Non-Hermitian gauge field theories and BPS limits

Andreas Fring

Solitons at Work, online seminar, 12th of May 2021

Based on:

A. Fring, T. Taira, Nucl. Phys. B, 950,(2020) 114834
A. Fring, T. Taira, Phys. Rev. D, 101 (2020) 045014
A. Fring, T. Taira, Phys. Lett. B, 807 (2020) 135583
A. Fring, T. Taira, J. Phys. A: Math. Theor., 53 (2020) 455701
A. Fring, T. Taira, arXiv:2004.00723
F. Correa, A. Fring, T. Taira, arXiv:2102.05781
A. Fring, T. Taira, arXiv:2103.13519, to appear special issue ed.
C. Bender, F. Correa, A. Fring: Journal of Physics: Conference Series, proceeding of online series https://vphhqp.com

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General motivation: shortcomings in the Standard Model

- theoretical:
  incomplete in many ways, at least 19 parameters,
  neutrino oscillations, dark matter/energy,...
- recent experiments:
  lepton universality (CERN), muon g-factor (Fermilab)

$\Rightarrow$ explore sectors in the Standard Model

Outline

- Short introduction to $\mathcal{PT}$-quantum mechanics

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- Spontaneous symmetry breaking in non-Hermitian field theory
  - Nambu-Goldstone bosons
  - Higgs mechanism

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- Conclusions

$\mathcal{PT}$-quantum mechanics (real eigenvalues)

- $\mathcal{PT}$-symmetry: $\quad \mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$

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$\mathcal{PT}$-symmetry is only an example of an antilinear involution
[E. Wigner, *J. Math. Phys.* 1 (1960) 409]
[C. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243]

$\mathcal{H}$ is Hermitian with respect to a new metric

- Assume pseudo-Hermiticity:

$$h = \eta \mathcal{H} \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger \mathcal{H}^\dagger \eta^\dagger \quad \Leftrightarrow \quad \mathcal{H}^\dagger \eta^\dagger \eta = \eta^\dagger \eta \mathcal{H}$$

$$\Phi = \eta^{-1} \phi \qquad \eta^\dagger = \eta$$

$\Rightarrow \mathcal{H}$ is Hermitian with respect to the new metric

Proof:

$$\langle \Psi | \mathcal{H} \Phi \rangle_\eta = \langle \Psi | \eta^2 \mathcal{H} \Phi \rangle = \langle \eta^{-1} \psi | \eta^2 \mathcal{H} \eta^{-1} \phi \rangle = \langle \psi | \eta \mathcal{H} \eta^{-1} \phi \rangle =$$
$$\langle \psi | h \phi \rangle = \langle h \psi | \phi \rangle = \langle \eta \mathcal{H} \eta^{-1} \psi | \phi \rangle = \langle \mathcal{H} \Psi | \eta \phi \rangle = \langle \mathcal{H} \Psi | \eta^2 \Phi \rangle$$
$$= \langle \mathcal{H} \Psi | \Phi \rangle_\eta$$

$\Rightarrow$ Eigenvalues of $\mathcal{H}$ are real, eigenstates are orthogonal

Problem with non-Hermitain field theory

Consider action of the general form

$$\mathcal{I} = \int d^4x \left[\partial_\mu \phi \partial^\mu \phi^* - V(\phi)\right],$$

complex scalar fields $\phi = (\phi_1, \ldots, \phi_n)$, potential $V(\phi) \neq V^\dagger(\phi)$

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$$\frac{\delta \mathcal{I}_n}{\delta \phi_i} = \frac{\partial \mathcal{L}_n}{\partial \phi_i} - \partial_\mu \left[\frac{\partial \mathcal{L}_n}{\partial(\partial_\mu \phi_i)}\right] = 0, \; \frac{\delta \mathcal{I}_n}{\delta \phi_i^*} = \frac{\partial \mathcal{L}_n}{\partial \phi_i^*} - \partial_\mu \left[\frac{\partial \mathcal{L}_n}{\partial(\partial_\mu \phi_i^*)}\right] = 0$$

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Resolutions:

- Keep surface terms
  [J. Alexandre, J. Ellis, P. Millington, D. Seynaeve]
- Seek similarity transformation
  [C. Bender, H. Jones, R. Rivers, P. Mannheim, ...
  A. Fring, T. Taira]

Goldstone theorem and Higgs mechanism

Key findings:

Goldstone theorem in non-Hermitian field theories

- The GT holds in the $\mathcal{PT}$-symmetric regime
- The GT breaks down in the broken $\mathcal{PT}$ regime
- At exceptional points the Goldstone boson can be identified
- At the zero EP the Goldstone boson can NOT be identified

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Non-Hermitian systems posses intricate physical parameter spaces

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Vacua Φ_0:

$$\left. \frac{\partial V(\Phi)}{\partial \Phi} \right|_{\Phi=\Phi_0} = 0$$

Symmetry $\Phi \to \Phi + \delta\Phi$: $V(\Phi) = V(\Phi) + \nabla V(\Phi)^T \delta\Phi$,

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Differentiating with respect to Φ_j at a vacuum Φ_0

$$\left. \frac{\partial^2 V(\Phi)}{\partial \Phi_j \partial \Phi_i} \right|_{\Phi = \Phi_0} \delta\Phi_i(\Phi_0) + \left. \frac{\partial V(\Phi)}{\partial \Phi_i} \right|_{\Phi = \Phi_0} \left. \frac{\partial \delta\Phi_i(\Phi)}{\partial \Phi_j} \right|_{\Phi = \Phi_0} = 0$$

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Non-Hermitian version:

$$\hat{\mathcal{I}} = \int d^4x \left[\frac{1}{2}\partial_\mu\Phi\hat{I}\partial^\mu\Phi^* - \hat{V}(\Phi)\right]$$

$$\hat{I}\hat{H}(\Phi_0)\delta\Phi_i(\Phi_0) = \hat{M}^2\delta\Phi_i(\Phi_0) = 0$$

$\hat{M}^2$ is no longer Hermitian

An Abelian model with three complex scalar fields

$$\mathcal{I}_3 = \int d^4x \sum\nolimits_{i=1}^{3} \partial_\mu \phi_i \partial^\mu \phi_i^* - V_3$$

$$V_3 = -\sum_{i=1}^{3} c_i m_i^2 \phi_i \phi_i^* + c_\mu \mu^2 \left(\phi_1^* \phi_2 - \phi_2^* \phi_1\right) + c_\nu \nu^2 \left(\phi_2 \phi_3^* - \phi_3 \phi_2^*\right) + \frac{g}{4} \left(\phi_1 \phi_1^*\right)^2$$

with $m_i, \mu, \nu, g \in \mathbb{R}$ and $c_i, c_\mu, c_\nu = \pm 1$

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Properties:

- discrete modified $\mathcal{CPT}$-transformations

$$\mathcal{CPT}_1 : \phi_i(x_\mu) \to (-1)^{i+1} \phi_i^*(-x_\mu)$$
$$\mathcal{CPT}_2 : \phi_i(x_\mu) \to (-1)^{i} \phi_i^*(-x_\mu), \qquad i = 1, 2, 3$$

- continuous global $U(1)$-symmetry

$$\phi_i \to e^{i\alpha} \phi_i, \qquad \phi_i^* \to e^{-i\alpha} \phi_i^*, \qquad i = 1, 2, 3,\ \alpha \in \mathbb{R}$$

- non-Hermitian potential $V_3 \neq V_3^\dagger$

(incompatible) equations of motion:

$$
\begin{aligned}
\Box\phi_1 - c_1 m_1^2\phi_1 - c_\mu\mu^2\phi_2 + \frac{g}{2}\phi_1^2\phi_1^* &= 0\\
\Box\phi_2 - c_2 m_2^2\phi_2 + c_\mu\mu^2\phi_1 + c_\nu\nu^2\phi_3 &= 0\\
\Box\phi_3 - c_3 m_3^2\phi_3 - c_\nu\nu^2\phi_2 &= 0\\
\Box\phi_1^* - c_1 m_1^2\phi_1^* + c_\mu\mu^2\phi_2^* + \frac{g}{2}\phi_1(\phi_1^*)^2 &= 0\\
\Box\phi_2^* - c_2 m_2^2\phi_2^* - c_\mu\mu^2\phi_1^* - c_\nu\nu^2\phi_3^* &= 0\\
\Box\phi_3^* - c_3 m_3^2\phi_3^* + c_\nu\nu^2\phi_2^* &= 0
\end{aligned}
$$

This can be fixed with a similarity transformation:

$$
\eta = \exp\left[\frac{\pi}{2}\int d^3x \Pi_2^\varphi(\mathrm{x},t)\varphi_2(\mathrm{x},t)\right]\exp\left[\frac{\pi}{2}\int d^3x \Pi_2^\chi(\mathrm{x},t)\chi_2(\mathrm{x},t)\right]
$$

$$
\eta\phi_i\eta^{-1} = (-i)^{\delta_{2i}}\phi_i, \qquad \eta\phi_i^*\eta^{-1} = (-i)^{\delta_{2i}}\phi_i^*
$$

Equivalent version ($\hat{\mathcal{I}}_3 = \eta \mathcal{I}_3 \eta^{-1}$) $\phi_i = 1/\sqrt{2}(\varphi_i + i\chi_i)$

$$\hat{\mathcal{I}}_3 = \int d^4x \sum_{i=1}^{3} \frac{1}{2}(-1)^{\delta_{2i}} \left[\partial_\mu \varphi_i \partial^\mu \varphi_i + \partial_\mu \chi_i \partial^\mu \chi_i + c_i m_i^2 \left(\varphi_i^2 + \chi_i^2\right)\right]$$
$$+ c_\mu \mu^2 \left(\varphi_1 \chi_2 - \varphi_2 \chi_1\right) + c_\nu \nu^2 \left(\varphi_3 \chi_2 - \varphi_2 \chi_3\right) - \frac{g}{16}(\varphi_1^2 + \chi_1^2)^2$$

(compatible) equations of motion:

$$\begin{aligned}
-\Box \varphi_1 &= -c_1 m_1^2 \varphi_1 - c_\mu \mu^2 \chi_2 + \frac{g}{4}\varphi_1(\varphi_1^2 + \chi_1^2) \\
-\Box \chi_2 &= -c_2 m_2^2 \chi_2 + c_\mu \mu^2 \varphi_1 + c_\nu \nu^2 \varphi_3 \\
-\Box \varphi_3 &= -c_3 m_3^2 \varphi_3 - c_\nu \nu^2 \chi_2 \\
-\Box \chi_1 &= -c_1 m_1^2 \chi_1 + c_\mu \mu^2 \varphi_2 + \frac{g}{4}\chi_1(\varphi_1^2 + \chi_1^2) \\
-\Box \varphi_2 &= -c_2 m_2^2 \varphi_2 - c_\mu \mu^2 \chi_1 - c_\nu \nu^2 \chi_3 \\
-\Box \chi_3 &= -c_3 m_3^2 \chi_3 + c_\nu \nu^2 \varphi_2
\end{aligned}$$

Hessian matrix $H\left(\Phi = (\varphi_1, \chi_2, \varphi_3, \chi_1, \varphi_2, \chi_3)^T\right)$:

$$
\begin{pmatrix}
\frac{g(3\varphi_1^2+\chi_1^2)}{4} - c_1 m_1^2 & -c_\mu \mu^2 & 0 & \frac{g}{2}\varphi_1\chi_1 & 0 & 0 \\
-c_\mu \mu^2 & c_2 m_2^2 & -c_\nu \nu^2 & 0 & 0 & 0 \\
0 & -c_\nu \nu^2 & -c_3 m_3^2 & 0 & 0 & 0 \\
\frac{g}{2}\varphi_1\chi_1 & 0 & 0 & \frac{g(\varphi_1^2+3\chi_1^2)}{4} - c_1 m_1^2 & c_\mu \mu^2 & 0 \\
0 & 0 & 0 & c_\mu \mu^2 & c_2 m_2^2 & c_\nu \nu^2 \\
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No Goldstone bosons for $U(1)$-invariant vacuum (no zero EV of M^2)

$$\Phi_s^0 = (0,0,0,0,0,0)$$

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No Goldstone bosons for $U(1)$-invariant vacuum (no zero EV of M^2)

$$\Phi_s^0 = (0,0,0,0,0,0)$$

One Goldstone bosons for $U(1)$-broken vacuum (one zero EV of M^2)

$$\begin{aligned} \Phi_b^0 \;\; = \;\; & \Bigg(\varphi_1^0, \frac{c_3 c_\mu m_3^2 \mu^2 \varphi_1^0}{\kappa}, -\frac{c_\nu c_\mu \nu^2 \mu^2 \varphi_1^0}{\kappa}, \\ & -K(\varphi_1^0), \frac{c_3 c_\mu m_3^2 \mu^2 K(\varphi_1^0)}{\kappa}, \frac{c_\nu c_\mu \nu^2 \mu^2 K(\varphi_1^0)}{\kappa} \Bigg) \end{aligned}$$

with $K(x) := \pm\sqrt{\frac{4c_3 m_3^2 \mu^4}{g\kappa} + \frac{4c_1 m_1^2}{g} - x^2}$, $\qquad \kappa := c_2 c_3 m_2^2 m_3^2 + \nu^4$

Identification of $\mathcal{C}, \mathcal{P}, \mathcal{T}$ $(H \equiv M^2)$

$\mathcal{PT}$-symmetric Hamiltonian:

$$[H, \mathcal{PT}] = H\mathcal{P} - \mathcal{P}H^* = 0 \ , \quad \mathcal{P}^T\mathcal{P} = 1$$

Bi-orthonormal basis: $\{v_n\}$, $\{u_n\}$

$$Hv_n = \epsilon_n v_n, \quad H^\dagger u_n = \epsilon u_n$$

$$\langle u_n | v_m \rangle = \delta_{nm}, \quad \sum_n |u_n\rangle \langle v_n| = \sum_n |v_n\rangle \langle u_n| = 1, \quad |u_n\rangle = s_n \mathcal{P} |v_n\rangle$$

$\mathcal{P}$ operator:

$$\mathcal{P} = \sum_n s_n |u_n\rangle \langle u_n| \ , \quad \mathcal{P}^T = \sum_n s_n |v_n\rangle \langle v_n| \ , \quad s_n = \pm 1$$

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Metric operator: $\rho := \eta^\dagger \eta$ with $\mathcal{C} = \rho^{-1}\mathcal{P}$

We find 8 different solutions:

$$\mathcal{P}' = \sum_{j=0,\pm} \frac{s_j}{N_j^2} \begin{pmatrix} \left(\Lambda_j^2\Lambda_j^3+\nu^4\right)^2 & i\mu^2\Lambda_j^3\left(\Lambda_j^2\Lambda_j^3+\nu^4\right) & \mu^2\nu^2\left(\Lambda_j^2\Lambda_j^3+\nu^4\right) \\ -i\mu^2\Lambda_j^3\left(\Lambda_j^2\Lambda_j^3+\nu^4\right) & \mu^4\left(\Lambda_j^3\right)^2 & -i\nu^2\mu^4\Lambda_j^3 \\ \mu^2\nu^2\left(\Lambda_j^2\Lambda_j^3+\nu^4\right) & i\nu^2\mu^4\Lambda_j^3 & \mu^4\nu^4 \end{pmatrix}$$

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The choices $\mathcal{P}(s_0 = \pm, s_- = \mp, s_+ = \pm)$ correspond to $\mathcal{CPT}_{1,2}$.

Note here η is a matrix multiplication whereas above it involves equal time commutators of canonical fields.

Non-Abelian models

$SU(N)$-symmetric model with n complex scalars:

$$\mathcal{L}_n^{SU(N)} = \sum_{i=1}^{n} \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + c_i m_i^2 \phi_i^\dagger \phi_i + \sum_{i=1}^{n-1} \kappa_i \mu_i^2 \left(\phi_i^\dagger \phi_{i+1} - \phi_{i+1}^\dagger \phi_i \right) - \frac{g_i}{4} \left(\phi_1^\dagger \phi_1 \right)^2$$

Properties:

$$SU(N) \quad : \quad \phi_j \to e^{i\alpha T^a} \phi_j$$

$$\mathcal{CPT}_{1/2} \quad : \quad \phi_i(x_\mu) \to \mp \phi_i^*(-x_\mu) \quad \text{for } \frac{i}{2} \in \mathbb{Z}$$

$$\phi_j(x_\mu) \to \pm \phi_j^*(-x_\mu) \quad \text{for } \frac{j+1}{2} \in \mathbb{Z}$$

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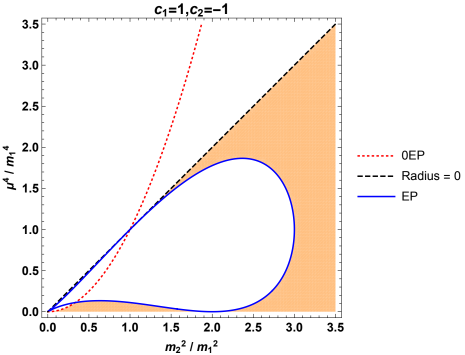
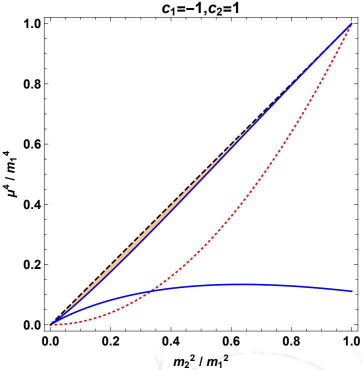
We discard models with ill-defined classical mass spectrum.

Physical regions in the parameter space for $\mathcal{L}_2^{SU(2)}$:

The choices $c_1 = c_2 = \pm 1$ are non-physical.





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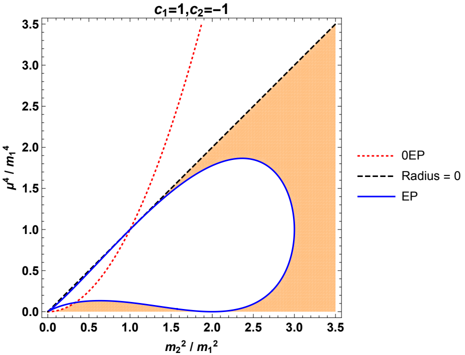

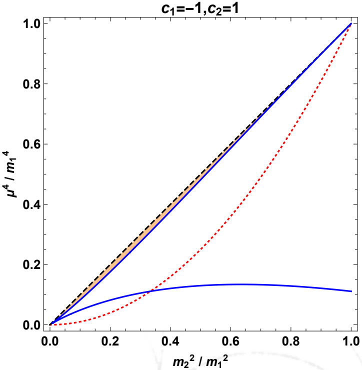

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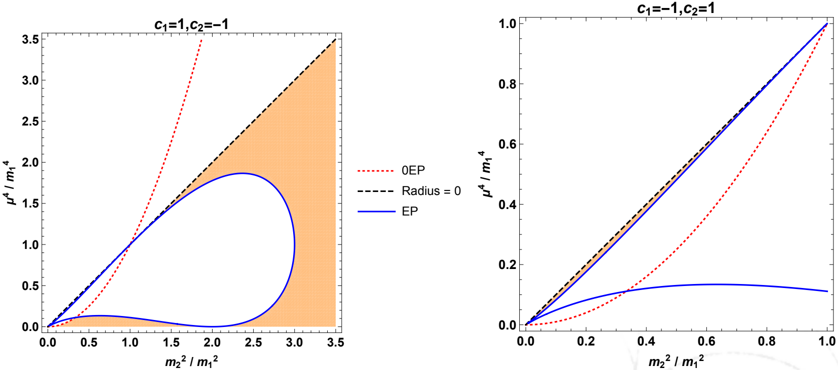

Physical region: expected # of Goldstone bosons

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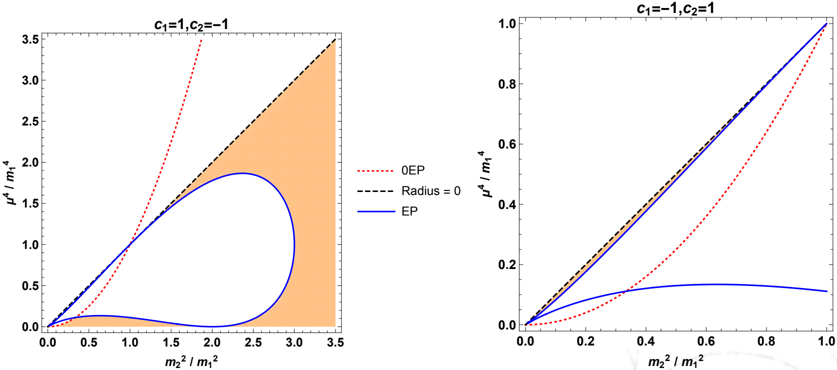

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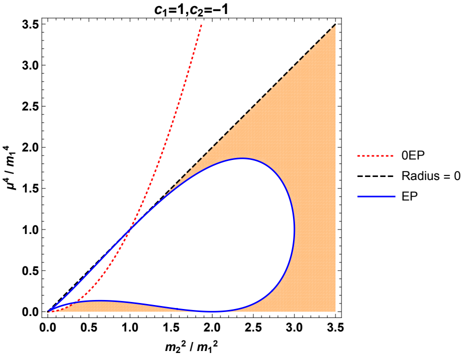

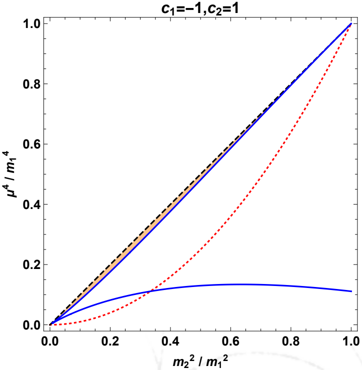

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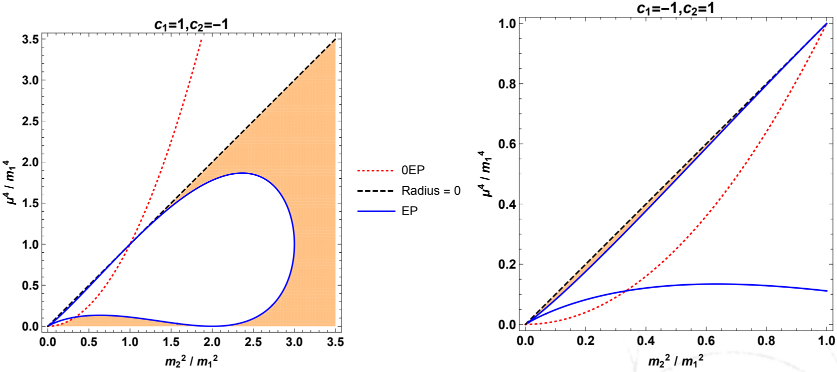

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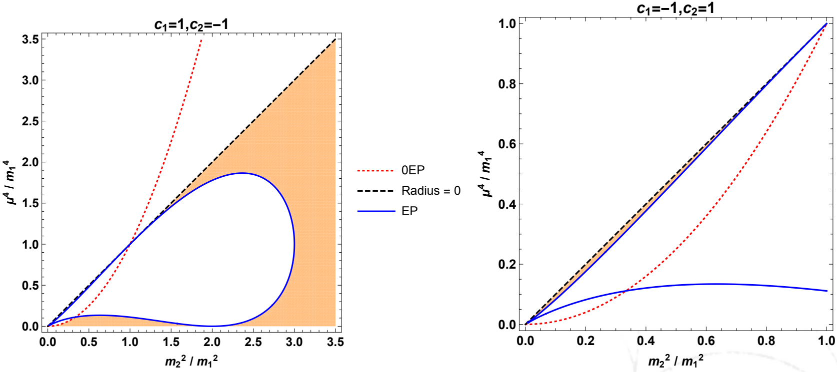

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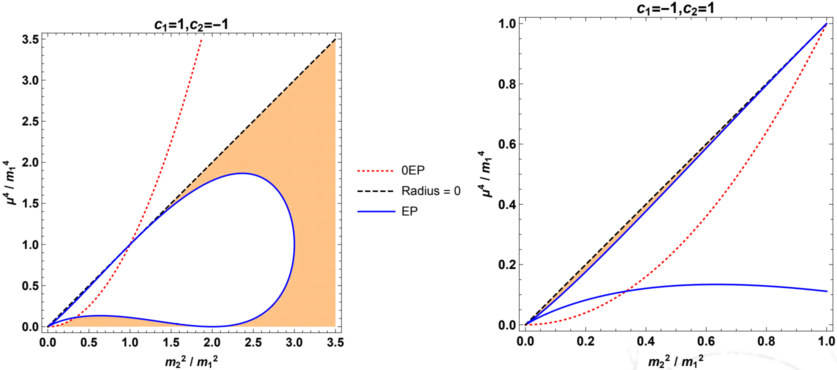

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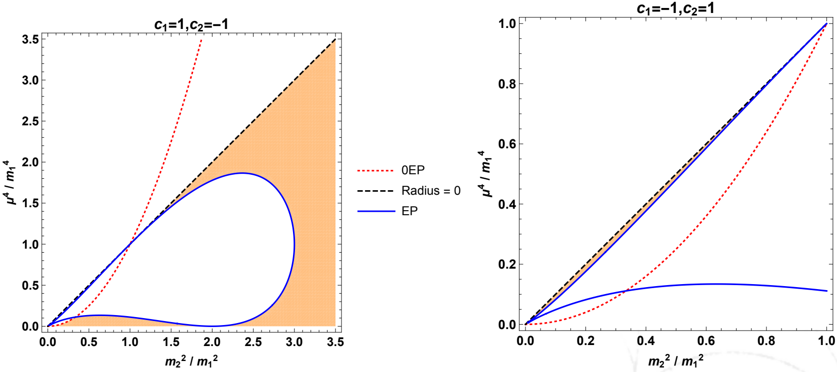

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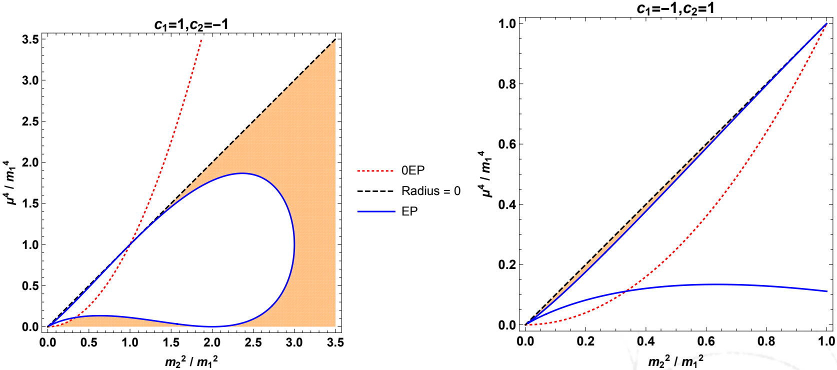

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Higgs mechanism

Global to local symmetry: $\phi_j \to e^{i\alpha T^a}\phi_j$ to $\phi_j \to e^{i\alpha T^a(x)}\phi_j$

$$\mathcal{L}_I = \sum_{i=1}^{2} |D_\mu \phi_i|^2 + m_i^2 |\phi_i|^2 - \mu^2 \left(\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1\right) - \frac{g}{4}\left(|\phi_1|^2\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

minimal coupling: $D_\mu = \partial_\mu - ieA_\mu$
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$$m_g = \frac{eR_f}{m_2^2}\sqrt{m_2^4 - \mu^4},$$

with $R_f = \sqrt{4(\mu^4 + c_1 c_2 m_1^2 m_2^2)/g m_2^2}$
Thus the Higgs mechanism fails for a) $R_f = 0$ or b) $m_2^4 = \mu^4$

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a) trivial vacuum with no GB

Higgs mechanism

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$$\mathcal{L}_l = \sum_{i=1}^{2} \left|D_\mu \phi_i\right|^2 + m_i^2 \left|\phi_i\right|^2 - \mu^2 \left(\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1\right) - \frac{g}{4}\left(\left|\phi_1\right|^2\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

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Thus the Higgs mechanism fails for a) $R_f = 0$ or b) $m_2^4 = \mu^4$

a) trivial vacuum with no GB

b) zero exception point with no identifiable GB

Kinetic term in the physical region:

$$\mathcal{L} = \sum_{a=1}^{3} \partial_\mu G^a \partial^\mu G^a - m_g A_\mu^1 \partial^\mu G^1 + m_g A_\mu^2 \partial^\mu G^1 + m_g A_\mu^3 \partial^\mu G^3 + \frac{1}{2} m_g^2 A_\mu^a A^{a\mu} + \ldots$$
$$= \frac{1}{2} m_g^2 \left(A_\mu^1 - \frac{1}{m_g} \partial_\mu G^1 \right)^2 + \frac{1}{2} m_g^2 \left(A_\mu^2 + \frac{1}{m_g} \partial_\mu G^2 \right)^2 + \frac{1}{2} m_g^2 \left(A_\mu^3 + \frac{1}{m_g} \partial_\mu G^3 \right)^2$$
$$= \frac{1}{2} m_g^2 \sum_{a=1}^{3} B_\mu^a B^{a\mu} + \ldots .$$

with Goldstone fields $\{G^a\}$
new gauge field $B_\mu^a = A_\mu^a \pm \frac{1}{m_g} \partial_\mu G^a$

Kinetic term in the physical region:

$$
\begin{aligned}
\mathcal{L} &= \sum_{a=1}^{3} \partial_\mu G^a \partial^\mu G^a - m_g A_\mu^1 \partial^\mu G^1 + m_g A_\mu^2 \partial^\mu G^1 + m_g A_\mu^3 \partial^\mu G^3 + \frac{1}{2} m_g^2 A_\mu^a A^{a\mu} + \ldots \\
&= \frac{1}{2} m_g^2 \left(A_\mu^1 - \frac{1}{m_g} \partial_\mu G^1 \right)^2 + \frac{1}{2} m_g^2 \left(A_\mu^2 + \frac{1}{m_g} \partial_\mu G^2 \right)^2 + \frac{1}{2} m_g^2 \left(A_\mu^3 + \frac{1}{m_g} \partial_\mu G^3 \right)^2 \\
&= \frac{1}{2} m_g^2 \sum_{a=1}^{3} B_\mu^a B^{a\mu} + \ldots .
\end{aligned}
$$

with Goldstone fields $\{G^a\}$

new gauge field $B_\mu^a = A_\mu^a \pm \frac{1}{m_g} \partial_\mu G^a$

The Higgs mechanism breaks down at the zero exceptional point with the Goldstone boson being unidentifiable and the gauge particle unable to acquire a mass.

t'Hooft-Polyakov magnetic monopoles

Non-Hermitian t'Hooft-Polyakov model

$$\begin{aligned}\mathcal{L}_{cm} &= \frac{1}{2} Tr\left(D\phi_1\right)^2 + \frac{1}{2} Tr\left(D\phi_2\right)^2 - c_1 m_1^2 Tr\left(\phi_1^2\right) + c_2 m_2^2 Tr\left(\phi_2^2\right) \\ &\quad - i\mu^2 Tr\left(\phi_1\phi_2\right) - \frac{g}{4} Tr\left(\phi_1^2\right)^2 - \frac{1}{4} Tr\left(F_{\mu\nu}F^{\mu\nu}\right)\end{aligned}$$

- ϕ_i in adjoint representation of $SU(2)$: $\phi_i(x) = \phi_i^a(x)\tau^a$

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- parametrization:

$$(\phi_\alpha^{cl})^a = h_\alpha(r)\hat{r}^a_{n_\alpha}\,, (A_i^{cl})^a = \epsilon^{iaj}\hat{r}^j_n\left(\frac{u(r)-1}{er}\right), \hat{r}^a_n = \begin{pmatrix}\sin(\theta)\cos(n\varphi)\\ \sin(\theta)\sin(n\varphi)\\ \cos(\theta)\end{pmatrix}$$

- boundary condition: $(E < \infty \equiv$ sol$^{\text{ns}}$ tend to vacuum at $\infty)$

$$\lim_{r\to\infty} h_1(r) = \pm R_a\,, \qquad \lim_{r\to\infty} h_2(r) = \mp\frac{c_2 c_3 \mu^2}{m_2^2}R_a,$$

with $R_a = \sqrt{(m_1^2 m_2^2 - \mu^4)/2g m_2^2}$

equations of motion:

$$
\begin{aligned}
u'' + \frac{u\left[1-u^2\right]}{r^2} + \frac{e^2 u}{2}\left\{h_2^2 - h_1^2\right\} &= 0 \\
h_1'' + \frac{2h_1'}{r} - \frac{2h_1 u^2}{r^2} + g\left\{c_1\frac{m_1^2}{g}h_1 + c_3\frac{\mu^2}{g}h_2 + 2h_1^3\right\} &= 0 \\
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$$

- difficult to solve

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$$

- difficult to solve
- simplify by fourfold limit

$$
\lim_{g,m_1,m_2,\mu\to 0} (\text{eom}) \quad \text{with} \quad X := \frac{m_1^2}{g} < \infty,\, Y := \frac{\mu^2}{g} < \infty,\, Z := \frac{\mu^2}{m_2^2} < \infty
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$\Rightarrow$ BPS equations

Solutions:

$$
\begin{aligned}
u(r) &= \pm\frac{erlR_a}{\sinh(erlR_a)} \\
h_1^{\pm}(r) &= \pm\mathrm{Sign}(n)\frac{1}{l}\left\{|lR_a|\coth\left(e|lR_a|r\right)-\frac{1}{er}\right\} \\
h_2^{\pm} &= \mp\mathrm{Sign}(n)\frac{c_2c_3Z}{l}\left\{|lR_a|\coth\left(e|lR_a|r\right)-\frac{1}{er}\right\}
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$$

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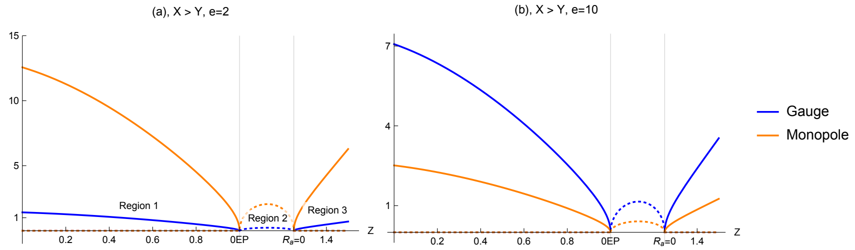
Energies:

$$
E = \int d^3x Tr\left(B^2\right) + Tr\left\{(D_i\phi_1)^2\right\} - Tr\left\{(D_i\phi_2)^2\right\} + V
$$

for BPS solutions

$$
E = \frac{8|n|\pi R_a}{e} \left(\frac{1 - Z^2}{\sqrt{1 - Z^2}} \right) = \frac{8|n|\pi R_a}{e} l
$$

Gauge mass versus monopole mass



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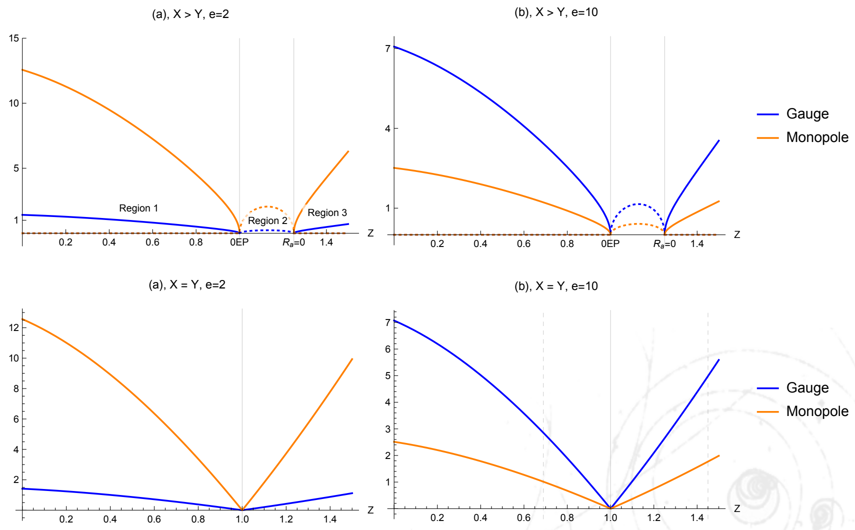

Reality of the energy for soliton solutions

The energy of solutions ϕ_1, ϕ_2 to the equation of motion is real if

(i) Hamiltonian transforms with modified $\mathcal{CPT}$-symmetry:

$$\mathcal{CPT} : \mathcal{H}[\phi(x_\mu)] \to \mathcal{H}^\dagger[\phi(-x_\mu)]$$

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(iii) The energies $E[\phi]$ are degenerate

$$E[\phi_1] = E[\phi_2]$$

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Condition (i):

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Condition (iii):

$$E[\phi_i^+] = E[\phi_i^-]$$

Bogomolny-Prasad-Sommerfield (BPS) solitons

Consider complex scalar field theory

$$\mathcal{L} = \frac{1}{2}\eta_{ab}\partial_\mu\phi_a\partial^\mu\phi_b - \mathcal{V}(\phi)$$

Taking the energy functional and topological charge of the form

$$E = \frac{1}{2}\int d^2x \left(A_\alpha^2 + \tilde{A}_\alpha^2\right) \qquad Q = \int d^2x A_\alpha \tilde{A}_\alpha,$$

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These two equations result as a compatibility equation between the Euler-Lagrange equations and $\delta Q = 0$.

Non-Hermitian sine-Gordon model (D=1+1)

$$\mathcal{V} = \frac{1}{2(1+\lambda^2)}\left[(\sin\phi_1 - \mu)^2 + 2i\lambda\,(\sin\phi_1 - \mu)\sin\phi_2 + \sin^2\phi_2\right]$$

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static BPS equations

$$BPS_1^{\pm} \quad : \quad \partial_x\phi_1 = \pm\frac{1}{1+\lambda^2}\left(\sin\phi_1 - \mu + i\lambda\sin\phi_2\right) =: G_1^{\pm}$$

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with modified $\mathcal{CPT}$-symmetry

$$\phi_1(x) \to [\phi_1(-x)]^\dagger\,, \phi_2(x) \to -[\phi_2(-x)]^\dagger\,, \Leftrightarrow\; BPS_i^{\pm} \to \left(BPS_i^{\mp}\right)^*$$

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Thus we have

$$\mathcal{V}\left[\phi_{\pm}(x)\right] = \mathcal{V}^{\dagger}\left[\phi_{\mp}(-x)\right],$$

which guarantees the reality of the energy.

Solutions and energy

Hermitian limit $\lambda = 0$:

$$\phi_1^{\pm(n)} = 2\arctan\left[\frac{1}{\mu} + \frac{\sqrt{(1-\mu^2)}}{\mu}\tanh\left[\frac{1}{2}\sqrt{(1-\mu^2)}(\kappa_1 \pm x)\right]\right] + 2\pi n$$

$$\phi_2^{\pm(n)} = 2\arctan\left(e^{\pm x+\kappa_2}\right) + 2\pi n$$

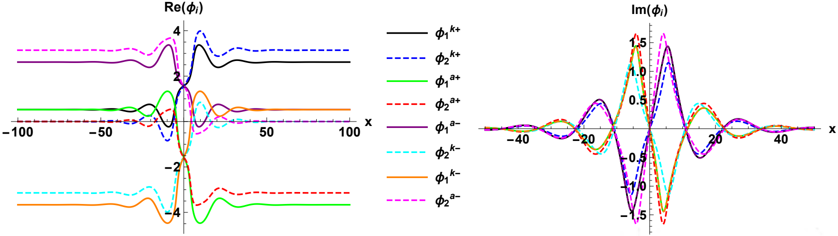
asymptotic limits:

$$\begin{aligned}
\lim_{x\to\infty}\phi_1^{+(n)}(x) &= \lim_{x\to-\infty}\phi_1^{-(n)}(x) = 2n\pi + \mathrm{sign}(\mu)\pi - \arcsin(\mu) \\
\lim_{x\to-\infty}\phi_1^{+(n)}(x) &= \lim_{x\to\infty}\phi_1^{-(n)}(x) = 2n\pi + \mathrm{sign}(\mu)\arcsin(\mu) \\
\lim_{x\to\pm\infty}\phi_2^{+(n)}(x) &= \lim_{x\to\mp\infty}\phi_2^{-(n)}(x) = 2n\pi + \frac{\pi \pm \pi}{2}
\end{aligned}$$

real energy for $|\mu| \leq 1$

$$E^{\pm}(\mu) = 2\left[1 + \sqrt{1-\mu^2} - \mu\arctan\left(\frac{\sqrt{1-\mu^2}}{\mu}\right)\right]$$

non-Hermitian case $\lambda \neq 0$ (numerical solution)



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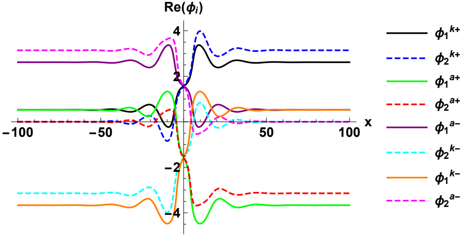

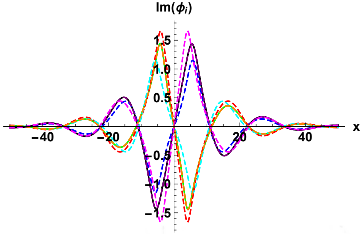

asymptotic limits are the same $\Rightarrow$ energies are the same

vacua:

$$v_1^{(n,m)} = (\arcsin \mu + 2\pi n, m\pi), \quad v_2^{(n,m)} = (\pi - \arcsin \mu + 2n\pi, m\pi)$$

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nature of the fixed points from eigenvalues of the Jacobian

$$J = \left. \begin{pmatrix} \partial_{\phi_1} G_1^{\pm} & \partial_{\phi_2} G_1^{\pm} \\ \partial_{\phi_1} G_2^{\pm} & \partial_{\phi_2} G_2^{\pm} \end{pmatrix} \right|_{v_j^{(n,m)}}$$

vacua:

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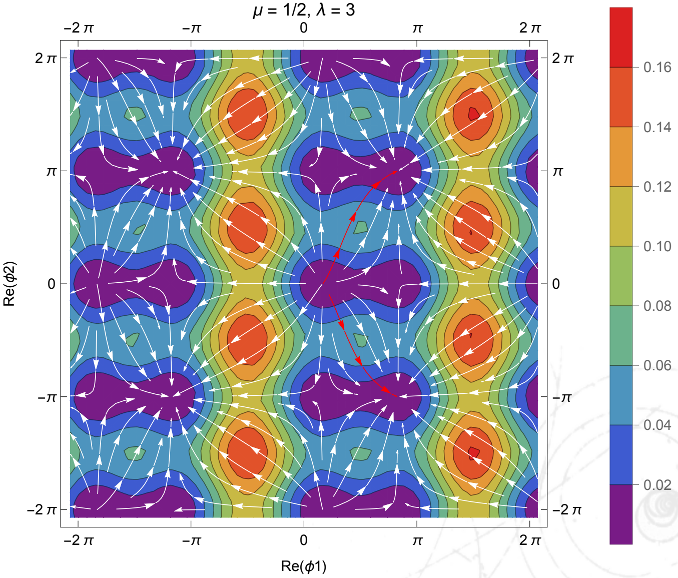
solutions interpolate between different vacua as

$$v_1^{(0,0)} \xrightarrow{\phi_1^{k+}\phi_2^{k+}} v_2^{(0,1)}, \quad v_1^{(0,0)} \xrightarrow{\phi_1^{a+}\phi_2^{a+}} v_2^{(-1,1)}$$

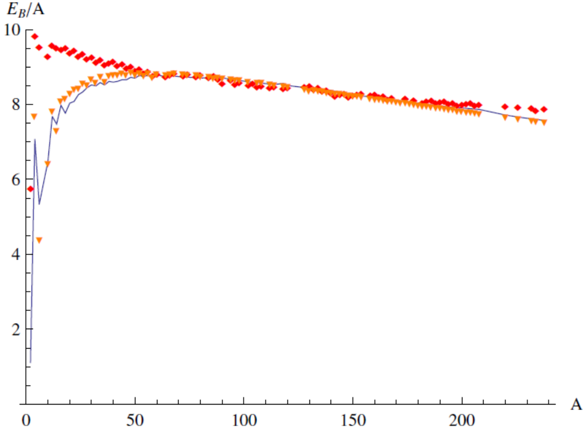
$$v_1^{(0,0)} \xrightarrow{\phi_1^{a-}\phi_2^{k-}} v_2^{(0,-1)}, \quad v_1^{(0,0)} \xrightarrow{\phi_1^{k-}\phi_2^{a-}} v_2^{(-1,1)}$$

gradient flow superimposed on the coupled sine-Gordon potential



Complex BPS Skyrme model (D=3+1)

Original model proposed in 1962 by Skyrme to describe QCD
BPS version: Adam, Sanchez-Guillen, Wereszczyński (2010)



exp. data (solid)
ASW (diamonds)
Weizsäcker (triangles)

from Adam et al. "The Skyrme model in the BPS limit"
The Multifaceted Skyrmion (2017): 193-232

$$\mathcal{L}_{\text{BPSS}} := -\tilde{\lambda}^2 N_0^2 B_\mu B^\mu \ - \tilde{\mu}^2 V$$

- $V = \frac{1}{2}\,\mathrm{Tr}\left(\mathbb{I} - U\right)$
- $U := e^{i\zeta(\sigma\cdot\vec{n})} \in SU(2)$, $\vec{n} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)$
- $L_\mu := U^\dagger \partial_\mu U$
- $B^\mu := \frac{1}{N_0}\epsilon^{\mu\nu\rho\tau}\, Tr\left(L_\nu L_\rho L_\tau\right) = \frac{1}{2N_0}\sin^2\zeta\sin\Theta\,\mathcal{B}^\mu$
- $\mathcal{B}^\mu := \varepsilon^{\mu\nu\rho\tau}\zeta_\nu \Theta_\rho \Phi_\tau$

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- $B^\mu := \frac{1}{N_0}\epsilon^{\mu\nu\rho\tau}\, Tr\,(L_\nu L_\rho L_\tau) = \frac{1}{2N_0}\sin^2\zeta \sin\Theta\, \mathcal{B}^\mu$
- $\mathcal{B}^\mu := \varepsilon^{\mu\nu\rho\tau}\zeta_\nu \Theta_\rho \Phi_\tau$

Dyson mapped version

$$\mathcal{L} = -\frac{\lambda^2}{4}\left(\sin\zeta - i\epsilon\cos\zeta\right)^4 \sin^2\Theta \mathcal{B}_\mu \mathcal{B}^\mu - \mu^2\left(\sqrt{1-\epsilon^2} - \cos\zeta - i\epsilon\sin\zeta\right)$$

$$\mathcal{H} = \eta^{-1}\mathcal{H}_{\mathrm{BPSS}}\eta, \quad \text{with} \quad \eta = \exp\left[-\text{arctanh}\;\epsilon \int d^3x \Pi^\zeta(t,r)\right]$$

same as boost with $\lambda \to \tilde{\lambda} = \lambda(1-\epsilon^2)$, $\mu \to \tilde{\mu} = \mu(1-\epsilon^2)^{1/4}$

Real solutions

$$\zeta(r)=\begin{cases} 2\arccos\left(\frac{1}{\sqrt{2}}\left|\frac{\tilde{\mu}}{n\tilde{\mu}}\right|^{1/3} r\right) & \text{for } r\in\left[0, r_c=\sqrt{2}\left|\frac{\tilde{\mu}}{n\tilde{\mu}}\right|^{1/3} r\right] \\ 0 & \text{otherwise}\end{cases}$$

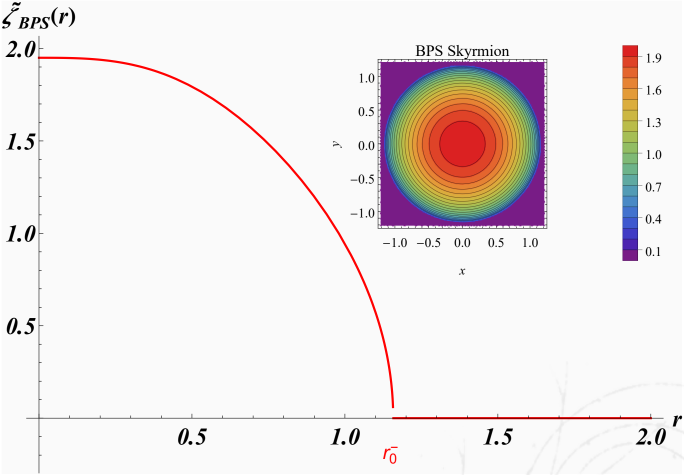
coordinates: (r,θ,ϕ), $r\in[0,\infty)$, $\theta\in[0,\pi)$, $\phi\in[0,2\pi)$
Skyrme fields: $\Theta=\theta$, $\Phi=n\phi$ with $n\in\mathbb{Z}$

Complex solutions:

$$\zeta^{\pm}_{\alpha,m}(r)=\tilde{\zeta}^{\pm}_{\alpha,m}(r)+i\operatorname{arctanh}\epsilon=2\arccos\left[\omega^{\alpha}\frac{(n\tilde{\lambda}c\mp\tilde{\mu}r^3)^{1/3}}{\sqrt{2}n^{1/3}\tilde{\lambda}^{1/3}}\right]+i\operatorname{arctan}$$

$\omega=\exp(i2\pi/3)$, $\alpha\in\{0,1,2\}$, $m\in\mathbb{Z}$, integration constant c

New type of solutions:



$$\zeta_{\text{BPS}} := \begin{cases} \tilde{\zeta}_{0,0}^{-} & \text{for} \quad 0 \leq r \leq r_0^- \\ 0 & \text{for} \quad r_0^- < r \end{cases}$$

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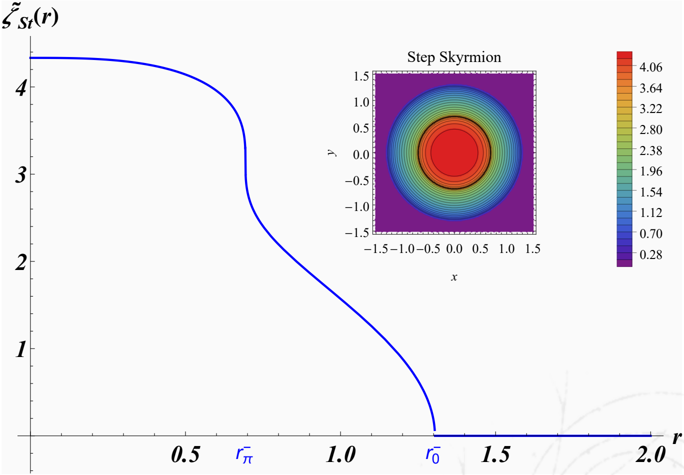
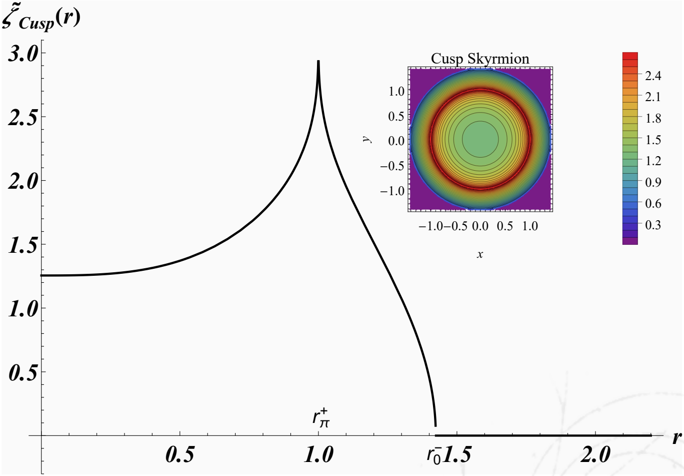


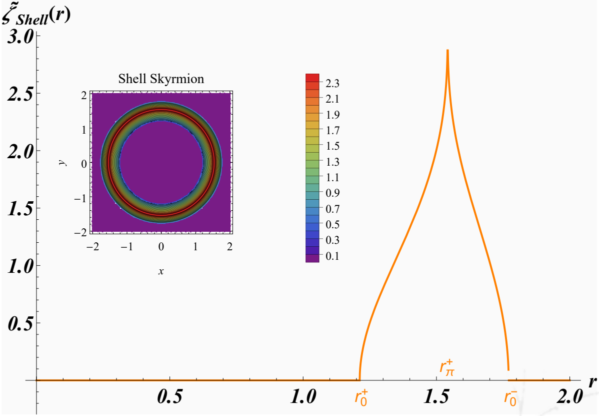
$$\zeta_{\text{St}} := \begin{cases} \tilde{\zeta}_{1,0}^- & \text{for} \quad 0 \leq r \leq r_\pi^- \\ \tilde{\zeta}_{0,0}^- & \text{for} \quad r_\pi^- \leq r \leq r_0^- \\ 0 & \text{for} \quad r_0^- < r \end{cases}$$

New type of solutions:



$$\zeta_{\text{Cusp}} := \begin{cases} \tilde{\zeta}^{+}_{0,0} & \text{for} \quad 0 \leq r \leq r_\pi^+ = r_\pi^- \\ \tilde{\zeta}^{-}_{0,0} & \text{for} \quad r_\pi^- \leq r \leq r_0^- \\ 0 & \text{for} \quad r_0^- < r \end{cases}$$

New type of solutions:



$$\zeta_{\text{Shell}} := \begin{cases} 0 & \text{for} \quad 0 \leq r \leq r_0^+ \\ \tilde{\zeta}_{0,0}^+ & \text{for} \quad r_0^+ \leq r \leq r_\pi^+ = r_\pi^- \\ \tilde{\zeta}_{0,0}^- & \text{for} \quad r_\pi^- \leq r \leq r_0^- \\ 0 & \text{for} \quad r_0^- < r \end{cases}$$

Energies:

$$E_{\text{BPS/St, Cusp}} = \frac{8}{15} n\tilde{\mu}\tilde{\lambda}\pi \left(8\sqrt{2} \mp 10c \pm 3c^{5/3}\right)$$
$$E_{\text{Shell}} = \frac{128}{15}\sqrt{2} n\tilde{\mu}\tilde{\lambda}\pi$$
$$E_{i\text{BPS}} = -E_{\text{BPS}}$$

Reality condition:

Condition (i):

$$\mathcal{CPT}' : \zeta(x_\mu) \to \zeta^*(-x_\mu) + 2i \operatorname{arctanh}\epsilon$$

Condition (ii):

$$\zeta^{\pm}_{\alpha,m}(r) \to \left[\zeta^{\pm}_{\alpha,m}(r)\right]^* + 2i \operatorname{arctanh}\epsilon = \zeta^{\pm}_{\alpha,m}(r). \tag{1}$$

Discussions of more potentials and different variants of $\mathcal{L}$ see:
F. Correa, A. Fring, T. Taira, arXiv:2102.05781

Some general conclusions

Goldstone theorem in non-Hermitian field theories

- The GT holds in the physical $\mathcal{PT}$-symmetric regime
- The GT holds at the standard exceptional point
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Complex BPS solitions and magnetic monopoles

- Complex t'Hooft-Polyakov solutions with real energies $\exists$
- Complex BPS solitons in 1+1 dim with real energies $\exists$
- Complex Skyrmions in 3+1 dim with real energies $\exists$

Check out the online seminar series on
Pseudo-Hermitian Hamiltonians in Quantum Physics
organizers: Francisco Correa and Andreas Fring
website: https://vphhqp.com