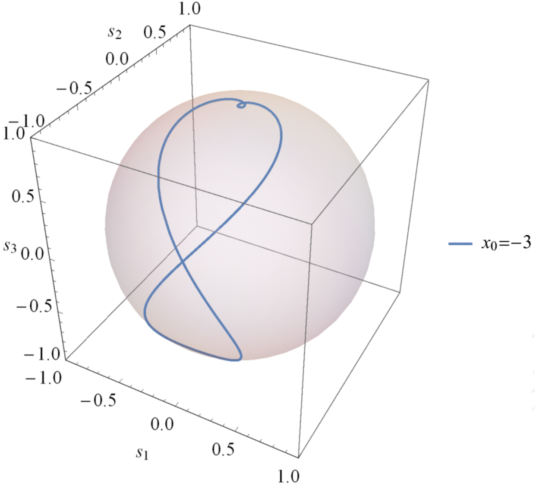

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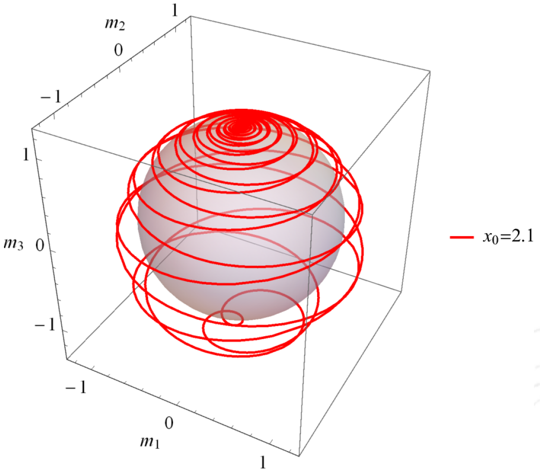
Trajectories of nonlocal extended version of the LL equation

Gauge equivalent version of nonlocal Hirota equation $\beta \neq 0$



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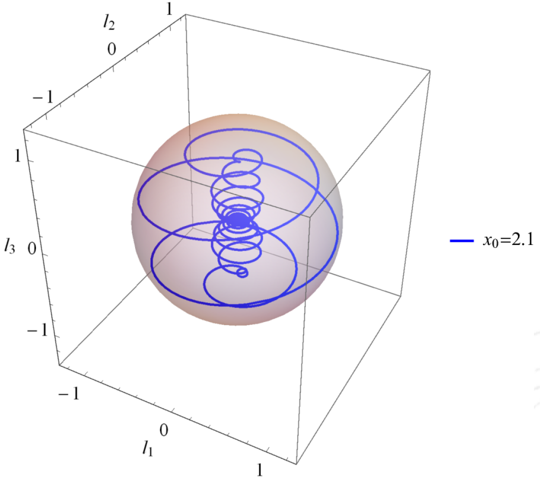

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Conclusions and Outlook

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- Study more physical applications.

Thank you for your attention