On integrable quantum spin chains with open boundaries

London, April 2005

Anastasia Doikou
LAPTH, Annecy-Le-Vieux

A.D., P.P. Martin, J. Phys. A36 (2003) 2203
A.D., math-ph/0402067, math-ph/0409060.

Relations/Applications

• High energy physics: HE QCD (Lipatov, Faddeev, Korchemsky...), super YMT (Minahan–Zarembo)

• String theory via CFT, and D–branes via BCFT (Polchinski, Schomerus...)

• Condensed matter, e.g. Kondo effect, quantum Hall effect,...(Affleck, Korepin, Saleur, Tsvelik, Wiegmann...)

• Mathematical aspects: quantum groups, braids, Lie and Hecke (TL) algebras, Virasoro algebras, ODEs,...(Drinfeld, Faddeev, Jimbo, Kulish, Sklyanin, Reshetikhin, Takhtajan...)

Aims

• Solving algebraically the YBE and RE ($U_q(\widehat{gl_n})$). *(Affine) Hecke* algebra realizations → solutions of the RE.

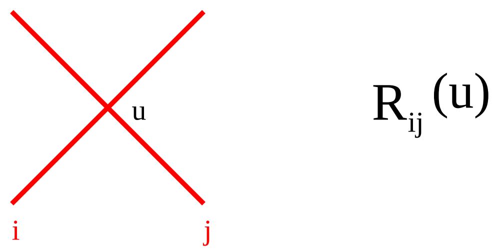
• Solutions of the RE build the integrable open spin chain → boundary non–local charges.

• Ultimate goal: Show the exact symmetry, remnant of $U_q(gl_n)$ (*study spectrum degeneracies*).

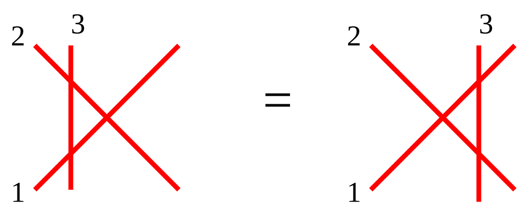
• *Important*: Here starting point (affine) Hecke algebra. Usually intertwining relations used to derive solutions of the YBE (Jimbo, Kulish, Reshetikhin,...) and RE (Delius, Mackay, Nepomechie).

The R matrix

The R matrix acts on $\mathbb{V}^{\otimes 2}$:



Satisfies the YBE (Baxter '72)



$$R_{12}(\lambda_1 - \lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

- Physical interpretation of R : scattering among excitations
- YBE factorization condition of multiparticle scattering

R matrices from the Hecke algebra

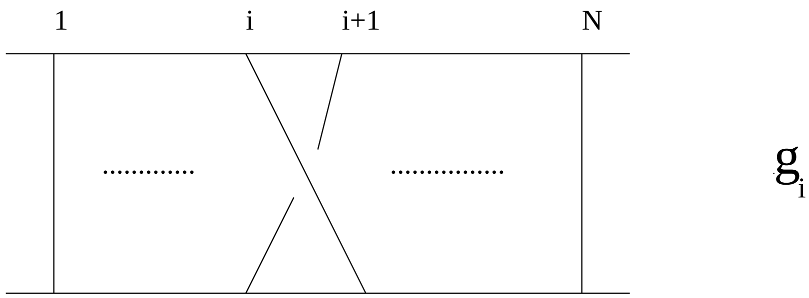
Let $\check{R}_{12}(\lambda) = \mathcal{P}_{12}\ R_{12}(\lambda), \quad \mathcal{P}(x \otimes y) = y \otimes x$, then YBE:

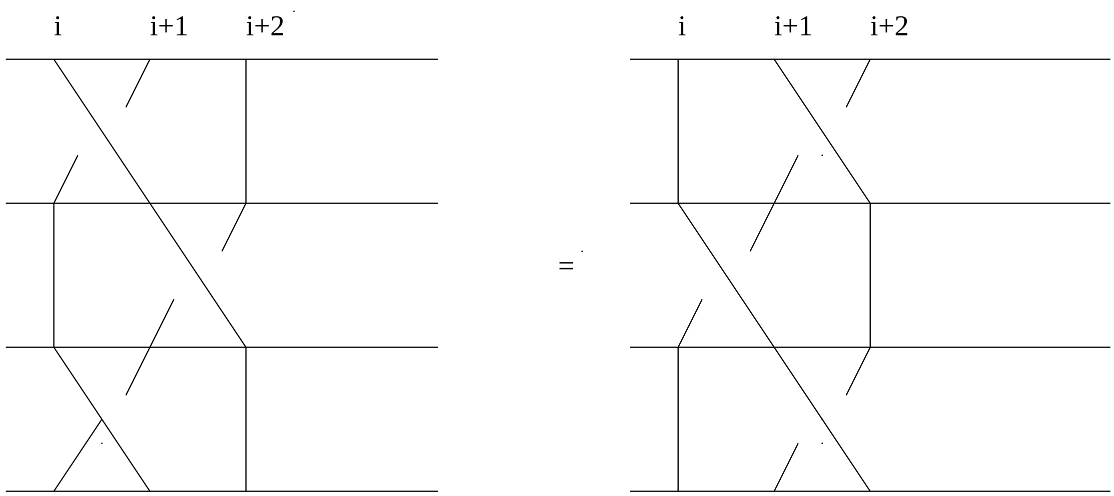
$$\check{R}_{12}(\lambda_1 - \lambda_2)\ \check{R}_{23}(\lambda_1)\ \check{R}_{12}(\lambda_2) = \check{R}_{23}(\lambda_2)\ \check{R}_{12}(\lambda_1)\ \check{R}_{23}(\lambda_1 - \lambda_2).$$

The Hecke algebra $\mathcal{H}_N(q)$ with g_i, $i = 1, \ldots, N-1$:

$$\begin{aligned}
&(g_i - q)\ (g_i + q^{-1}) = 0 \\
&g_i\ g_{i+1}\ g_i = g_{i+1}\ g_i\ g_{i+1}, \\
&[g_i,\ g_j] = 0, \quad |i - j| > 1.
\end{aligned}$$

The structural similarity between the *braid* relation and YBE:
$\mathcal{H}_N \longrightarrow$ candidate solutions of YBE.


1
i
i+1
N
....
....
g_i


i
i+1
i+2
=
i
i+1
i+2

Let $\mathcal{U}_i = g_i - q$ then reps of $\mathcal{H}_N \longrightarrow$ solution YBE (Jimbo '86)

$$\check{R}_{i\ i+1}(\lambda) = \sinh(\lambda + i\mu)\ \mathbb{I} + \sinh\lambda\ \rho(\mathcal{U}_i).$$

In particular, let $(e_{ij})_{kl} = \delta_{ik}\ \delta_{jl}$, and U on $(\mathbb{C}^n)^{\otimes 2}$:

$$U = \sum_{i \neq j} (e_{ij} \otimes e_{ji} - q^{-sgn(i-j)} e_{ii} \otimes e_{jj}).$$

R matrix in the fundamental representation of the $U_q(gl_n)$.

$U_q(sl_2)$: The XXZ model

U on $(\mathbb{C}^2)^{\otimes 2}$

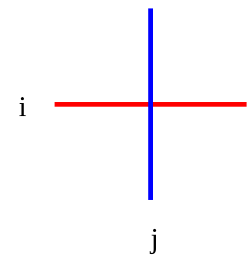
$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding R (spin $\frac{1}{2}$ rep of sl_2) matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$ (rep of Temperley–Lieb algebra)

$$R(\lambda) = \begin{pmatrix} \sinh \mu(\lambda + i) & 0 & 0 & 0 \\ 0 & \sinh \mu\lambda & e^{\mu\lambda} \sinh i\mu & 0 \\ 0 & e^{-\mu\lambda} \sinh i\mu & \sinh \mu\lambda & 0 \\ 0 & 0 & 0 & \sinh \mu(\lambda + i) \end{pmatrix}$$

The Lax operator

The $\mathcal{L}$ matrix acts on $\mathbb{V} \otimes \mathcal{A}$:



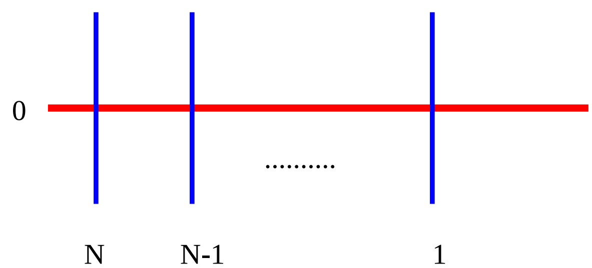
Satisfies the defining relation of $\mathcal{A} = U_q(\widehat{gl_n})$

$$R_{12}(\lambda_1 - \lambda_2)\ \mathcal{L}_{13}(\lambda_1)\ \mathcal{L}_{23}(\lambda_2) = \mathcal{L}_{23}(\lambda_2)\ \mathcal{L}_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

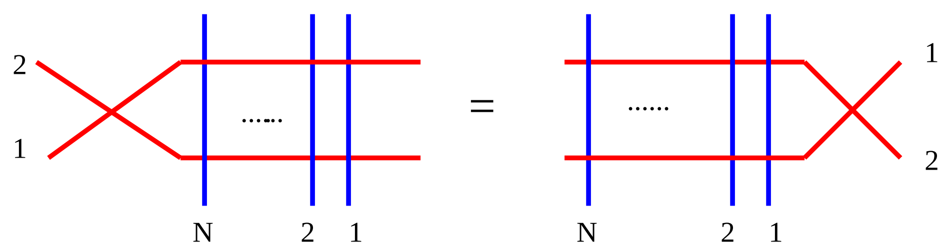
The periodic spin chain

The monodromy matrix $T \in \text{End}(\mathbb{V}) \otimes \mathcal{A}^{\otimes(N)}$ (QISM: Faddeev, Takhtajan '81):

$$T_0(\lambda) = \mathcal{L}_{0N}(\lambda)\ \mathcal{L}_{0\ N-1}(\lambda) \dots \mathcal{L}_{01}(\lambda)$$



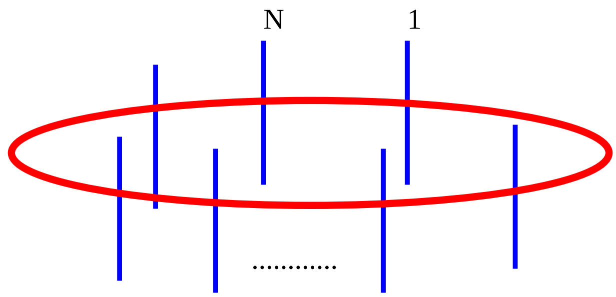
Satisfies the fundamental algebraic relation:



$$R_{12}(\lambda_1 - \lambda_2)\ T_1(\lambda_1)\ T_2(\lambda_2) = T_2(\lambda_2)\ T_1(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

The transfer matrix $t \in \mathcal{A}^{\otimes N}$ (Faddeev, Takhtajan '81):

$$t(\lambda) = Tr_0 \; T_0(\lambda)$$



Provides a family of commuting operators

$$\Big[t(\lambda), \; t(\lambda')\Big] = 0$$

Latter commutation relation ensures Integrability

$$\mathcal{H} \propto \frac{d}{d\lambda}(ln \; t(\lambda))|_{\lambda=0}$$

Quantum algebras

$$\mathcal{L}(\lambda) = e^{\mu\lambda}\mathcal{L}^{+} - e^{-\mu\lambda}\mathcal{L}^{-}$$

$\mathcal{L}^{\pm}_{ab} \in U_q(gl_n)$

As $\lambda \rightarrow \pm\infty$ $\mathcal{L}$ and consequently T reduce to upper, lower triangular matrices.

$$T(\lambda \rightarrow \pm\infty) \propto T^{\pm},$$

entries of $T^{\pm} \in U_q(gl_n)^{\otimes N}$

e.g. $U_q(sl_2)$

$$\mathcal{L}(\lambda) = \begin{pmatrix} \sinh \mu(\lambda + \frac{i}{2} + i s^z) & s^- e^{\mu\lambda} \sinh i\mu \\ s^+ e^{-\mu\lambda} \sinh i\mu & \sinh \mu(\lambda + \frac{i}{2} - i s^z) \end{pmatrix}$$

The asymptotics of the $\mathcal{L}$ matrix $\lambda \to \pm\infty$ (upper, lower triang), $(q = e^{i\mu})$

$$T^+ \propto \begin{pmatrix} q^{S^z} & c^+ S^- \\ 0 & q^{-S^z} \end{pmatrix}, \quad T^- \propto \begin{pmatrix} q^{-S^z} & 0 \\ c^- S^+ & q^{S^z} \end{pmatrix}$$

$$S^z = \sum_{k=1}^{N} \mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes s_k^z \otimes \mathbb{I} \ldots \otimes \mathbb{I},$$

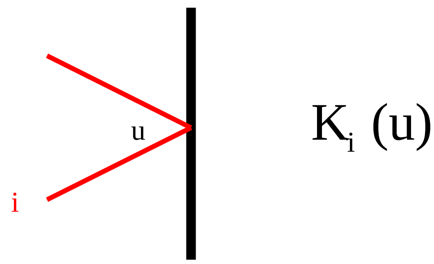
$$S^\pm = \sum_{k=1}^{N} q^{-s_1^z} \otimes \ldots \otimes q^{-s_{k-1}^z} \otimes s_k^\pm \otimes q^{s_{k+1}^z} \otimes \ldots \otimes q^{s_N^z}$$

S^z, $S^\pm$ tensor product realizations of $U_q(sl_2)$ (Jimbo '85)

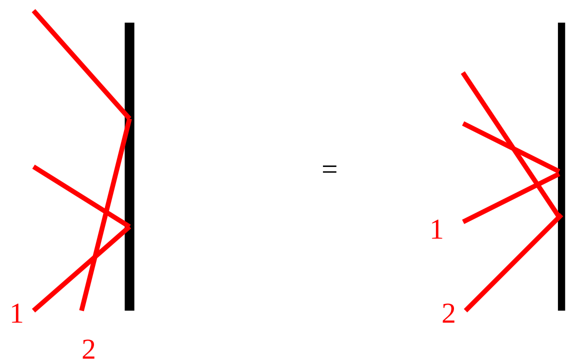
$$[S^+,\ S^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}}, \quad [S^z,\ S^\pm] = \pm S^\pm$$

Open boundaries

The K matrix acts on $\mathbb{V}$:



Satisfies the reflection equation (Cherednik '84)



$$R_{12}(\lambda_1 - \lambda_2)\ K_1(\lambda_1)\ R_{21}(\lambda_1 + \lambda_2)\ K_2(\lambda_2)$$
$$= K_2(\lambda_2)\ R_{12}(\lambda_1 + \lambda_2)\ K_1(\lambda_1)\ R_{21}(\lambda_1 - \lambda_2)$$

• Solutions of RE (e.g. via Hecke algebras: Levy and Martin '94, Doikou and Martin '02, Doikou '04) $\longrightarrow$ build open spin chains (Sklyanin '88)

K matrices from the affine Hecke algebra

The RE equation reads (Cherednik '84):

$$\check{R}_{12}(\lambda_1 - \lambda_2)\ K_1(\lambda_1)\ \check{R}_{12}(\lambda_1 + \lambda_2)\ K_1(\lambda_2)$$
$$= K_1(\lambda_2)\ \check{R}_{12}(\lambda_1 + \lambda_2)\ K_1(\lambda_1)\ \check{R}_{12}(\lambda_1 - \lambda_2)$$

The algebra $\mathcal{H}_N^0(q,Q)$, g_i, $i \in \{1, \ldots, N-1\}$ *and* g_0*:*

$$g_1\ g_0\ g_1\ g_0 = g_0\ g_1\ g_0\ g_1$$
$$[g_0,\ g_i] = 0, \quad i > 1$$

$\mathcal{H}_N^0(q,Q) \rightarrow$ candidate solutions of RE (Levy and Martin '94).
Quotients of $\mathcal{H}_N^0$, B-type $\mathcal{B}_N(q,Q)$:

$$(g_0 - Q)(g_0 + Q^{-1}) = 0.$$

Let $\mathcal{U}_0 = g_0 - Q$ then reps of $\mathcal{B}_N(q, Q) \to$ solutions to RE

$$K(\lambda) = x(\lambda)\mathbb{I} + y(\lambda)\rho(\mathcal{U}_0).$$

$\mathbb{U} \to$ rep of $\mathcal{U}_0$ on $\mathbb{C}^n$ (Doikou '04)

$$\mathbb{U} = \frac{1}{2i\sinh i\mu}(-Q^{-1}\hat{e}_{11} - Q\hat{e}_{nn} + \hat{e}_{1n} + \hat{e}_{n1}).$$

(Blob case: Martin and Saleur '94). K matrix $\to$ (Abad and Rios '95) for the $U_q(\widehat{gl_n})$ case.

e.g. $U_q(\hat{s}l_2)$

$$\mathbb{U} = \begin{pmatrix} -Q^{-1} & 1 \\ 1 & -Q \end{pmatrix}$$

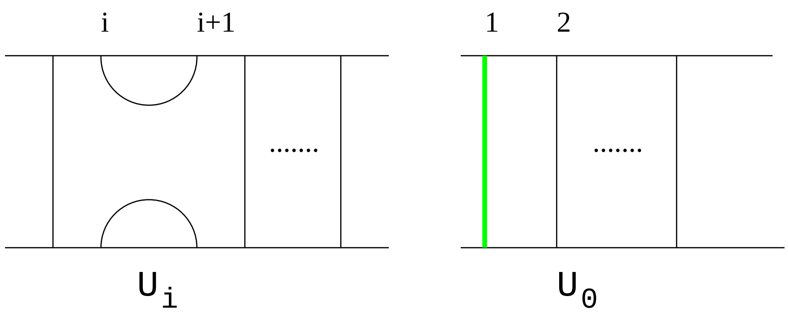
(rep of the blob algebra)

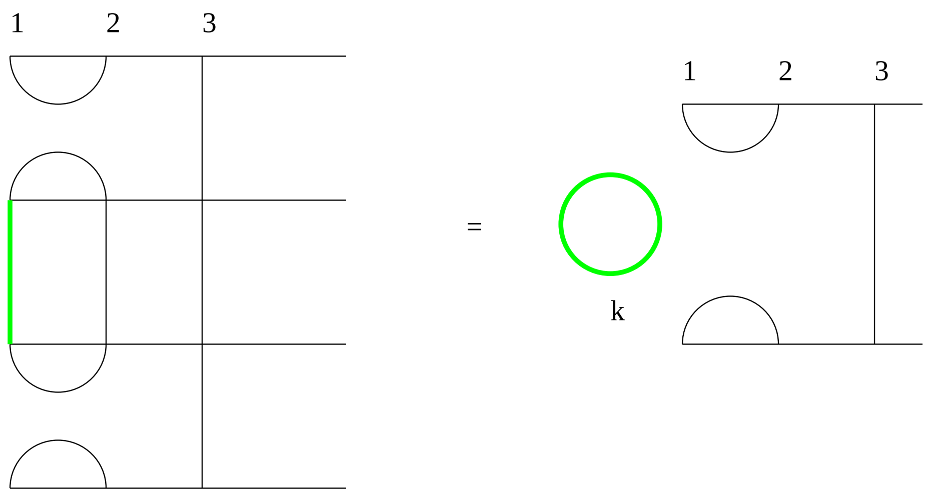
The blob algebra

Extension of the Temperley–Lieb algebra:

$$\mathcal{U}_{i\pm1}\,\mathcal{U}_i\,\mathcal{U}_{i\pm1} = \mathcal{U}_{i\pm1}$$

$$\mathcal{U}_1\,\mathcal{U}_0\,\mathcal{U}_1 = \kappa\,\mathcal{U}_1$$

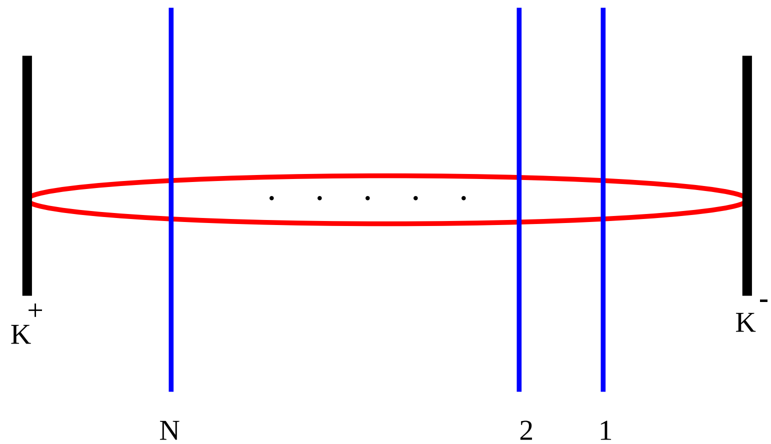




The open spin chain

Integrable boundary conditions (Sklyanin '88)

$$t(\lambda) = tr_0\ K_0^{(l)}(\lambda)\ \underbrace{T_0(\lambda)\ K_0^{(r)}(\lambda)\ T_0^{-1}(-\lambda)}_{\mathcal{T}_0(\lambda)}$$



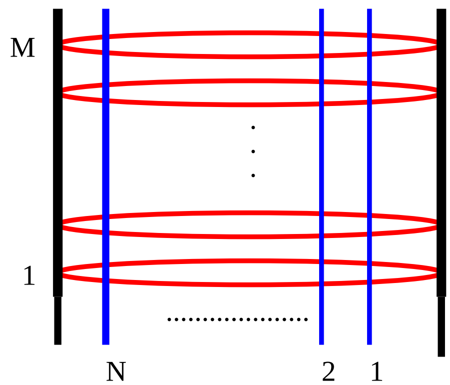
$$\Big[t(\lambda),\ t(\lambda')\Big] = 0$$

Integrability ensured.
Boundary S matrices (Doikou, Mezincescu and Nepomechie '97)
Boundary symmetries (Doikou '04).

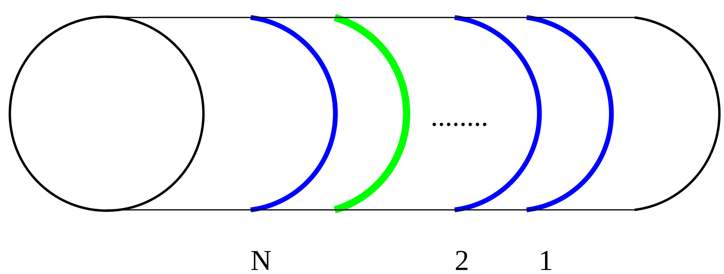
The partition function

The partition function for the open spin chain:

$$Z(\lambda) = Tr\ t(\lambda)^M$$



t^M



Z

The Hamiltonian

Open Hamiltonian: $\mathcal{H} \propto -\frac{d}{d\lambda} t(\lambda)|_{\lambda=0}$

$$\mathcal{H} \propto -\frac{1}{2} \sum_{i=1}^{N-1} U_i - \frac{\sinh i\mu \; y'(0)}{4\mu x(0)} \mathbb{U}_1 + c$$

For the spin $\frac{1}{2}$ XXZ in particular

$$\mathcal{H} = -\frac{1}{4} \sum_{j=1}^{N-1} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cosh i\mu \; \sigma_j^z \sigma_{j+1}^z \right)$$

$$+ c_x \sigma_1^x + c_y \sigma_1^y + c_z \sigma_1^z$$

The boundary quantum algebra

(Mezincescu and Nepomechie '97, Delius and Mackay '01, Doikou '04)
As $\lambda \to \infty$ T ($\hat{T}$) upper (lower) triangular ($K^{(l)} = \mathbb{I}$)

$$T(\lambda \to \infty) \propto T^{+}, \quad \hat{T}(\lambda \to \infty) \propto \hat{T}^{+}$$

entries of T^{+}, $\hat{T}^{+} \in U_q(gl_n)^{\otimes N}$:

$$K^{(r)}(\lambda \to \infty) \propto K^{+}$$

Then:

$$\mathcal{T}(\lambda \to \infty) \propto T^{+}\ K^{+}\ \hat{T}^{+} = \mathcal{T}^{+}$$

The entries $\mathcal{T}^{+}_{ab}$ the boundary non-local charges!

$$\Big[t(\lambda),\ \mathcal{T}^{+}_{ab}\Big] = 0.$$

Example $U_q(sl_2)$:The asymptotics of the $\mathcal{L}$, $\hat{\mathcal{L}}(\lambda) = \mathcal{L}^{-1}(-\lambda)$, $\lambda \to \pm\infty$ (upper, lower triang),

$$T^+ = \begin{pmatrix} q^{S^z} & cS^- \\ 0 & q^{-S^z} \end{pmatrix}, \qquad \hat{T}^+ = \begin{pmatrix} q^{S^z} & 0 \\ c^- S^+ & q^{-S^z} \end{pmatrix}$$

recall S^z, $S^\pm$ coproducts of $U_q(sl_2)$.

$$K^+ = \begin{pmatrix} -e^{-i\mu\xi} & \kappa \\ \kappa & 0 \end{pmatrix}$$

$$\mathcal{T}(\lambda \to \infty) \propto T^{+}\ K^{+}\ \hat{T}^{+} = \mathcal{T}^{+}$$

$\mathcal{T}^{+}_{ab}$ the boundary non-local charges:

$$\mathcal{T}^{+} = \begin{pmatrix} c\mathcal{T}^{+}_{11} & \kappa \\ \kappa & 0 \end{pmatrix}$$

$$\mathcal{T}^{+}_{11} = xq^{2S^z} + q^{-\frac{1}{2}}q^{S^z}S^{+} + q^{-\frac{1}{2}}q^{S^z}S^{-}$$

The symmetry of the open spin chain (Doikou '04)

$$\left[U_l,\ \mathcal{T}^{+}_{11}\right] = 0 \to \left[\mathcal{H},\ \mathcal{T}^{+}_{11}\right] = 0$$

$$\left[t(\lambda),\ \mathcal{T}^{+}_{11}\right] = 0.$$

Symmetries

- K non-diagonal: $[t(\lambda),\ \mathcal{T}_{ab}^{+}] = 0$

(Doikou '04)

- K diagonal: $[t(\lambda),\ U_q(gl_l) \otimes U_q(gl_{n-l}))] = 0$

(Doikou and Nepomechie '98)

- $K = \mathbb{I}$: $[t(\lambda),\ U_q(gl_n)] = 0$

(Kulish and Sklyanin '91, Mezincescu and Nepomechie '91)

Comments

• Other reps of $\mathcal{H}_N^0 \rightarrow$ solutions to RE,e.g. 'Twin' repr. (Martin and Woodcock '03, Doikou and Martin '02).

• Here SP b.c. Algebra for SNP b.c.? A Birman–Wenzl–Murakami algebra with gens correspond to boundaries.

• Symmetry for SNP b.c. 'deformed' twisted Yangian (rational case: *Twisted Yangian* (Arnaudon, Avan, Crampe, Doikou, Frappat, Ragoucy '04). Higher non-local charges (Baseilhac '04, Doikou '04).

• QFT with SP b.c. Known b.c. from FT are SNP (Bowcock, Corrigan, Dorey, Rietdijk '95). Non-local charges (Delius and Mackay '01) and K matrices were derived (Gandenberger '99).

• Extension to the dynamical YBE and RE (in progress with: J. Avan, Z. Nagy and G. Rollet).