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Non-Hermitian multi-particle systems from complex root spaces

Andreas Fring

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25-th-28-th of September 2011

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2. From constraint field equations
3. Deformed Calogero-Moser-Sutherland models

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Calogero-Moser-Sutherland models (extended)

$$\mathcal{H}_{BK} = \frac{p^2}{2} + \frac{\omega^2}{2} \sum_i q_i^2 + \frac{g^2}{2} \sum_{i \neq k} \frac{1}{(q_i - q_k)^2} + i\tilde{g} \sum_{i \neq k} \frac{1}{(q_i - q_k)} p_i$$

with $g, \tilde{g} \in \mathbb{R}, q, p \in \mathbb{R}^{\ell+1}$
[B. Basu-Mallick, A. Kundu, Phys. Rev. B62 (2000) 9927]

1. Representation independent formulation?
2. Other potentials apart from the rational one?
3. Other algebras apart from A_n, B_n or Coxeter groups?
4. Is it possible to include more coupling constants?
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- Generalize Hamiltonian to:

$$\mathcal{H}_\mu = \frac{1}{2}p^2 + \frac{1}{2}\sum_{\alpha\in\Delta} g_\alpha^2 V(\alpha\cdot q) + i\mu\cdot p$$

· Now Δ is any root system

· $\mu = 1/2\sum_{\alpha\in\Delta}\tilde{g}_\alpha f(\alpha\cdot q)\alpha$, $f(x) = 1/x$ $V(x) = f^2(x)$

[A. F., Mod. Phys. Lett. A21 (2006) 691, Acta P. 47 (2007) 44]

- Not so obvious that one can re-write

$$\mathcal{H}_\mu = \frac{1}{2}(p+i\mu)^2 + \frac{1}{2}\sum_{\alpha\in\Delta}\hat{g}_\alpha^2 V(\alpha\cdot q), \quad \hat{g}_\alpha^2 = \begin{cases} g_s^2 + \alpha_s^2\tilde{g}_s^2 & \alpha\in\Delta_s \\ g_l^2 + \alpha_l^2\tilde{g}_l^2 & \alpha\in\Delta_l \end{cases}$$

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- From real fields to complex particle systems

  i) No restrictions
  e.g. Benjamin-Ono equation

$$u_t + uu_x + \lambda H u_{xx} = 0 \qquad (*)$$

$H \equiv$ Hilbert transform, i.e. $Hu(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{u(x)}{z-x} dz$
Then

$$u(x,t) = \frac{\lambda}{2}\sum_{k=1}^{\ell}\left(\frac{i}{x-z_k} - \frac{i}{x-z_k^*}\right) \in \mathbb{R}$$

satisfies (*) iff z_k obeys the A_n-Calogero equ. of motion

$$\ddot{z}_k = \frac{\lambda^2}{2}\sum_{k\neq j}(z_j - z_k)^{-3}$$

[H. Chen, N. Pereira, Phys. Fluids 22 (1979) 187]
[talk by J. Feinberg, PHHQP workshop VI, 2007, London]

ii) restrict to submanifold

Theorem: [Airault, McKean, Moser, CPAM, (1977) 95]
Given a Hamiltonian $H(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n)$ with flow

$$x_i = \partial H / \partial \dot{x}_i \qquad \text{and} \qquad \ddot{x}_i = -\partial H / \partial x_i \qquad i = 1, \ldots, n$$

and conserved charges I_j in involution with H,i.e. $\{I_j, H\} = 0$. Then the locus of $\mathrm{grad}\, I = 0$ is invariant.

Example: Boussinesq equation

$$v_{tt} = a(v^2)_{xx} + bv_{xxxx} + v_{xx} \qquad (^{**})$$

Then

$$v(x,t) = c \sum\nolimits_{k=1}^{\ell} (x - z_k)^{-2}$$

satisfies (**) iff b=1/12, c=-a/2 and z_k obeys

$$\ddot{z}_k = 2 \sum\nolimits_{j \neq k} (z_j - z_k)^{-3} \qquad \Leftrightarrow \qquad \ddot{z}_k = -\frac{\partial H}{\partial z_i}$$

$$\dot{z}_k = 1 - \sum\nolimits_{j \neq k} (z_j - z_k)^{-2} \qquad \Leftrightarrow \qquad \mathrm{grad}(I_3 - I_1) = 0$$

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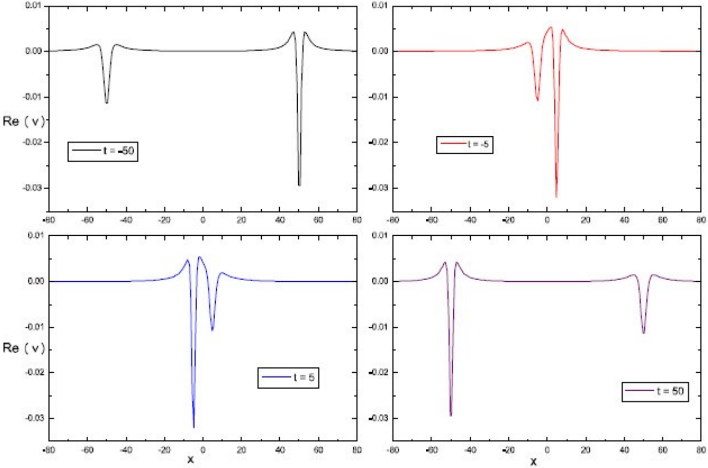
Constrained field equations → complex Calogero models



[P. Assis and A.F., J. Phys. A42 (2009) 425206]

Consider

Antilinearly invariant deformed Calogero model

$$\mathcal{H}_{\mathcal{PT}\text{CMS}} = \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha\in\Delta_s} (\alpha \cdot \tilde{q})^2 + \frac{1}{2} \sum_{\alpha\in\Delta} g_\alpha V(\alpha \cdot \tilde{q}), \quad m, g_\alpha \in \mathbb{R}$$

Define deformed coordinates (A_2)

$$\begin{aligned} q_1 &\to \tilde{q}_1 = q_1 \cosh\varepsilon + i\sqrt{3}(q_2 - q_3)\sinh\varepsilon \\ q_2 &\to \tilde{q}_2 = q_2 \cosh\varepsilon + i\sqrt{3}(q_3 - q_1)\sinh\varepsilon \\ q_3 &\to \tilde{q}_3 = q_3 \cosh\varepsilon + i\sqrt{3}(q_1 - q_2)\sinh\varepsilon \end{aligned}$$

With standard 3D representation for the simple A_2-roots $\alpha_1 = \{1, -1, 0\}$, $\alpha_2 = \{0, 1, -1\}$, $q_{ij} := q_i - q_j$ compute

$$\begin{aligned} \alpha_1 \cdot \tilde{q} &= q_{12} \cosh\varepsilon - \frac{\imath}{\sqrt{3}}(q_{13} + q_{23})\sinh\varepsilon, \\ \alpha_2 \cdot \tilde{q} &= q_{23} \cosh\varepsilon - \frac{\imath}{\sqrt{3}}(q_{21} + q_{31})\sinh\varepsilon, \\ (\alpha_1 + \alpha_2) \cdot \tilde{q} &= q_{13} \cosh\varepsilon + \frac{\imath}{\sqrt{3}}(q_{12} + q_{32})\sinh\varepsilon. \end{aligned}$$

Symmetries:

$$\begin{aligned} \mathcal{S}_1 &: \quad q_1 \leftrightarrow q_2,\ q_3 \leftrightarrow q_3,\ \imath \to -\imath, \\ \mathcal{S}_2 &: \quad q_2 \leftrightarrow q_3,\ q_1 \leftrightarrow q_1,\ \imath \to -\imath. \end{aligned}$$

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Unbroken $\mathcal{PT}$-symmetry guarantees real eigenvalues

- $\mathcal{PT}$-symmetry: $\mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$
  $(\mathcal{P}: x \to -x, p \to -p;\ \mathcal{T}: x \to x, p \to -p, i \to -i)$
- $\mathcal{PT}$ is an anti-linear operator:
  $$\mathcal{PT}(\lambda\Phi + \mu\Psi) = \lambda^* \mathcal{PT}\Phi + \mu^* \mathcal{PT}\Psi \qquad \lambda, \mu \in \mathbb{C}$$
- Real eigenvalues from unbroken $\mathcal{PT}$-symmetry:
  $$[\mathcal{H}, \mathcal{PT}] = 0 \quad \wedge \quad \mathcal{PT}\Phi = \Phi \qquad \Rightarrow \varepsilon = \varepsilon^* \ \text{ for } \mathcal{H}\Phi = \varepsilon\Phi$$
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PT-symmetry is only an example of an antilinear involution
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  $(\mathcal{P}: x \to -x, p \to -p; \ \mathcal{T}: x \to x, p \to -p, i \to -i)$
- $\mathcal{PT}$ is an anti-linear operator:

$$\mathcal{PT}(\lambda\Phi + \mu\Psi) = \lambda^*\mathcal{PT}\Phi + \mu^*\mathcal{PT}\Psi \qquad \lambda, \mu \in \mathbb{C}$$

- Real eigenvalues from unbroken $\mathcal{PT}$-symmetry:

$$[\mathcal{H}, \mathcal{PT}] = 0 \quad \wedge \quad \mathcal{PT}\Phi = \Phi \quad \Rightarrow \varepsilon = \varepsilon^* \text{ for } \mathcal{H}\Phi = \varepsilon\Phi$$

- *Proof*:
  $\varepsilon\Phi = \mathcal{H}\Phi = \mathcal{HPT}\Phi = \mathcal{PTH}\Phi = \mathcal{PT}\varepsilon\Phi = \varepsilon^*\mathcal{PT}\Phi = \varepsilon^*\Phi$

PT-symmetry is only an example of an antilinear involution
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Note, this Hamiltonian also results from deforming the roots:

$$\alpha_1 \;\rightarrow\; \tilde{\alpha}_1 = \alpha_1 \cosh\varepsilon \; + i\sqrt{3}\sinh\varepsilon\lambda_2$$
$$\alpha_2 \;\rightarrow\; \tilde{\alpha}_2 = \alpha_2 \cosh\varepsilon \; - i\sqrt{3}\sinh\varepsilon\lambda_1$$

Thus

$$\mathcal{H}_{\mathcal{PT}\text{CMS}} = \frac{p^2}{2} + \frac{m^2}{16}\sum_{\tilde{\alpha}\in\tilde{\Delta}_s}(\tilde{\alpha}\cdot q)^2 + \frac{1}{2}\sum_{\tilde{\alpha}\in\tilde{\Delta}} g_{\tilde{\alpha}} V(\tilde{\alpha}\cdot q), \;\; m, g_{\tilde{\alpha}} \in \mathbb{R}$$
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Symmetries:

$$\sigma_1^{\varepsilon} \;:\; \tilde{\alpha}_1 \leftrightarrow -\tilde{\alpha}_1, \tilde{\alpha}_2 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 \quad\Leftrightarrow\quad q_1 \leftrightarrow q_2, q_3 \leftrightarrow q_3, \imath \rightarrow -\imath$$
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Construction of antilinear deformations

- Involution $\in \mathcal{W} \equiv$ Coxeter group $\Rightarrow$ deform in antilinear way
- Find a linear deformation map:

$$\delta : \; \Delta \to \tilde{\Delta}(\varepsilon) \qquad \alpha \mapsto \tilde{\alpha} = \theta_\varepsilon \alpha$$

$$\alpha_i \in \Delta \subset \mathbb{R}^n, \quad \tilde{\alpha}_i(\varepsilon) \in \tilde{\Delta}(\varepsilon) \subset \mathbb{R}^n \oplus \imath\mathbb{R}^n, \quad \varepsilon \in \mathbb{R}$$

- Find a second map that leaves $\tilde{\Delta}(\varepsilon)$ invariant

$$\varpi : \tilde{\Delta}(\varepsilon) \to \tilde{\Delta}(\varepsilon), \qquad \tilde{\alpha} \mapsto \omega\tilde{\alpha}$$

(i) $\varpi : \tilde{\alpha} = \mu_1\alpha_1 + \mu_2\alpha_2 \mapsto \mu_1^*\omega\alpha_1 + \mu_2^*\omega\alpha_2$ for $\mu_1, \mu_2 \in \mathbb{C}$
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Make the following assumptions

(i) ω decomposes as

$$\omega = \tau\hat{\omega} = \hat{\omega}\tau$$

with $\hat{\omega} \in \mathcal{W}$, $\hat{\omega}^2 = \mathbb{I}$ and complex conjugation τ

(ii) there are at least two different ω_i with $i = 1, 2, \ldots$

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$$\omega_i := \theta_\varepsilon \hat{\omega}_i \theta_\varepsilon^{-1} = \tau\hat{\omega}_i, \qquad \text{for } i = 1, \ldots, \kappa \geq 2$$

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$$\theta_\varepsilon^* = \theta_\varepsilon^{-1} \qquad \text{and} \qquad \det \theta_\varepsilon = \pm 1$$

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Many solutions were constructed

$\tilde{\Delta}(\varepsilon)$ for A_3

$$\theta_\varepsilon = r_0\mathbb{I} + r_2\sigma^2 + \imath r_1\left(\sigma - \sigma^3\right)$$

with explicit representation

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \sigma = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix},$$

$\sigma_- = \sigma_1\sigma_3,\ \sigma_+ = \sigma_2,\ \sigma = \sigma_-\sigma_+$

$$\theta_\varepsilon = \begin{pmatrix} r_0 - \imath r_1 & -2\imath r_1 & -\imath r_1 - r_2 \\ 2\imath r_1 & r_0 - r_2 + 2\imath r_1 & 2\imath r_1 \\ -\imath r_1 - r_2 & -2\imath r_1 & r_0 - \imath r_1 \end{pmatrix}$$

all constraints require

$$\begin{aligned}(r_0 + r_2)\left[(r_0 + r_2)^2 - 4r_1^2\right] &= 1 \\ r_0 - r_2 + 2r_1 &= (r_0 - r_2 + 2r_1)(r_0 + r_2) \\ (r_0 + r_2) &= (r_0 - r_2)^2 - 4r_1^2\end{aligned}$$

these are solved by

$$r_0(\varepsilon) = \cosh\varepsilon, \quad r_1(\varepsilon) = \pm\sqrt{\cosh^2\varepsilon - \cosh\varepsilon}, \quad r_2(\varepsilon) = 1 - \cosh\varepsilon$$

$\Rightarrow$ simple deformed roots

$$\tilde{\alpha}_1 = \cosh\varepsilon\alpha_1 + (\cosh\varepsilon - 1)\alpha_3 - \imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1 + 2\alpha_2 + \alpha_3),$$

$$\tilde{\alpha}_2 = (2\cosh\varepsilon - 1)\alpha_2 + 2\imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1 + \alpha_2 + \alpha_3),$$

$$\tilde{\alpha}_3 = \cosh\varepsilon\alpha_3 + (\cosh\varepsilon - 1)\alpha_1 - \imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1 + 2\alpha_2 + \alpha_3).$$

remaining positive roots

$\tilde{\alpha}_4 := \tilde{\alpha}_1 + \tilde{\alpha}_2$, $\tilde{\alpha}_5 := \tilde{\alpha}_2 + \tilde{\alpha}_3$, $\tilde{\alpha}_6 := \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$.

$\tilde{\Delta}(\varepsilon)$ for A_{4n-1}-subseries

closed solution

$$\theta_\varepsilon = r_0 \mathbb{I} + r_{2n}\sigma^{2n} + \imath r_n \left(\sigma^n - \sigma^{-n}\right),$$

- with $r_{2n} = 1 - r_0$, $r_n = \pm\sqrt{r_0^2 - r_0}$
- useful choice $r_0 = \cosh\varepsilon$

$\tilde{\Delta}(\varepsilon)$ for E_6

$$\theta_\varepsilon = \begin{pmatrix} r_0 & -2\imath r_2 & 0 & -2\imath r_2 & -2\imath r_2 & -\imath r_2 \\ 2\imath r_2 & r_0 + \imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 \\ 0 & 2\imath r_2 & r_0 + 2\imath r_2 & 4\imath r_2 & 3\imath r_2 & 2\imath r_2 \\ -2\imath r_2 & -2\imath r_2 & -4\imath r_2 & r_0 - 5\imath r_2 & -4\imath r_2 & -2\imath r_2 \\ 2\imath r_2 & 2\imath r_2 & 3\imath r_2 & 4\imath r_2 & r_0 + 2\imath r_2 & 0 \\ -\imath r_2 & -2\imath r_2 & -2\imath r_2 & -2\imath r_2 & 0 & r_0 \end{pmatrix}$$

$r_2 = \pm 1/\sqrt{3}\sqrt{r_0^2 - 1}$, $r_0 = \cosh\varepsilon$

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no solution based on factorisation of the Coxeter element

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$\tilde{\Delta}(\varepsilon)$ for E_6

$$\theta_\varepsilon = \begin{pmatrix} r_0 & -2\imath r_2 & 0 & -2\imath r_2 & -2\imath r_2 & -\imath r_2 \\ 2\imath r_2 & r_0 + \imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 \\ 0 & 2\imath r_2 & r_0 + 2\imath r_2 & 4\imath r_2 & 3\imath r_2 & 2\imath r_2 \\ -2\imath r_2 & -2\imath r_2 & -4\imath r_2 & r_0 - 5\imath r_2 & -4\imath r_2 & -2\imath r_2 \\ 2\imath r_2 & 2\imath r_2 & 3\imath r_2 & 4\imath r_2 & r_0 + 2\imath r_2 & 0 \\ -\imath r_2 & -2\imath r_2 & -2\imath r_2 & -2\imath r_2 & 0 & r_0 \end{pmatrix}$$

$r_2 = \pm 1/\sqrt{3}\sqrt{r_0^2 - 1}$, $r_0 = \cosh\varepsilon$

$\tilde{\Delta}(\varepsilon)$ for B_{2n+1}-subseries

no solution based on factorisation of the Coxeter element

with different ω_i we find for instance for B_{2n+1}

$$\begin{aligned}
\tilde{\alpha}_{2j-1} &= \cosh\varepsilon\alpha_{2j-1} + i\sinh\varepsilon\left(\alpha_{2j-1} + 2\sum_{k=2j}^{\ell}\alpha_k\right) \quad \text{for } j = 1,\ldots, \\
\tilde{\alpha}_{2j} &= \cosh\varepsilon\alpha_{2j} - i\sinh\varepsilon\left(\sum_{k=2j}^{2j+2}\alpha_k + 2\sum_{k=2j+3}^{\ell}2\alpha_k\right) \quad \text{for } j = 1,\ldots \\
\tilde{\alpha}_{\ell-1} &= \cosh\varepsilon(\alpha_{\ell-1}+\alpha_\ell) - \alpha_\ell - i\sinh\varepsilon\left(\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell\right), \\
\tilde{\alpha}_\ell &= \alpha_\ell.
\end{aligned}$$

in dual space

$$\theta_\varepsilon^\star = \begin{pmatrix} R & & & & \\ & R & & 0 & \\ & & R & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad \text{with } R = \begin{pmatrix} \cosh\varepsilon & i\sinh\varepsilon \\ -i\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}$$

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$$

For **any** model based on roots, these deformed roots can be used to define new invariant models simply by

$$\alpha \longrightarrow \tilde{\alpha}.$$

For instance Calogero models:

- Physical properties (A_2, G_2)
  - The deformed model can be solved by separation of variables as the undeformed case.
  - Some restrictions cease to exist, as the wavefunctions are now regularized.
  - $\Rightarrow$ modified energy spectrum:

$$E = 2\left|\omega\right|\left(2n + \lambda + 1\right)$$

becomes

$$E^{\pm}_{n\ell} = 2|\omega|\left[2n + 6(\kappa^{\pm}_{s} + \kappa^{\pm}_{l} + \ell) + 1\right] \qquad \text{for } n, \ell \in \mathbb{N}_0,$$

with $\kappa^{\pm}_{s/l} = (1 \pm \sqrt{1 + 4g_{s/l}})/4$

[A. Fring and M. Znojil, J. Phys. A41 (2008) 194010]

- generalized Calogero Hamiltionian (undeformed)

$$\mathcal{H}_C(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2 + \sum_{\alpha\in\Delta^+}\frac{g_\alpha}{(\alpha\cdot q)^2},$$

- define the variables

$$z := \prod_{\alpha\in\Delta^+}(\alpha\cdot q) \qquad \text{and} \qquad r^2 := \frac{1}{\hat{h}t_\ell}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2,$$

$\hat{h} \equiv$ dual Coxeter number, $t_\ell \equiv \ell$-th symmetrizer of I

- Ansatz:

$$\psi(q) \to \psi(z,r) = z^{\kappa+1/2}\varphi(r)$$

$\Rightarrow$ solution for $\kappa = 1/2\sqrt{1+4g}$.

$$\varphi_n(r) = c_n \exp\left(-\sqrt{\frac{\hat{h}t_\ell}{2}}\frac{\omega}{2}r^2\right) L_n^a\left(\sqrt{\frac{\hat{h}t_\ell}{2}}\omega r^2\right).$$

$L_n^a(x) \equiv$ Laguerre polynomial, $a = \left(2+h+h\sqrt{1+4g}\right)l/4 - 1$

The generic case

- generalized Calogero Hamiltonian (undeformed)

$$\mathcal{H}_C(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2 + \sum_{\alpha\in\Delta^+}\frac{g_\alpha}{(\alpha\cdot q)^2},$$

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- eigenenergies

$$E_n = \frac{1}{4}\left[\left(2 + h + h\sqrt{1+4g}\right) l + 8n\right]\sqrt{\frac{\hat{h}t_\ell}{2}}\omega$$

- anyonic exchange factors

$$\psi(q_1, \ldots, q_i, q_j, \ldots q_n) = e^{\imath\pi s}\psi(q_1, \ldots, q_j, q_i, \ldots q_n), \quad \text{for } 1 \leq i, j \leq n,$$

with

$$s = \frac{1}{2} + \frac{1}{2}\sqrt{1+4g}$$

$\because$ r is symmetric and z antisymmetric

The construction is based on the identities:

$$\sum_{\alpha,\beta\in\Delta^+} \frac{\alpha\cdot\beta}{(\alpha\cdot q)(\beta\cdot q)} = \sum_{\alpha\in\Delta^+} \frac{\alpha^2}{(\alpha\cdot q)^2},$$

$$\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\frac{(\alpha\cdot q)}{(\beta\cdot q)} = \frac{\hat{h}h\ell}{2}t_\ell,$$

$$\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\,(\alpha\cdot q)(\beta\cdot q) = \hat{h}t_\ell \sum_{\alpha\in\Delta^+} (\alpha\cdot q)^2,$$

$$\sum_{\alpha\in\Delta^+} \alpha^2 = \ell\hat{h}t_\ell.$$

Strong evidence on a case-by-case level, but no rigorous proof.

- antilinearly deformed Calogero Hamiltionian

$$\mathcal{H}_{adC}(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q)^2 + \sum_{\tilde{\alpha}\in\Delta^+} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha}\cdot q)^2}$$

- define the variables

$$\tilde{z} := \prod_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q) \qquad \text{and} \qquad \tilde{r}^2 := \frac{1}{\hat{h}t_\ell} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q)^2$$

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$$\psi(q) \to \psi(\tilde{z},\tilde{r}) = \tilde{z}^s \varphi(\tilde{r})$$

when identies still hold ⇒

$$\psi(q) = \psi(\tilde{z},r) = \tilde{z}^s \varphi_n(r)$$

eigenenergies with different constraints (only performed for ground state)

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Deformed A_3-models

- potential from deformed Coxeter group factors

$\alpha_1 = \{1, -1, 0, 0\}$, $\alpha_2 = \{0, 1, -1, 0\}$, $\alpha_3 = \{0, 0, 1, -1\}$

$$
\begin{aligned}
\tilde{\alpha}_1 \cdot q &= q_{43} + \cosh\varepsilon(q_{12} + q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13} + q_{24}) \\
\tilde{\alpha}_2 \cdot q &= q_{23}(2\cosh\varepsilon - 1) + \imath 2\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2} q_{14} \\
\tilde{\alpha}_3 \cdot q &= q_{21} + \cosh\varepsilon(q_{12} + q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13} + q_{24}) \\
\tilde{\alpha}_4 \cdot q &= q_{42} + \cosh\varepsilon(q_{13} + q_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{12} + q_{34}) \\
\tilde{\alpha}_5 \cdot q &= q_{31} + \cosh\varepsilon(q_{13} + q_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{12} + q_{34}) \\
\tilde{\alpha}_6 \cdot q &= q_{14}(2\cosh\varepsilon - 1) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2} q_{23}
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notation $q_{ij} = q_i - q_j$, **No longer singular for $q_{ij} = 0$**

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- $\mathcal{PT}$-symmetry for $\tilde{\alpha}$

$$\sigma_-^\varepsilon : \tilde{\alpha}_1 \to -\tilde{\alpha}_1, \tilde{\alpha}_2 \to \tilde{\alpha}_6, \tilde{\alpha}_3 \to -\tilde{\alpha}_3, \tilde{\alpha}_4 \to \tilde{\alpha}_5, \tilde{\alpha}_5 \to \tilde{\alpha}_4, \tilde{\alpha}_6 \to \tilde{\alpha}_2$$
$$\sigma_+^\varepsilon : \tilde{\alpha}_1 \to \tilde{\alpha}_4, \tilde{\alpha}_2 \to -\tilde{\alpha}_2, \tilde{\alpha}_3 \to \tilde{\alpha}_5, \tilde{\alpha}_4 \to \tilde{\alpha}_1, \tilde{\alpha}_5 \to \tilde{\alpha}_3, \tilde{\alpha}_6 \to \tilde{\alpha}_6$$

- $\mathcal{PT}$-symmetry in dual space

$$\sigma_-^\varepsilon : q_1 \to q_2, q_2 \to q_1, q_3 \to q_4, q_4 \to q_3, \imath \to -\imath$$
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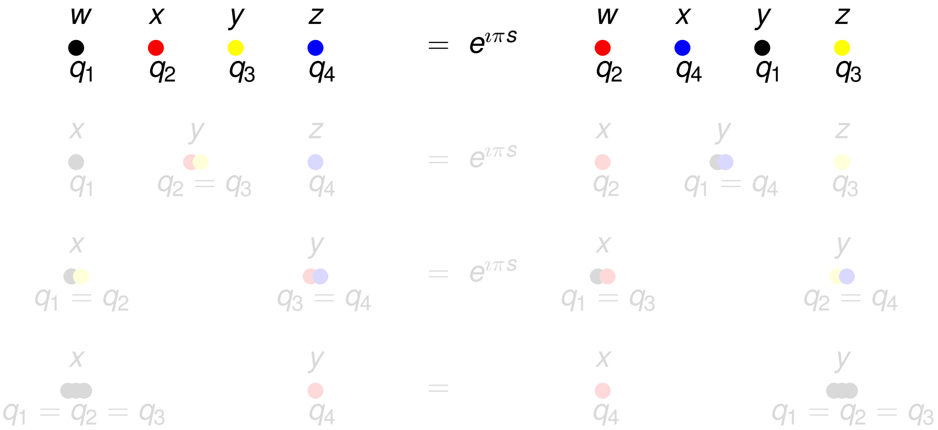
Anyonic exchange factors in the 4-particle scattering process



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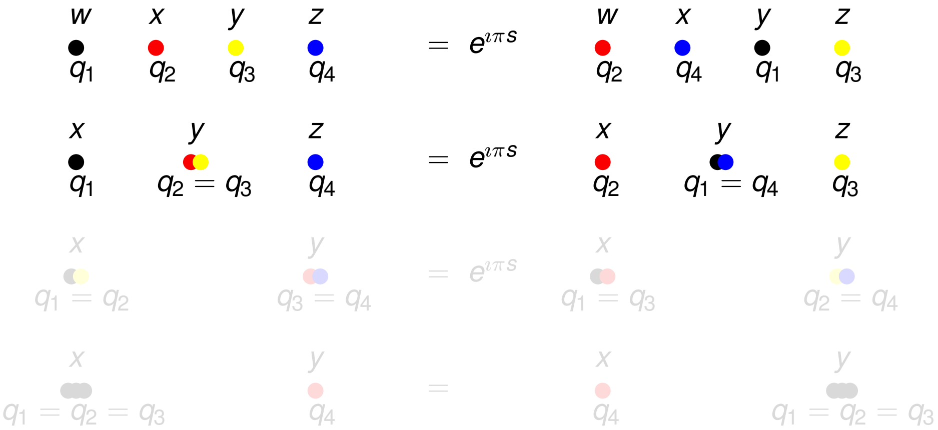

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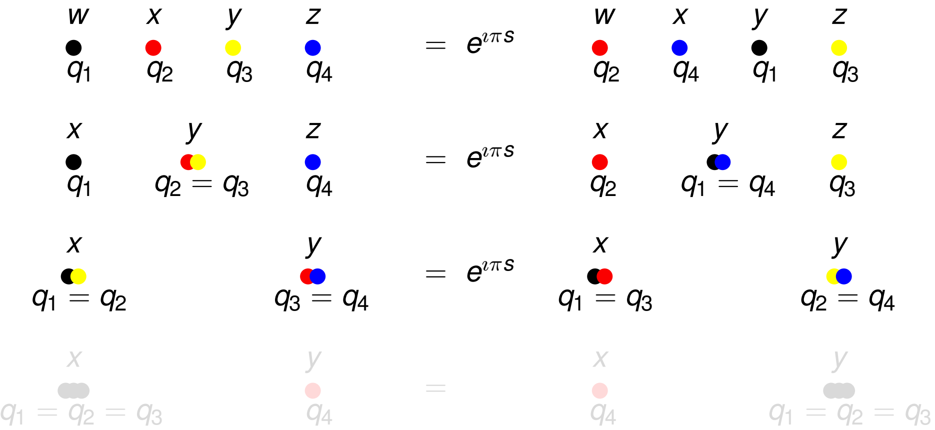

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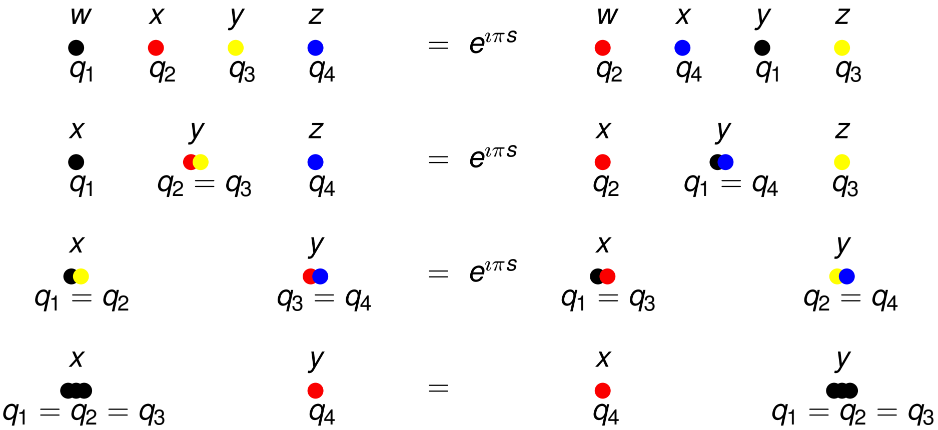

Find Hermitian counterpart h, Dyson map η and metric ρ:

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger \rho = \rho H \;\text{ with } \rho = \eta^\dagger \eta$$

Some B_ℓ-models correspond to complex rotations

$$\begin{pmatrix} \tilde{z}_i \\ \tilde{z}_j \end{pmatrix} = R_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} = \eta_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} \eta_{ij}^{-1}, \qquad \text{for } z \in \{x, p\},\ \eta_{ij} = e^{\varepsilon(x_i p_j - x_j p_i)}$$

For instance for:

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we have

$$\mathcal{H}_0(p, x) = \eta \mathcal{H}_\varepsilon(p, x) \eta^{-1}$$

with

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with

$$\eta = \eta_{12}^{-1} \eta_{34}^{-1} \eta_{56}^{-1} \cdots \eta_{(\ell-2)(\ell-1)}^{-1}$$

Find Hermitian counterpart h, Dyson map η and metric ρ:

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger \rho = \rho H \text{ with } \rho = \eta^\dagger \eta$$

Some B_ℓ-models correspond to complex rotations

$$\begin{pmatrix} \tilde{z}_i \\ \tilde{z}_j \end{pmatrix} = R_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} = \eta_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} \eta_{ij}^{-1}, \qquad \text{for } z \in \{x, p\},\ \eta_{ij} = e^{\varepsilon(x_i p_j - x_j p_i)}$$

For instance for:

$$\theta_\varepsilon^\star = \begin{pmatrix} R & & & & \\ & R & & 0 & \\ & & R & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad \text{with } R = \begin{pmatrix} \cosh\varepsilon & i\sinh\varepsilon \\ -i\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}$$

we have

$$\mathcal{H}_0(p, x) = \eta \mathcal{H}_\varepsilon(p, x) \eta^{-1}$$

with

$$\eta = \eta_{12}^{-1} \eta_{34}^{-1} \eta_{56}^{-1} \cdots \eta_{(\ell-2)(\ell-1)}^{-1}$$

For B_5

$$\theta_\varepsilon^\star = \begin{pmatrix} r_0 & -i\vartheta & i\vartheta & 1-r_0 & 0 \\ i\vartheta & r_0 & 1-r_0 & -i\vartheta & 0 \\ -i\vartheta & 1-r_0 & r_0 & i\vartheta & 0 \\ 1-r_0 & i\vartheta & -i\vartheta & r_0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

we find

$$\tilde{x} = \theta_\varepsilon^\star x = R_{24}^{-1} R_{13} R_{34} R_{12}^{-1} x = \eta x \eta^{-1}, \qquad \text{with } \eta = \eta_{24}^{-1} \eta_{13} \eta_{34} \eta_{12}^{-1}.$$

In general this is an open problem.

Conclusions

Deformed CMS models have interesting new properties

- less singular $\Rightarrow$ new energy spectral
- configuration space is not separated $\Rightarrow$ exchange factors

Open problems

- construction based on different assumptions
- solve generic case
- proof of identities involved
- generic h, Dyson map η and metric ρ
- different types of models, e.g. Toda [A.F.,M. Smith arXiv:1108.1719]

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Thank you for your attention