Eternal life of entropy in time-dependent non-Hermitian quantum systems

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EPJP (2018) 133:57(9); Mod. Phys. Lett. A (2019) 2050041

$\mathcal{PT}$-quantum mechanics (real eigenvalues)

- $\mathcal{PT}$-symmetry: $\mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$
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$\mathcal{PT}$-symmetry is only an example of an antilinear involution
[E. Wigner, *J. Math. Phys.* 1 (1960) 409]
[C. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243]

$\mathcal{H}$ is Hermitian with respect to a new metric

- Assume pseudo-Hermiticity:

$$h = \eta \mathcal{H} \eta^{-1} = h^{\dagger} = (\eta^{-1})^{\dagger} \mathcal{H}^{\dagger} \eta^{\dagger} \quad \Leftrightarrow \quad \mathcal{H}^{\dagger} \eta^{\dagger} \eta = \eta^{\dagger} \eta \mathcal{H}$$

$$\Phi = \eta^{-1} \phi \qquad \eta^{\dagger} = \eta$$

$\Rightarrow$ $\mathcal{H}$ is Hermitian with respect to the new metric

Proof:

$$\langle \Psi | \mathcal{H} \Phi \rangle_{\eta} = \langle \Psi | \eta^2 \mathcal{H} \Phi \rangle = \langle \eta^{-1} \psi | \eta^2 \mathcal{H} \eta^{-1} \phi \rangle = \langle \psi | \eta \mathcal{H} \eta^{-1} \phi \rangle =$$
$$\langle \psi | h \phi \rangle = \langle h \psi | \phi \rangle = \langle \eta \mathcal{H} \eta^{-1} \psi | \phi \rangle = \langle \mathcal{H} \Psi | \eta \phi \rangle = \langle \mathcal{H} \Psi | \eta^2 \Phi \rangle$$
$$= \langle \mathcal{H} \Psi | \Phi \rangle_{\eta}$$

$\Rightarrow$ Eigenvalues of $\mathcal{H}$ are real, eigenstates are orthogonal

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Time-dependent Dyson operator

$$\phi(t) = \eta(t)\Psi(t)$$

$\Rightarrow$ Time-dependent Dyson relation

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t)$$

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$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t)$$

$\Rightarrow$ Time-dependent quasi-Hermiticity relation

$$H^\dagger\rho(t) - \rho(t)H = i\hbar\partial_t\rho(t)$$

[from conjugating Dyson relation and $\rho(t) := \eta^\dagger(t)\eta(t)$)]

$H(t)$ governs unitary time-evolution:

Hermitian:

$$\phi(t) = u(t,t')\phi(t'), \qquad u(t,t') = T\exp\left[-i\int_{t'}^{t} ds\, h(s)\right]$$

with

$$h(t)u(t,t') = i\hbar\partial_t u(t,t'), \quad u(t,t')u(t',t'') = u(t,t''), \quad u(t,t) = \mathbb{I}$$

$$\left\langle u(t,t')\phi(t') \middle| u(t,t')\tilde{\phi}(t') \right\rangle = \left\langle \phi(t) \middle| \tilde{\phi}(t) \right\rangle$$

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Non-Hermitian:

$$\Psi(t) = U(t,t')\Psi(t'), \qquad U(t,t') = T\exp\left[-i\int_{t'}^{t} ds H(s)\right]$$

$$H(t)U(t,t') = i\hbar\partial_t U(t,t'),\ U(t,t')U(t',t'') = U(t,t''),\ U(t,t) = \mathbb{I}$$

$$\left\langle U(t,t')\Psi(t') \middle| U(t,t')\tilde{\Psi}(t') \right\rangle_\rho = \left\langle \Psi(t) \middle| \tilde{\Psi}(t) \right\rangle_\rho$$

Relation between $u(t, t')$ and $U(t, t')$:

$$U(t, t') = \eta^{-1}(t) u(t, t') \eta(t')$$

or the generalized Duhamel's formula

$$\begin{aligned} U(t, t') &= u(t, t') - \int_{t'}^{t} \frac{d}{ds} \left[U(t, s) u(s, t') \right] ds \\ &= u(t, t') - i\hbar \int_{t'}^{t} U(t, s) \left[H(s) - h(s) \right] u(s, t') ds \end{aligned}$$

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Relation between Green's functions:

$$G_h(t,t') := -iu(t,t')\theta(t-t') \quad G_H(t,t') := -iU(t,t')\theta(t-t')$$

$$G_U(t,t') = G_u(t,t') + i \int_{-\infty}^{\infty} G_U(t,s) \left[H(s) - h(s) \right] G_u(s,t') ds$$

$H(t)$ is nonobservable and not the energy operator

Observables $o(t)$ in the Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian $\mathcal{O}(t)$ are quasi Hermitian

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Since $H(t)$ is not quasi/pseudo Hermitian it is not an observable.
The observable energy operator is

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\hbar\eta^{-1}(t)\partial_t\eta(t).$$

Three scenarios:

1. $\partial_t \eta = 0$, $\partial_t H \neq 0$, $\partial_t h \neq 0$

   Technically reduces to time-independent case.

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3. $\partial_t \eta \neq 0$, $\partial_t H \neq 0$, $\partial_t h \neq 0$

   - Solve full quasi-Hermiticity relation for $\rho(t)$
     $\Rightarrow \eta(t)$ from $\rho(t) := \eta^\dagger(t)\eta(t)$
   - Solve full time-dependent Dyson equation $\eta(t)$
     $\Rightarrow \rho(t)$ from $\rho(t) := \eta^\dagger(t)\eta(t)$

Making sense of the broken $\mathcal{PT}$-regime:

Two-level system

$$H = -\frac{1}{2}\left[\omega\mathbb{I} + \lambda\sigma_z + i\kappa\sigma_x\right]$$

with eigensystem

$$E_\pm = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}, \qquad \varphi_\pm = \begin{pmatrix} i(-\lambda \pm \sqrt{\lambda^2 - \kappa^2}) \\ \kappa \end{pmatrix}$$

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Claim: This system has real energies for $|\lambda(t)| < |\kappa(t)|$!

Two-dimensional system with infinite dimensional Hilbert space

$$H_K = aK_1 + bK_2 + i\lambda K_3, \qquad a, b, \lambda \in \mathbb{R}$$

with Lie algebraic generators

$$\begin{aligned} K_1 &= \frac{1}{2}\left(p_x^2 + x^2\right), \quad K_2 = \frac{1}{2}\left(p_y^2 + y^2\right), \quad K_3 = \frac{1}{2}\left(xy + p_x p_y\right) \\ K_4 &= \frac{1}{2}\left(xp_y - yp_x\right) \end{aligned}$$

$$\begin{array}{lll} [K_1, K_2] = 0, & [K_1, K_3] = iK_4, & [K_1, K_4] = -iK_3, \\ [K_2, K_3] = -iK_4, & [K_2, K_4] = iK_3, & [K_3, K_4] = i(K_1 - K_2)/2 \end{array}$$

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- H_K is $\mathcal{PT}$-symmetric: $[\mathcal{PT}_\pm, H_{xy}] = 0$

  $\mathcal{PT}_\pm : x \to \pm x,\ y \to \mp y,\ p_x \to \mp p_x,\ p_y \to \pm p_y,\ i \to -i$

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$$K_4 = \frac{1}{2}\left(xp_y - yp_x\right)$$

$$[K_1, K_2] = 0, \qquad [K_1, K_3] = iK_4, \qquad [K_1, K_4] = -iK_3,$$
$$[K_2, K_3] = -iK_4, \qquad [K_2, K_4] = iK_3, \qquad [K_3, K_4] = i(K_1 - K_2)/2$$

- H_K is $\mathcal{PT}$-symmetric: $[\mathcal{PT}_\pm, H_{xy}] = 0$

  $\mathcal{PT}_\pm : x \to \pm x,\ y \to \mp y,\ p_x \to \mp p_x,\ p_y \to \pm p_y,\ i \to -i$
- H_K is quasi-Hermitian: $h_K = \eta H_K \eta^{-1}$

$$h_K = \frac{1}{2}(a+b)\left(K_1 + K_2\right) + \frac{1}{2}\sqrt{(a-b)^2 - \lambda^2}\left(K_1 - K_2\right)$$

with $\eta = e^{2\theta K_4}$, $\theta = \operatorname{arctanh}[\lambda/(b-a)]$

Spontaneously broken $\mathcal{PT}$-symmetry for $a = b$:

Eigenenergies:

$$E_{n,m} = E^*_{m,n} = a(1 + n + m) + i\frac{\lambda}{2}(n - m)$$

Eigenfunctions:

$$\varphi_{n,m}(x,y) = \frac{e^{-\frac{x^2}{2}-\frac{y^2}{2}}}{2^{n+m}\sqrt{n!m!\pi}}\left[\sum_{k=0}^{n}\binom{n}{k}H_k(x)H_{n-k}(y)\right] \times \left[\sum_{l=0}^{m}(-1)^l\binom{m}{l}H_l(y)H_{m-l}(x)\right]$$

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Claim: This system has real energies for $a(t)$, $\lambda(t)$!

Time-dependent system:

$$H(t) = \frac{a(t)}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right) + i\frac{\lambda(t)}{2}\left(xy + p_x p_y\right), \;\; a(t), \lambda(t) \in \mathbb{R}$$

Ansatz:

$$\eta(t) = \prod_{i=1}^{4} e^{\gamma_i(t) K_i}, \qquad \gamma_i \in \mathbb{R}$$

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Time-dependent Dyson equations is satisfied when

Constraint:

$$\gamma_1 = \gamma_2 = q_1, \quad \dot{\gamma}_3 = -\lambda \cosh \gamma_4, \quad \dot{\gamma}_4 = \lambda \tanh \gamma_3 \sinh \gamma_4,$$

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Solution: $\gamma_4 = \operatorname{arcsinh}\left(\kappa \operatorname{sech}\gamma_3\right), \; \chi(t) := \cosh\gamma_3, \; \kappa = const$
with dissipative Ermakov-Pinney equation

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\dot{\chi} - \lambda^2\chi = \frac{\kappa^2\lambda^2}{\chi^3}$$

Instanteneous energies are real even in the broken $\mathcal{PT}$ regime !

Systems solved so far:

- non-Hermitian Swanson model
- one-site lattice Yang-Lee model
- non-Hermitian spin 1/2, 1 and 3/2 models
- two dimensional systems with infinite Hilbert space
- general Lie algebraic Hamiltonians (quasi-exactly solvable)

Von Neumann entropy in $\mathcal{PT}$-symmetric systems

statitical ensemble of states:

$$\varrho_h = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right|$$

partial traces (for subsystems)

$$\begin{aligned}\varrho_{h,A} &= \mathrm{Tr}_B(\varrho_h) = \sum_i \left\langle n_{i,B}\right| \varrho_h \left|n_{i,B}\right\rangle \\ \varrho_{h,B} &= \mathrm{Tr}_A(\varrho_h) = \sum_i \left\langle n_{i,A}\right| \varrho_h \left|n_{i,A}\right\rangle\end{aligned}$$

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Time evolution:

$$i\partial_t \varrho_h = [h, \varrho_h]$$

It follows

$$i\partial_t \varrho_H = [h, \varrho_H]$$

with

$$\varrho_h = \eta \varrho_H \eta^{-1}, \quad h = \eta H \eta^{-1} + i\partial_t \eta \eta^{-1}$$

Therefore

$$\varrho_H = \sum\nolimits_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \rho$$

recalling that $\rho = \eta^\dagger \eta$, $\left|\phi_i\right\rangle = \eta \left|\psi_i\right\rangle$
Von Neumann entropy

$$S_h = -\operatorname{tr}\left[\rho_h \ln \rho_h\right] = -\sum\nolimits_i \lambda_i \ln \lambda_i = S_H$$

Entropy of a subsystem

$$S_{h,A} = -\operatorname{tr}\left[\varrho_{h,A} \ln \varrho_{h,A}\right] = -\sum\nolimits_i \lambda_{i,A} \ln \lambda_{i,A} = S_{H,A}$$

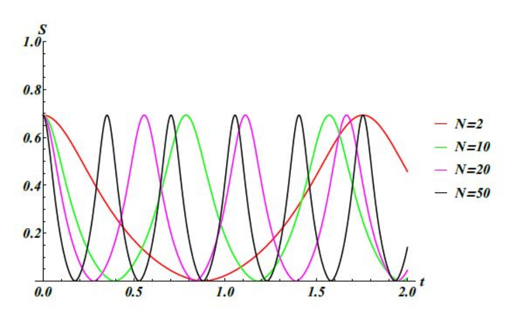
An example: bosonic system coupled to a bath

$$H = \nu a^\dagger a + \nu \sum_{n=1}^{N} q_n^\dagger q_n + (g+\kappa) a^\dagger \sum_{n=1}^{N} q_n + (g-\kappa) a \sum_{n=1}^{N} q_n^\dagger$$

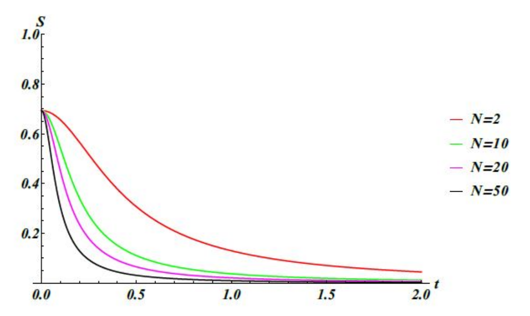
energy eigenvalues

$$E_{m,N}^{\pm} = m\left(\nu \pm \sqrt{N}\sqrt{g^2-\kappa^2}\right)$$

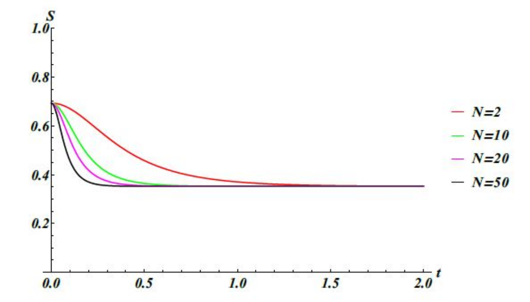
Von-Neumann entropy in the $\mathcal{PT}$ symmetric regime



Von-Neumann entropy at the exceptional point



Von-Neumann entropy in the broken $\mathcal{PT}$ regime



Conclusions

- The broken $\mathcal{PT}$ becomes physical

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- There are new physical effects in this regime