CITY UNIVERSITY LONDON

Minimal lengths, areas and volumes in noncommutative quasi-Hermitian systems

Andreas Fring

Coherent States and their Applications:
A Contemporary Panorama
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Outline:

- Introduction to PT/quasi-Hermitian quantum mechanics

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- Time-dependent quasi-Hermitian systems
- Squeezed states for minimal lengths, areas and volumes

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Hermiticity is only sufficient but not necessary

- Operators $\mathcal{O}$ which are left invariant under an antilinear involution $\mathcal{I}$ and whose eigenfunctions Φ also respect this symmetry,

$$[\mathcal{O}, \mathcal{I}] = 0 \quad \wedge \quad \mathcal{I}\Phi = \Phi$$

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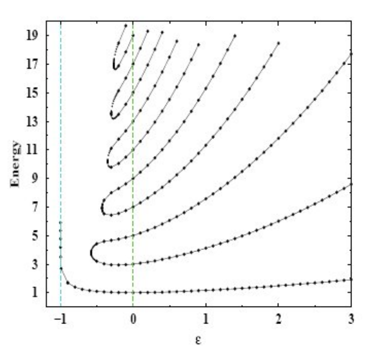
In particular this also holds for $\mathcal{O}$ being non-Hermitian.

Examples for non-Hermitian systems from the literature:

"Recent" classical example

$$\mathcal{H} = \frac{1}{2}p^2 + x^2(ix)^\varepsilon \qquad \text{for } \varepsilon \geq 0$$



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An older classical example

- Lattice Reggeon field theory:

$$\mathcal{H} = \sum\nolimits_{\vec{\imath}} \left[\Delta a^\dagger_{\vec{\imath}} a_{\vec{\imath}} + ig a^\dagger_{\vec{\imath}} (a_{\vec{\imath}} + a^\dagger_{\vec{\imath}}) a_{\vec{\imath}} + \tilde{g} \sum\nolimits_{\vec{\jmath}} (a^\dagger_{\vec{\imath}+\vec{\jmath}} - a^\dagger_{\vec{\imath}})(a_{\vec{\imath}+\vec{\jmath}} - a_{\vec{\imath}}) \right]$$

- $a^\dagger_{\vec{\imath}}, a_{\vec{\imath}}$ are creation and annihilation operators, $\Delta, g, \tilde{g} \in \mathbb{R}$
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- for one site this is almost ix^3

$$\begin{aligned} \mathcal{H} &= \Delta a^{\dagger} a + iga^{\dagger}\left(a + a^{\dagger}\right) a \\ &= \frac{1}{2}\left(\hat{p}^2 + \hat{x}^2 - 1\right) + i\frac{g}{\sqrt{2}}(\hat{x}^3 + \hat{p}^2\hat{x} - 2\hat{x} + i\hat{p}) \end{aligned}$$

with $a = (\omega\hat{x} + i\hat{p})/\sqrt{2\omega}$, $a^{\dagger} = (\omega\hat{x} - i\hat{p})/\sqrt{2\omega}$
[P. Assis and A.F., J. Phys. A41 (2008) 244001]

Examples for non-Hermitian systems from the literature:

- quantum spin chains: (c=-22/5 CFT)

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^{N} \sigma_i^x + \lambda \sigma_i^z \sigma_{i+1}^z + ih\sigma_i^z \qquad \lambda, h \in \mathbb{R}$$

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$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{\beta^2} \sum_{k=\mathbf{a}}^{\ell} n_k \exp(\beta \alpha_k \cdot \phi)$$

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- strings on $AdS_5 \times S^5$-background

[A. Das, A. Melikyan, V. Rivelles, *JHEP* 09 (2007) 104]

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  - deformed Heisenberg canonical commutation relations

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- representation: $X = (1 + \tau p_0^2)x_0$, $P = p_0$, $[x_0, p_0] = i\hbar$

Examples for non-Hermitian systems from the literature:

- with the standard inner product X is not Hermitian

$$X^\dagger = X + 2\tau i\hbar P \quad \text{and} \quad P^\dagger = P$$

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- example harmonic oscillator:

$$\begin{aligned} H_{ho} &= \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}(1+\tau p_0^2)x_0(1+\tau p_0^2)x_0, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}\left[(1+\tau p_0^2)^2 x_0^2 + 2i\hbar\tau p_0(1+\tau p_0^2)x_0\right]. \end{aligned}$$

[B. Bagchi and A. Fring, Phys. Lett. A373 (2009) 4307] q-dependent coherent states are constructed for:

$$[X,P]=i\hbar+i\frac{q^2-1}{q^2+1}\left(m\omega X^2+\frac{1}{m\omega}P^2\right)$$

Take

$$X=\alpha\left(A^\dagger+A\right),\qquad \text{and}\qquad P=i\beta\left(A^\dagger-A\right)$$

with $\alpha=1/2\sqrt{1+q^2}\sqrt{\hbar/(m\omega)}$, $\beta=1/2\sqrt{1+q^2}\sqrt{\hbar m\omega}$

Non-Hermitian representation:

$$A=\frac{1}{1-q^2}D_q,\qquad \text{and}\qquad A^\dagger=(1-x)-x(1-q^2)D_q$$

Jackson derivatives $D_q f(x):=[f(x)-f(q^2x)]/[x(1-q^2)]$

S. Dey, A. Fring, L. Gouba; J. Phys. A 45 (2012) 385302

S. Dey, A. Fring; Phys. Rev. D86 (2012) 064038

S. Dey, A. Fring, L. Gouba, P. Castro; Phys. Rev. D 87 (2013) 084033

How to explain the reality of the spectra?

1. Pseudo/Quasi-Hermiticity
2. Supersymmetry (Darboux transformations)
3. $\mathcal{PT}$-symmetry

Pseudo/Quasi-Hermiticity

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger \rho = \rho H \qquad \rho = \eta^\dagger \eta$$

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positivity of ρ	✓	✓	×
ρ Hermitian	✓	✓	✓
ρ invertible	✓	×	✓
terminology	q and p	quasi-Herm.	pseudo-Herm.
spectrum of H	real	could be real	real
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Supersymmetry (Darboux transformation)

Decompose Hamiltonian $\mathcal{H}$ as:

$$\mathcal{H} = H_+ \oplus H_- = Q\tilde{Q} \oplus \tilde{Q}Q$$

• intertwining operators: $QH_- = H_+Q$ and $\tilde{Q}H_+ = H_-\tilde{Q}$

$$\Rightarrow \qquad [\mathcal{H}, Q] = [\mathcal{H}, \tilde{Q}] = 0$$

• realization: $Q = \frac{d}{dx} + W$ and $\tilde{Q} = -\frac{d}{dx} + W$

$$\Rightarrow \qquad H_\pm = -\Delta + W^2 \pm W' = -\Delta + V_\pm$$

• ground state: $H_-\Phi_n^- = \varepsilon_n \Phi_n^-$ and $H_-\Phi_m^- = 0$
$\Rightarrow$isospectral Hamiltonians

$$H_\pm^m = -\Delta + V_\pm^m + E_m \quad H_\pm^m \Phi_n^\pm = E_n \Phi_n^\pm \quad \text{for } n > m$$

Unbroken $\mathcal{PT}$-symmetry guarantees real eigenvalues

- $\mathcal{PT}$-symmetry: $\mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$
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- $\mathcal{PT}$-symmetry: $\quad \mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$
  $(\mathcal{P}: x \to -x, p \to -p; \ \mathcal{T}: x \to x, p \to -p, i \to -i)$
- $\mathcal{PT}$ is an anti-linear operator:

$$\mathcal{PT}(\lambda\Phi + \mu\Psi) = \lambda^*\mathcal{PT}\Phi + \mu^*\mathcal{PT}\Psi \qquad \lambda, \mu \in \mathbb{C}$$

- Real eigenvalues from unbroken $\mathcal{PT}$-symmetry:

$$[\mathcal{H}, \mathcal{PT}] = 0 \quad \wedge \quad \mathcal{PT}\Phi = \Phi \quad \Rightarrow \varepsilon = \varepsilon^* \ \text{ for } \mathcal{H}\Phi = \varepsilon\Phi$$

- *Proof*:
  $\varepsilon\Phi = \mathcal{H}\Phi = \mathcal{HPT}\Phi = \mathcal{PTH}\Phi = \mathcal{PT}\varepsilon\Phi = \varepsilon^*\mathcal{PT}\Phi = \varepsilon^*\Phi$

PT-symmetry is only an example of an antilinear involution

General deformation prescription:

$\mathcal{PT}$-anti-symmetric quantities:

$$\mathcal{PT} : \phi(x,t) \mapsto -\phi(x,t) \quad \Rightarrow \quad \delta_\varepsilon : \phi(x,t) \mapsto -i[i\phi(x,t)]^\varepsilon$$

Two possibilities to deform the KdV Hamiltonian

$$\mathcal{H}_{KdV} = -\frac{\beta}{6}u^3 - \frac{\gamma}{2}(u_x)^2$$

$$\delta_\varepsilon^+ : u_x \mapsto u_{x,\varepsilon} := -i(iu_x)^\varepsilon \qquad \text{or} \qquad \delta_\varepsilon^- : u \mapsto u_\varepsilon := -i(iu)^\varepsilon,$$

such that

$$\mathcal{H}_\varepsilon^+ = -\frac{\beta}{6}u^3 - \frac{\gamma}{1+\varepsilon}(iu_x)^{\varepsilon+1} \qquad \mathcal{H}_\varepsilon^- = \frac{\beta}{(1+\varepsilon)(2+\varepsilon)}(iu)^{\varepsilon+2} + \frac{\gamma}{2}u_x^2$$

with equations of motion

$$u_t + \beta u u_x + \gamma u_{xxx,\varepsilon} = 0 \qquad u_t + i\beta u_\varepsilon u_x + \gamma u_{xxx} = 0$$

Calogero-Moser-Sutherland models (PT-extended)

$$\mathcal{H}_{ext} = \frac{p^2}{2} + \frac{\omega^2}{2} \sum_i q_i^2 + \frac{g^2}{2} \sum_{i \neq k} \frac{1}{(q_i - q_k)^2} + i\tilde{g} \sum_{i \neq k} \frac{1}{(q_i - q_k)} p_i$$

with $g, \tilde{g} \in \mathbb{R}, q, p \in \mathbb{R}^{\ell+1}$

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$$\mathcal{H}_{def} = \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\alpha \cdot q)^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha V(\alpha \cdot q), \;\; m, g_\alpha \in \mathbb{R}$$

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Define deformed coordinates (A_2)

$$\begin{aligned}
q_1 &\rightarrow \tilde{q}_1 = q_1 \cosh \varepsilon + i\sqrt{3}(q_2 - q_3) \sinh \varepsilon \\
q_2 &\rightarrow \tilde{q}_2 = q_2 \cosh \varepsilon + i\sqrt{3}(q_3 - q_1) \sinh \varepsilon \\
q_3 &\rightarrow \tilde{q}_3 = q_3 \cosh \varepsilon + i\sqrt{3}(q_1 - q_2) \sinh \varepsilon
\end{aligned}$$

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or deformed roots:

$$\begin{aligned} \alpha_1 &\rightarrow \tilde{\alpha}_1 = \alpha_1 \cosh\varepsilon + i\sqrt{3}\sinh\varepsilon\lambda_2 \\ \alpha_2 &\rightarrow \tilde{\alpha}_2 = \alpha_2 \cosh\varepsilon - i\sqrt{3}\sinh\varepsilon\lambda_1 \end{aligned}$$

H is Hermitian with respect to new metric

- Assume pseudo-Hermiticity:

$$h = \eta H \eta^{-1} = h^{\dagger} = (\eta^{-1})^{\dagger} H^{\dagger} \eta^{\dagger} \quad \Leftrightarrow \quad H^{\dagger} \eta^{\dagger} \eta = \eta^{\dagger} \eta H$$

$$\Phi = \eta^{-1} \phi \qquad \eta^{\dagger} = \eta$$

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$$\langle \Psi | H\Phi \rangle_\eta = \langle \Psi | \eta^2 H \Phi \rangle$$

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$\Rightarrow$ Eigenvalues of H are real, eigenstates are orthogonal

Observables

- Observables are Hermitian with respect to the new metric

$$\langle \Phi | \mathcal{O}\Phi \rangle_\eta = \langle \mathcal{O}\Phi | \Phi \rangle_\eta$$

$$\mathcal{O} = \eta^{-1} o \eta \qquad \Leftrightarrow \qquad \mathcal{O}^\dagger = \rho \mathcal{O} \rho^{-1}$$

- o is an observable in the Hermitian system
- $\mathcal{O}$ is an observable in the non-Hermitian system

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  - o is an observable in the Hermitian system
  - $\mathcal{O}$ is an observable in the non-Hermitian system

- Ambiguities:
  Given H the metric is not uniquely defined for unknown h.
  $\Rightarrow$ Given only H the observables are not uniquely defined.
  This is different in the Hermitian case.
  - Fixing one more observable achieves uniqueness.
  [Scholtz, Geyer, Hahne, , *Ann. Phys.* 213 (1992) 74]

General procedure:

- Given $H \begin{cases} \text{either} & \text{solve } \eta H \eta^{-1} = h \quad \text{for} \quad \eta \;\Rightarrow \rho = \eta^\dagger \eta \\ \text{or} & \text{solve } H^\dagger = \rho H \rho^{-1} \quad \text{for} \quad \rho \;\Rightarrow \eta = \sqrt{\rho} \end{cases}$

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Note:

- Thus, this is not re-inventing or disputing the validity of quantum mechanics.
- We only give up the restrictive requirement that Hamiltonians have to be Hermitian.

[C. Bender, *Rep. Prog. Phys.* 70 (2007) 947]
[A. Mostafazadeh, Int. J. Geom. Meth. Phys. 7 (2010) 1191]
[C Bender, A Fring, U Günther, H Jones, J.Phys. A45 (2012) 440301]
[A. Fring, Phil. Trans. R. Soc. A 371 (2013) 20120046]

Time-dependent Schrödinger equations for $h = h^\dagger$ and $H \neq H^\dagger$

$$h(t)\phi(t) = i\hbar\partial_t\phi(t) \qquad H(t)\Psi(t) = i\hbar\partial_t\Psi(t)$$

with time-dependent Dyson map $\eta(t)$

$$\phi(t) = \eta(t)\Psi(t)$$

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$$\phi(t) = \eta(t)\Psi(t)$$

$\Rightarrow$ time-dependent Dyson relation

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t)$$

$\Rightarrow$ time-dependent quasi-Hermiticity relation ($\rho(t) := \eta^\dagger(t)\eta(t)$)

$$H^\dagger(t)\rho(t) - \rho(t)H(t) = i\hbar\partial_t\rho(t)$$

H is not quasi-Hermitian $\Rightarrow$ No-go theorem?

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- There exist non-trivial solutions.
- Energy operator: $\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\hbar\eta^{-1}(t)\partial_t\eta(t)$

Solutions time-dependent quasi-Hermiticity relation

Time-dependent harmonic oscillator with linear terms

$$H_h(t) = \omega(t)a^\dagger a + \alpha(t)a + \beta(t)a^\dagger$$

Time-dependent lattice Yang-Lee model

$$H_N(t) = -\frac{1}{2}\sum_{j=1}^{N}(\sigma_j^z + \lambda(t)\sigma_j^x\sigma_{j+1}^x + i\kappa(t)\sigma_j^x)$$

Time-dependent Swanson Hamiltonian

$$H_S(t) = \omega(t)\left(a^\dagger a + 1/2\right) + \alpha(t)a^2 + \beta(t)a^{\dagger 2}$$

[A. Fring, M.H.Y. Moussa, Phys. Rev. A 93, 042114 (2016)]
[A. Fring, M.H.Y. Moussa,, Phys. Rev. A 94, 042128 (2016)]

Time-independent Hamiltonian, time-dependent metric

Instead of solving time-dependent Schrödinger equation:

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Solve time-independent Schrödinger equation:

$$H\Psi(t) = i\hbar\partial_t\Psi(t)$$

and time-dependent quasi-Hermiticity relation

$$\begin{aligned} H^\dagger\rho(t) - \rho(t)H &= i\hbar\partial_t\rho(t) \\ \rho(t) &= \eta^\dagger(t)\eta(t) \end{aligned}$$

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$\Rightarrow \qquad \phi(t) = \eta(t)\Psi(t)$

[A. Fring and T. Frith, arXiv:1610.07537]

Rabi-type Hamiltonian ⇔ One-site lattice Yang-Lee model

$$h(t) = -\frac{1}{2}\left[\omega\mathbb{I}+\frac{2\phi^2}{2+\gamma^2\sin(t\phi)-\gamma^2}\sigma_z\right], \quad \phi=\sqrt{1-\gamma^2}$$

$$H_1 = -\frac{1}{2}\left[\omega\mathbb{I}+\sigma_z+i\gamma\sigma_x\right]$$

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Solution of time-independent Schrödinger equation:

$$\Psi_\pm(t) = \frac{\sqrt{\gamma}}{\sqrt{2}\phi\sqrt{1\pm\phi}}\begin{pmatrix}\gamma \\ i(1\pm\phi)\end{pmatrix}e^{-iE_\pm t} \quad E_\pm = \frac{1}{2}(-\omega\pm\phi)$$

Solution of time-dependent quasi-Hermiticity relation

$$\rho(t) = \left[\frac{1}{\gamma}+\gamma\sin(\phi t)\right]\mathbb{I}+\phi\cos(\phi t)\sigma_x-\left[1+\sin(\phi t)\right]\sigma_y.$$

Time-dependent Dyson map:

$$\eta(t)=\frac{1}{2}\left[p_+(t)+p_-(t)\right]\mathbb{I}+\frac{p_+(t)-p_-(t)}{2\left|p_0(t)\right|}\left[Im\left[p_0(t)\right]\sigma_x-Re\left[p_0(t)\right]\sigma_y\right]$$

with

$$\begin{aligned} p_\pm(t) &= \sqrt{\gamma^{-1}+\gamma\sin(\phi t)\pm\left|p_0(t)\right|} \\ p_0(t) &= 1+\sin(\phi t)+i\phi\cos(\phi t) \end{aligned}$$

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Time evolution operator ($\phi(t)=u(t,t_0)\phi(t_0)$):

$$u(t,t_0)=\begin{pmatrix} e^{i\theta(t)} & 0 \\ 0 & e^{\frac{i\pi}{2}\left(\frac{\omega}{\phi}+\frac{2t\omega}{\pi}\right)-i\theta(t)} \end{pmatrix}$$

with $t_0=-\frac{\pi}{2\phi}$

$$\theta(t)=\frac{\pi}{4}+\frac{\omega}{2}(t-t_0)+\arctan\left[\frac{(1-\phi)^2+\gamma\tan\left(\frac{t\phi}{2}\right)}{\gamma+(1-\phi)^2\tan\left(\frac{t\phi}{2}\right)}\right]$$

Consider

$$(\Delta A)\,(\Delta B) \geq \frac{1}{2}\,|\langle [A, B]\rangle|$$

with $(\Delta A)^2 := \langle A^2\rangle - \langle A\rangle^2 = \left\langle \hat{A}^2 \right\rangle$, $\hat{A} = A - \langle A\rangle$

- Minimal length: Smallest value for ΔA or ΔB?

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- Minimal length: Smallest value for ΔA or ΔB?
  - commutative space: $[A, B] = const$
    give up information B, i.e. $\Delta B \to \infty \Rightarrow \Delta A = 0$

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    give up information B, i.e. $\Delta B \to \infty \Rightarrow \Delta A = 0$
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Consider

$$(\Delta A)\,(\Delta B) \geq \frac{1}{2}\,|\langle [A,B]\rangle|$$

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  - generalization to triple uncertainty relations

Direct minimization:

Define

$$f(\Delta A, \Delta B) := \Delta A \Delta B - \frac{1}{2} \left| \langle [A, B] \rangle \right|^2$$

Solve

$$f(\Delta A, \Delta B) = 0, \quad \partial_{\Delta B} f(\Delta A, \Delta B) = 0, \quad \Rightarrow \Delta A_{\text{min}}$$

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Examples:

$$[X, P] = i\hbar \left(1 + \tau P^2 \right) \Rightarrow \Delta X_{\min} = \hbar \sqrt{\tau} \sqrt{1 + \tau \langle P \rangle^2} = \hbar \sqrt{\tau}$$

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$$[X, P] = i\hbar \left(1 + \tau P^2 + \frac{\tau^2 P^4}{2}\right) \Rightarrow \Delta X_{\text{min}} = \sqrt{\frac{17+7\sqrt{7}}{27}}\hbar\sqrt{\tau} \approx 1.147\hbar\sqrt{\tau}$$

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Do the corresponding squeezed states exist?

Direct method

Assume minimality for equality and solve for $|\psi\rangle$

$$\left[A-\alpha+\frac{\langle[A,B]\rangle}{2b^2}(B-\beta)\right]|\psi\rangle=0$$

with three free parameters $\alpha=\langle A\rangle$, $\beta=\langle B\rangle$, $b^2=\left\langle B^2\right\rangle-\langle B\rangle^2$

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Analytic method

Minimize LHS with Lagrange multiplier for normalization

$$\frac{\delta}{\delta\langle\psi|}\left[\left(\langle\psi|A^2|\psi\rangle-\langle\psi|A|\psi\rangle^2\right)\left(\langle\psi|B^2|\psi\rangle-\langle\psi|B|\psi\rangle^2\right)-m\left(\langle\psi|\psi\rangle-1\right)\right]$$

$\Rightarrow$ eigenvalue problem

$$\left[\frac{(A-\alpha)^2}{a^2}+\frac{(B-\beta)^2}{b^2}\right]|\psi\rangle=2\,|\psi\rangle$$

with four free parameter α, β, b^2, $a^2=\left\langle A^2\right\rangle-\langle A\rangle^2$

Re-express the direct method

$$\left[\frac{(A-\alpha)^2}{a^2} + \frac{(B-\beta)^2}{b^2} \right] |\psi\rangle = 2 \frac{[A,B]}{\langle [A,B] \rangle} |\psi\rangle$$

$\Rightarrow$ two schemes agree if and only if $|\psi\rangle$ is an eigenstate of $[A, B]$

Direct versus analytic method [R. Jackiw, J. Math. Phys. 9 (1968) 339]

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The analytic method

- is valid when $[A, B]$ is not a c-number
- is valid when minimum is not reached at equality
- allows generalization to $\Delta A \Delta B \Delta C$

$$\left[\frac{(A-\alpha)^2}{a^2}+\frac{(B-\beta)^2}{b^2}+\frac{(C-\gamma)^2}{c^2}\right]|\psi\rangle = 3\,|\psi\rangle$$

two additional free parameters $\gamma = \langle C\rangle$, $c = \langle C^2\rangle - \langle C\rangle^2$

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Recall:

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$$\begin{aligned}
X_{(1)} &= (1+\tau p^2)x, & P_{(1)} &= p, \\
X_{(2)} &= (1+\tau p^2)^{1/2} x (1+\tau p^2)^{1/2}, & P_{(2)} &= p, \\
X_{(3)} &= x, & P_{(3)} &= \frac{1}{\sqrt{\tau}}\tan\left(\sqrt{\tau}p\right) \\
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For $\Pi_{(1)}$ in momentum space

$$\left[i\hbar \left(1+\tau p^2\right) \partial_p + i\hbar \frac{1+\tau b^2+\tau\beta^2}{2b^2}(p-\beta) - \alpha \right] \psi_d(p) = 0$$

Normalized solution ($\beta = 0$)

$$\psi_d(p) = \left[\frac{\sqrt{\tau}\Gamma\left(\frac{3}{2} + \frac{1}{2\tau b^2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2} + \frac{1}{2\tau b^2}\right)} \right]^{1/2} \left(1 + \tau p^2\right)^{-\frac{1}{4\tau b^2} - \frac{1}{4}} \exp\left[-\frac{i\alpha \arctan\left(p\sqrt{\tau}\right)}{\hbar\sqrt{\tau}} \right]$$

for quasi-Hermitian inner product

$$\langle \psi | \psi \rangle_\rho := \int_{-\infty}^{\infty} \rho(p)\psi^*(p)\psi(p)dp = 1,$$

with metric operator $\rho = \left(1 + \tau p^2\right)^{-1}$

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Minimizing $(\Delta X)^2$ we find $b = 1/\sqrt{\tau}$, such that

$$\Delta X_{\min} = \hbar\sqrt{\tau} \qquad\qquad \Delta X_{\min}\Delta P_{\min} = \hbar$$

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Agrees with previous result.

Minimal area from analytic method:

$$\left[\frac{\hbar^2\left(1+\tau p^2\right)^2\partial_p^2}{a^2}+\frac{2\hbar(i\alpha+\hbar p\tau)\left(1+\tau p^2\right)\partial_p}{a^2}-\frac{\alpha^2}{a^2}-\frac{(p-\beta)^2}{b^2}+2\right]\psi_a(p)=0$$

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Solution ($\beta=0$)

$$\psi_a(p)=\exp\left[-\frac{i\alpha\arctan\left(p\sqrt{\tau}\right)}{\hbar\sqrt{\tau}}\right]\left[c_1P_\ell^m(ip\sqrt{\tau})+c_2Q_\ell^m(ip\sqrt{\tau})\right]$$

$P_\ell^m(x)\equiv$ associated Legendre polynomials
$Q_\ell^m(x)\equiv$ Legendre functions of the second kind
with $\ell=\sqrt{4a^2+\hbar^2\tau^2b^2}/(2b\tau\hbar)-1/2$ and $m=a\sqrt{1+2\tau b^2}/(b\tau\hbar)$

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First meaningful solutions for small integers $\ell=1$, $m=2$ fixes a, b
Suitably normalized

$$\psi_a(p)=\sqrt{\frac{8}{3\pi}\frac{\tau^{1/4}}{1+\tau p^2}}\exp\left[-\frac{i\alpha\arctan\left(p\sqrt{\tau}\right)}{\hbar\sqrt{\tau}}\right]$$

We compute

$$\langle X \rangle_\rho = \alpha, \quad \left\langle X^2 \right\rangle_\rho = \alpha^2 + \frac{4\hbar^2\tau}{3}, \quad \langle P \rangle_\rho = 0, \quad \left\langle P^2 \right\rangle_\rho = \frac{1}{3\tau},$$

so that

$$(\Delta X \Delta P)_{min} = 2\hbar/3 < \Delta X_{\text{min}} \Delta P_{\text{min}}$$

$$\Delta X = \lambda\hbar\sqrt{\tau} > \Delta X_{\text{min}}$$

$$\Delta P = \frac{1}{\sqrt{3\tau}}$$

with $\lambda = 2/\sqrt{3} \approx 1.15$

Minimal volume from analytic method:

Consider 3D flat noncommutative space

$$\begin{aligned}[X,Y] &= i\theta_1, \qquad [Z,X]=i\theta_2, \qquad [Y,Z]=i\theta_3, \\ [X,P_X] &= [Y,P_Y]=[Z,P_Z]=i\hbar\end{aligned}$$

Bopp shifted representation

$$X=x-\frac{\theta_1}{\hbar}p_y,\ Y=y-\frac{\theta_3}{\hbar}p_z,\ Z=z+\frac{\theta_2}{\hbar}p_x,\ P_X=p_x,\ P_Y=p_y,\ P_Z=p_z$$

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Bopp shifted representation

$$X = x - \frac{\theta_1}{\hbar} p_y, \; Y = y - \frac{\theta_3}{\hbar} p_z, \; Z = z + \frac{\theta_2}{\hbar} p_x, \; P_X = p_x, \; P_Y = p_y, \; P_Z = p_z$$

Simplify to one noncommutative constant θ:

$$\left[\frac{b^2 + c^2}{b^2 c^2} \partial_x^2 - \frac{2i}{\theta} \left(\frac{\beta}{b^2} + \frac{\alpha + \beta - x}{c^2} \right) \partial_x - \frac{(x - \alpha)^2}{a^2 \theta^2} - \frac{\beta^2}{b^2 \theta^2} - \frac{(\alpha + \beta - x)^2 - i\theta}{c^2 \theta^2} + \frac{3}{\theta^2} \right] \psi(x) = 0$$

$$\psi(x) = c_1 e^{f(x)} H_n \left[\frac{\sqrt{bc}(x-\alpha)(a^2+b^2+c^2)^{1/4}}{\sqrt{a\theta}\sqrt{b^2+c^2}} \right].$$

$$f(x) = -\frac{bc\sqrt{a^2+b^2+c^2}+iab^2}{2ab^2\theta+2ac^2\theta}x^2 + \frac{\alpha bc\sqrt{a^2+b^2+c^2}+ia\left(b^2(\alpha+\beta)+\beta c^2\right)}{a\theta\left(b^2+c^2\right)}x$$

$H_n(x) \equiv$ Hermite polynomials with

$n = 3abc/(2\theta\sqrt{a^2+b^2+c^2}) - 1/2$

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$n = 0$ fixes one constant, e.g. $c = \theta\sqrt{a^2+b^2}/\sqrt{9a^2b^2-\theta^2}$

$$\psi(x) = \left[\frac{3b^2(a^2+b^2)}{\pi a^2(9b^4+\theta^2)}\right]^{1/4} e^{f(x)}$$

$$f(x) = -\frac{b^2\left(\theta+3ia^2\right)}{2a^2\theta\left(3b^2+i\theta\right)}x^2 + i\left(\frac{\alpha+\beta}{\theta} - \frac{\alpha\left(a^2+b^2\right)}{a^2\left(\theta-3ib^2\right)}\right)x - \frac{3\alpha^2b^2\left(a^2+b^2\right)}{2a^2\left(9b^4+\theta^2\right)}$$

We compute

$$\langle X\rangle_\rho = \alpha, \qquad \left\langle X^2\right\rangle_\rho = \frac{a^2\left(9b^4+\theta^2\right)}{6b^2\left(a^2+b^2\right)} + \alpha^2,$$

$$\langle Y\rangle_\rho = \beta, \qquad \left\langle Y^2\right\rangle_\rho = \frac{b^2\left(9a^4+\theta^2\right)}{6a^2\left(a^2+b^2\right)} + \beta^2,$$

$$\langle Z\rangle_\rho = -\alpha-\beta, \qquad \left\langle Z^2\right\rangle_\rho = \frac{\theta^2\left(a^2+b^2\right)}{6a^2b^2} + (\alpha+\beta)^2.$$

Minimizing $\Delta X \Delta Y \Delta Z \Rightarrow a = b = \sqrt{\theta}/3^{1/4}$
so that

$$(\Delta X \Delta Y \Delta Z)_{min} = \left(\lambda\frac{\theta}{2}\right)^{3/2} \qquad (\Delta X)^2 = (\Delta Y)^2 = (\Delta Z)^2 = \theta/\sqrt{3}$$

For $\theta = \hbar$ this agrees with
[S. Kechrimparis, S. Weigert, J. Math. Phys. 9 (2014) 062118]

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