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Noncommutative quantum mechanics in a time-dependent background

Andreas Fring

14th International Workshop on Pseudo-Hermitian Hamiltonians
University of Ferhat Abbas (Setif, Algeria), 5-10/09/2014

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based on: Sanjib Dey and Andreas Fring; arXiv:1407.4843

Content

- Introduction to noncommutative spaces
- The 2D harmonic oscillator in a time-dependent background
- The Ermakov-Pinney equation
- The generalized uncertainty relations
- Conclusions and outlook

Noncommutative spaces

- Flat (abelian) noncommutative space:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

In 3D:

$$\begin{array}{llll} [x_0, y_0] = i\theta_1, & [x_0, z_0] = i\theta_2, & [y_0, z_0] = i\theta_3, & \theta_1, \theta_2, \theta_3 \in \mathbb{R} \\ [x_0, p_{x_0}] = i\hbar, & [y_0, p_{y_0}] = i\hbar, & [z_0, p_{z_0}] = i\hbar, & \end{array}$$

Noncommutative spaces

- Flat (abelian) noncommutative space:

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- Snyder spaces, from twists:

$$[x_i, x_j] = i\theta(x_i p_j - x_j p_i)$$

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$$[x_i, x_j] = i\theta(x_i p_j - x_j p_i)$$

- Minimal length spaces, from q-deformed algebras:

$$[x_i, x_j] \approx i\theta(x_j)^2$$

Minimal lengths, areas and volumes

Uncertainty relation:

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle_{\rho} \right|$$

- Standard case:
  $[A, B] = const$; give up knowledge about $B \Rightarrow \Delta A = 0$

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  $[A, B] \approx B^2$; even give up knowledge about $B \Rightarrow \Delta A \neq 0$
- For instance:

$$[X, P] = i\hbar \left(1 + \tau P^2\right)$$

$\Rightarrow$ minimal length

$$\Delta X_{\mathrm{min}} = \hbar \sqrt{\tau} \sqrt{1 + \tau \langle P^2 \rangle_\rho}$$

from minimizing with $(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2$

$\mathcal{PT}$-symmetric noncommutative spaces

P. Giri, P. Roy, Eur. Phys. C60 (2009) 157: $\nexists$ $\mathcal{PT}$-symmetry

$$[x_0, y_0] = i\theta_1, \qquad [x_0, z_0] = i\theta_2, \qquad [y_0, z_0] = i\theta_3, \qquad \theta_1, \theta_2, \theta_3 \in \mathbb{R}$$
$$[x_0, p_{x_0}] = i\hbar, \qquad [y_0, p_{y_0}] = i\hbar, \qquad [z_0, p_{z_0}] = i\hbar,$$

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$$\mathcal{PT}_{xz}: \quad x_0 \to z_0, \quad y_0 \to y_0, \quad z_0 \to x_0, \quad i \to -i,$$
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- Useful to reduce number of free parameters.

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- Useful to reduce number of free parameters.
- Models on these spaces will have the usual nice properties.

A particular $\mathcal{PT}_{\pm}$-symmetric solution

from q-deformed oscillator algebra
[S. Dey, A. Fring, L. Gouba, J. Phys. A45 (2012) 385302]

$$[X, Y] = i\theta_1 + i\frac{q^2-1}{q^2+1}\frac{\theta_1}{\hbar}\left[\frac{m\omega}{2\kappa_6^2}Y^2 + \frac{2\kappa_6^2}{m\omega}P_y^2\right]$$

$$[Y, Z] = i\theta_3 + i\frac{q^2-1}{q^2+1}\frac{\theta_3}{\hbar}\left[\frac{m\omega}{2\kappa_6^2}Y^2 + \frac{2\kappa_6^2}{m\omega}P_y^2\right]$$

$$[X, P_x] = i\hbar + i\frac{q^2-1}{q^2+1}2m\omega\left[\kappa_{11}^2 X^2 + \frac{P_x^2/4}{m^2\omega^2\kappa_{11}^2} + \frac{\theta_1^2\kappa_{11}^2 P_y^2}{\hbar^2} + \frac{\theta_1\kappa_{11}^2 X P_y}{\hbar/2}\right]$$

$$[Y, P_y] = i\hbar + i\frac{q^2-1}{q^2+1}2m\omega\left[\frac{1}{4\kappa_6^2}Y^2 + \frac{\kappa_6^2}{m^2\omega^2}P_y^2\right]$$

$$[Z, P_z] = i\hbar + i\frac{q^2-1}{q^2+1}2m\omega\left[\frac{Z^2}{4\kappa_7^2} + \frac{\kappa_7^2}{m^2\omega^2}P_z^2 + \frac{\theta_3^2}{4\hbar^2\kappa_7^2}P_y^2 - \frac{\theta_3 Z P_y}{2\hbar^2\kappa_7^2}\right]$$

- Reduced three dimensional solution for $q \to 1$

$$
\begin{aligned}
[X, Y] &= i\theta_1 \left(1 + \hat{\tau} Y^2\right), \quad [Y, Z] = i\theta_3 \left(1 + \hat{\tau} Y^2\right), \\
[X, P_x] &= i\hbar \left(1 + \check{\tau} P_x^2\right), \quad [Y, P_y] = i\hbar \left(1 + \hat{\tau} Y^2\right) \\
[Z, P_z] &= i\hbar \left(1 + \check{\tau} P_z^2\right)
\end{aligned}
$$

where $\hat{\tau} = \tau m\omega/\hbar$, $\check{\tau} = \tau/(m\omega\hbar)$

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where $\hat{\tau} = \tau m\omega/\hbar$, $\check{\tau} = \tau/(m\omega\hbar)$

- Representation in flat noncommutative space:

$$\begin{aligned} X &= (1 + \check{\tau} p_{x_0}^2) x_0 + \frac{\theta_1}{\hbar} \left(\check{\tau} p_{x_0}^2 - \hat{\tau} y_0^2\right) p_{y_0}, \qquad & P_x &= p_{x_0}, \\ Z &= (1 + \check{\tau} p_{z_0}^2) z_0 + \frac{\theta_3}{\hbar} \left(\hat{\tau} y_0^2 - \check{\tau} p_{z_0}^2\right) p_{y_0}, \qquad & P_z &= p_{z_0}, \\ P_y &= (1 + \hat{\tau} y_0^2) p_{y_0}, \qquad & Y &= y_0. \end{aligned}$$

- Reduced three dimensional solution for $q \to 1$

$$\begin{aligned} [X,Y] &= i\theta_1\left(1+\hat{\tau}Y^2\right), \quad [Y,Z]=i\theta_3\left(1+\hat{\tau}Y^2\right), \\ [X,P_x] &= i\hbar\left(1+\check{\tau}P_x^2\right), \quad [Y,P_y]=i\hbar\left(1+\hat{\tau}Y^2\right) \\ [Z,P_z] &= i\hbar\left(1+\check{\tau}P_z^2\right) \end{aligned}$$

where $\hat{\tau}=\tau m\omega/\hbar$, $\check{\tau}=\tau/(m\omega\hbar)$

- Representation in flat noncommutative space:

$$\begin{aligned} &X=(1+\check{\tau}p_{x_0}^2)x_0+\frac{\theta_1}{\hbar}\left(\check{\tau}p_{x_0}^2-\hat{\tau}y_0^2\right)p_{y_0}, && P_x=p_{x_0}, \\ &Z=(1+\check{\tau}p_{z_0}^2)z_0+\frac{\theta_3}{\hbar}\left(\hat{\tau}y_0^2-\check{\tau}p_{z_0}^2\right)p_{y_0}, && P_z=p_{z_0}, \\ &P_y=(1+\hat{\tau}y_0^2)p_{y_0}, && Y=y_0. \end{aligned}$$

- Bopp-shift to standard canonical variables:

$x_0 \to x_s - \frac{\theta_1}{\hbar}p_{y_s}$, $y_0 \to y_s$, $z_0 \to z_s + \frac{\theta_3}{\hbar}p_{y_s}$,
$p_{x_0} \to p_{x_s}$, $p_{y_0} \to p_{y_s}$, $p_{z_0} \to p_{z_s}$

- Dyson map: $\eta = \eta_{y_0}\eta_{p_{x_0}}\eta_{p_{z_0}}$

$$\eta_{y_0} = (1+\hat{\tau}y_0^2)^{-1/2},\ \eta_{p_{x_0}} = (1+\check{\tau}p_{x_0}^2)^{-1/2},\ \eta_{p_{z_0}} = (1+\check{\tau}p_{z_0}^2)^{-1/2}$$

- Hermitian variables:

$$\begin{aligned}
x &:= \eta X \eta^{-1} = \eta_{p_{x_0}}^{-1}\left(x_0 + \frac{\theta_1}{\hbar}\right)\eta_{p_{x_0}}^{-1} - \frac{\theta_1}{\hbar}\eta_{y_0}^{-1}p_{y_0}\eta_{y_0}^{-1} = x^\dagger \\
y &:= \eta Y \eta^{-1} = y_0 = y^\dagger \\
z &:= \eta Z \eta^{-1} = \eta_{p_{z_0}}^{-1}\left(z_0 - \frac{\theta_3}{\hbar}\right)\eta_{p_{z_0}}^{-1} + \frac{\theta_3}{\hbar}\eta_{y_0}^{-1}p_{y_0}\eta_{y_0}^{-1} = z^\dagger \\
p_x &:= \eta P_x \eta^{-1} = p_{x_0} = p_x^\dagger \\
p_y &:= \eta P_y \eta^{-1} = \eta_{y_0}^{-1}p_{y_0}\eta_{y_0}^{-1} = p_y^\dagger \\
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\end{aligned}$$

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\end{aligned}$$

- Isospectral Hermitian counterpart:

$$H(X,Y,Z,P_x,P_y,P_z) \neq H^\dagger(X,Y,Z,P_x,P_y,P_z) \Rightarrow h = \eta H \eta^{-1} = h^\dagger$$

- Metric: $\rho = \eta^2$

Different types of representations (1D)

$$[X,P] = i\hbar\left(1+\check{\tau}P^2\right)$$

non-Hermitian: $X_{(1)} = (1+\check{\tau}p^2)x, \quad P_{(1)} = p$

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non-Hermitian: $X_{(4)} = ix(1 + \check{\tau} p^2)^{1/2}, \; P_{(4)} = -ip(1 + \check{\tau} p^2)^{-1/2}$

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Hermitian: $X_{(2)} = (1 + \check{\tau} p^2)^{1/2} x (1 + \check{\tau} p^2)^{1/2}, \quad P_{(2)} = p$

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The physics is the same for all representations

$$\left\langle \psi_{(i)} \right| F\left(P_{(i)}, X_{(i)}\right) \left. \psi_{(i)} \right\rangle_{\rho_{(i)}}$$

$$= \frac{1}{N}\int_{-1}^{1} F\left[\frac{z}{\sqrt{\check{\tau}(1-z^2)}}, i\hbar\sqrt{\check{\tau}(1-z^2)}\partial_z\right]\left|P_{m-\mu_-}^{\mu_-}(z)\right|^2 dz$$

[S. Dey, A. Fring, B. Khantoul, J. Phys. A46 (2013) 335304]

Time-dependent noncommutativity (2D)

$$
\begin{aligned}
[X, Y] &= i\theta(t) \\
[P_x, P_y] &= i\Omega(t), \\
[X, P_x] &= i\hbar + i\frac{\theta(t)\Omega(t)}{4\hbar} \\
[Y, P_y] &= i\hbar + i\frac{\theta(t)\Omega(t)}{4\hbar}
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$\Rightarrow$ time-dependent Hamiltonians $H(X, Y, P_x, P_y) \rightarrow H(t)$.

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$\Rightarrow$ time-dependent Hamiltonians $H(X, Y, P_x, P_y) \to H(t)$.
Representation:

$$
\begin{aligned}
X &= x - \frac{\theta(t)}{2\hbar}p_y, \quad Y = y + \frac{\theta(t)}{2\hbar}p_x, \\
P_x &= p_x + \frac{\Omega(t)}{2\hbar}y, \quad P_y = p_y - \frac{\Omega(t)}{2\hbar}x.
\end{aligned}
$$

with nonvanishing commutators $[x, p_x] = [y, p_y] = i\hbar$

Lewis-Riesenfeld theory of invariants

Aim: solve time-dependent Schrödinger equation

$$i\hbar\partial_t |\psi_n\rangle = H(t)|\psi_n\rangle$$

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[H. Lewis, W. Riesenfeld, J. Math. Phys. 10, 1458 (1969)]

Example: The 2D harmonic oscillator

$$H(X,Y,P_x,P_y) = \frac{1}{2m}\left(P_x^2 + P_y^2\right) + \frac{m\omega^2}{2}(X^2+Y^2),$$

Using the above representation

$$\begin{aligned} H(t) &= \frac{1}{2}a(t)\left(p_x^2+p_y^2\right) + \frac{1}{2}b(t)\left(x^2+y^2\right) + c(t)\left(p_x y - x p_y\right) \\ &= \frac{1}{2}a(t)\left(p_r^2 + \frac{p_\theta^2}{r^2} - \frac{\hbar^2}{4r^2}\right) + \frac{1}{2}b(t)r^2 - c(t)p_\theta \end{aligned}$$

with coefficients

$$a(t) = \frac{1}{m} + \frac{m\omega^2}{4\hbar^2}\theta^2(t),\ \ b(t) = m\omega^2 + \frac{\Omega^2(t)}{4m\hbar^2},\ \ c(t) = \frac{m\omega^2\theta(t)}{2\hbar} + \frac{\Omega(t)}{2\hbar m}$$

Step 1 in LR-theory

The Ansatz:

$$I(t) = \alpha(t)p_r^2 + \beta(t)r^2 + \gamma(t)\{r, p_r\} + \delta(t)\frac{p_\theta^2}{r^2} + \varepsilon(t)\frac{p_\theta}{r^2} + \phi(t)\frac{1}{r^2}$$

leads to the set of coupled differential equations

$$\dot{\alpha} = -2a\gamma, \quad \dot{\beta} = 2b\gamma, \quad \dot{\gamma} = b\alpha - a\beta$$

$$\dot{\delta}p_\theta^2 + \dot{\varepsilon}p_\theta + \dot{\phi} = \hbar^2 a\gamma - 2a\gamma p_\theta^2, \quad (\delta - \alpha)\, p_\theta^2 + \varepsilon p_\theta + \phi + \frac{\alpha\hbar^2}{4} = 0$$

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which are solved by

$$I(t) = \frac{\tau}{\sigma^2}r^2 + \left(\sigma p_r - \frac{\dot{\sigma}}{a}r\right)^2 + \frac{\sigma^2 p_\theta^2}{r^2} - \frac{\sigma^2\hbar^2}{4r^2}$$

$\tau = const$, $\sigma(t)$ solves the Ermakov-Pinney equation

$$\ddot{\sigma} - \frac{\dot{a}}{a}\dot{\sigma} + ab\sigma = \tau\frac{a^2}{\sigma^3}$$

Step 2 in LR-theory

Rewrite $I(t)$ in terms of time-dependent creation and annihilation operators

$$\hbar\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)-p_{\theta}=\frac{1}{4}I(t)-\frac{1}{2}p_{\theta}=:\hat{I}(t)$$

$$\begin{aligned}
\hat{a}(t) &= \frac{1}{2\sqrt{\hbar}}\left[\left(\sigma p_r-\frac{\dot{\sigma}}{a}r\right)-i\left(\frac{r}{\sigma}+\frac{\sigma}{r}(p_\theta+\frac{\hbar}{2})\right)\right]e^{-i\theta} \\
\hat{a}^{\dagger}(t) &= \frac{1}{2\sqrt{\hbar}}e^{i\theta}\left[\left(\sigma p_r-\frac{\dot{\sigma}}{a}r\right)+i\left(\frac{r}{\sigma}+\frac{\sigma}{r}(p_\theta+\frac{\hbar}{2})\right)\right]
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From standard arguments:

$$\psi_{n,m-n}=\lambda_n\frac{\left(i\hbar^{1/2}\sigma\right)^m}{\sqrt{m!}}r^{n-m}e^{i\theta(m-n)-\frac{a-i\sigma\dot\sigma}{2a\hbar\sigma^2}r^2}U\left(-m,1-m+n,\frac{r^2}{\hbar\sigma^2}\right)$$

with normalization constant

$$\lambda_n^2=\frac{1}{\pi n!(\hbar\sigma^2)^{(1+n)}}$$

Step 3 in LR-theory

We fix the phase by solving:

$$\dot{\alpha}_{n,\ell} = \frac{1}{\hbar} \langle n, \ell | \, i\hbar\partial_t - H \, | n, \ell \rangle$$

to

$$\alpha_{n,\ell}(t) = (n+\ell) \int^t \left(c(s) - \frac{a(s)}{\sigma^2(s)} \right) ds$$

The Ermakov-Pinney equation

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- For $\dot{a} = 0$, i.e. $\theta(t) = const$ particular solutions are known

$$\sigma = \sqrt{u_1^2 + \tau a^2 \frac{u_2^2}{W^2}},$$

where u_1, u_2 solve $\ddot{u} + ab(t)u = 0$ and $W = u_1\dot{u}_2 - \dot{u}_1 u_2$
[E. Pinney, Proc. Amer. Math. Soc. 1, 681(1) (1950)]

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[E. Pinney, Proc. Amer. Math. Soc. 1, 681(1) (1950)]
For instance for $a(t) = \alpha$ and $b(t) = \beta e^{\gamma t}$, $\alpha, \beta, \gamma \in \mathbb{R}$

$$\sigma(t) = \sqrt{\frac{\pi^2\alpha^2\tau}{\gamma^2 c_1^2} Y_0^2\left(\frac{2\sqrt{\alpha\beta}e^{\gamma t/2}}{\gamma}\right) + c_1^2 J_0^2\left(\frac{2\sqrt{\alpha\beta}e^{\gamma t/2}}{\gamma}\right)},$$

with J_0, Y_0 Bessel functions of first and second kind.

- When $\dot{a} \neq 0$ no general solution is known.
  Special solution:

$$\frac{1}{\lambda_k} \int^{\sigma} \frac{\dot{a} s^3}{\tau a^3 - a^2 b s^4} ds = t \qquad \lambda_\kappa^\pm = \frac{-1 \pm \sqrt{1 - 4\kappa}}{2\kappa}$$

when Chiellini integrability condition holds

$$\frac{d}{d\sigma} \left(\frac{\dot{a} s^3}{\tau a^3 - a^2 b s^4} \right) = -\kappa \frac{\dot{a}}{a} \qquad \kappa \in \mathbb{R}$$

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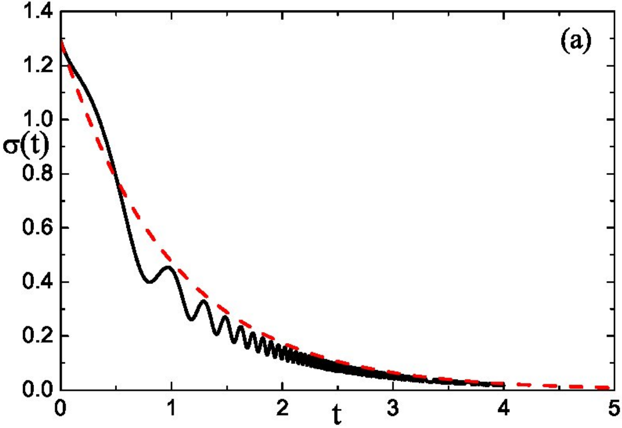
  $\Rightarrow$ Does not allow to pre-select $\Theta(t)$ and $\Omega(t)$.
  $\Rightarrow$ Resort to numerical solutions.

Sample solutions:



(a) Exactly integrable solution (red, dashed) versus a non-Chiellini integrable solution for pre-selected exponential backgrounds $\theta(t) = \alpha e^{-\gamma t}$ and $\Omega(t) = \beta e^{\gamma t}$ (black, solid). $\alpha = 5$, $\beta = 2$, $\gamma = 2$, $m = \hbar = \tau = \omega = 1$, $\kappa = 1/4$, $\mu = \sqrt{5/3}$

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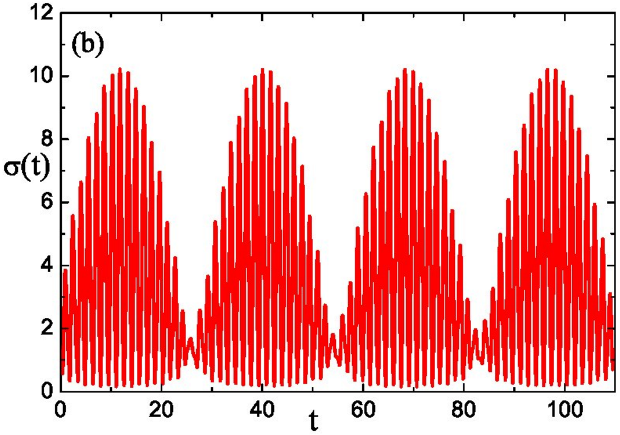

(b) Non-Chiellini integrable solution for pre-selected sinusoidal background $\theta(t) = \alpha \sin(\gamma t)$ and $\Omega(t) = \beta \sin(\gamma t/2)$.
$\alpha = 5$, $\beta = 2$, $\gamma = 2$, $m = \hbar = \tau = \omega = 1$, $\kappa = 1/4$, $\mu = \sqrt{5/3}$

Generalized uncertainty relations

In general we have:

$$\Delta A \Delta B\big|_{\psi} \geq \frac{1}{2} \left| \langle\psi| \left[A, B\right] |\psi\rangle \right|$$

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$$\Delta X \Delta Y|_{\psi_{n,m-n}} = \frac{n-m}{2}\theta(t) + \frac{n+m+1}{8\hbar}\left[4\hbar\sigma^2 + \left(\frac{1}{\sigma^2} + \frac{\dot{\sigma}^2}{a^2}\right)\theta^2(t)\right]$$

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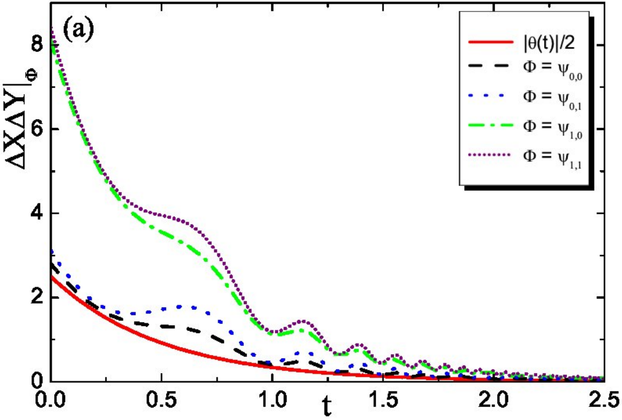
&\geq \frac{\theta(t)}{2} \\
\Delta P_x \Delta P_y|_{\psi_{n,m-n}} &= \frac{n-m}{2}\Omega(t) + \frac{\hbar}{2}(n+m+1)\left[\frac{\sigma^2\Omega^2(t)}{4} + \left(\frac{1}{\sigma^2} + \frac{\dot{\sigma}^2}{a^2}\right)\right] \\
&\geq \frac{\Omega(t)}{2}
\end{aligned}$$

$$\Delta X \Delta P_x|_{\psi_{n,m-n}} = \Delta Y \Delta P_y|_{\psi_{n,m-n}} \geq \frac{\hbar}{2} + \frac{\theta(t)\Omega(t)}{8\hbar}$$

Examples:



(a) for background fields $\theta(t) = \alpha e^{-\gamma t}$ and $\Omega(t) = \beta e^{\gamma t}$
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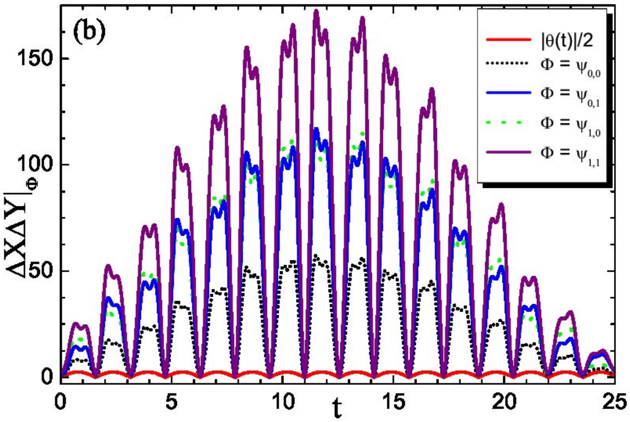

(b) for background fields $\theta(t) = \alpha \sin(\gamma t)$, $\Omega(t) = \beta \sin(\gamma t/2)$
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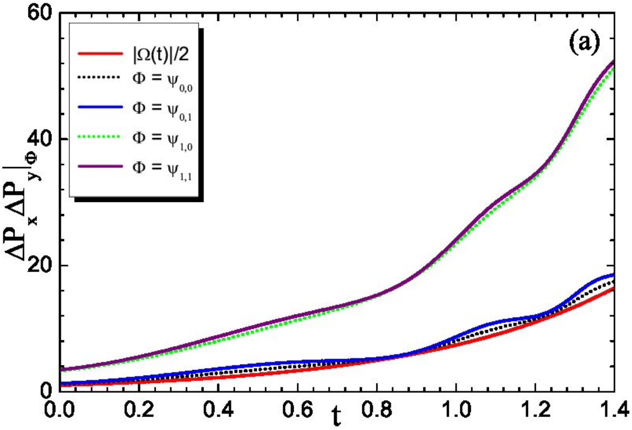

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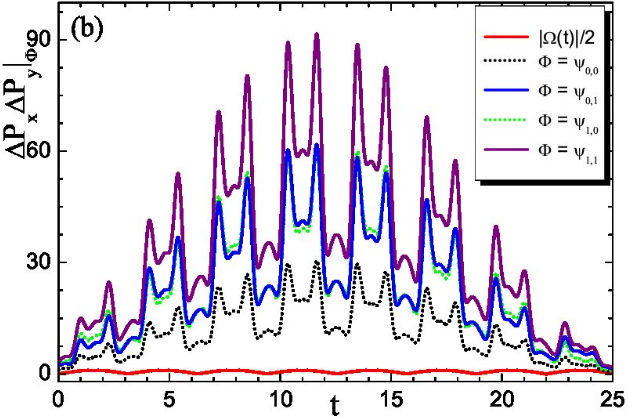

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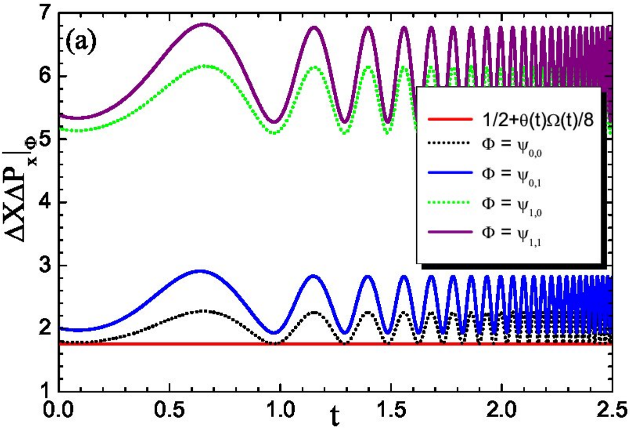

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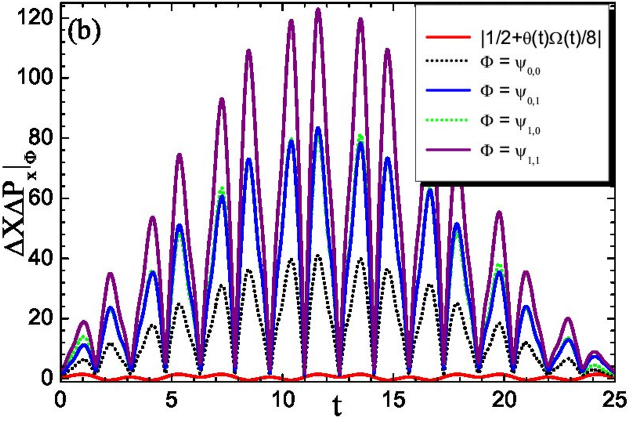

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GUR for coherent states

Glauber coherent states

$$|\alpha, t\rangle := D(\alpha, t)\,|0,0\rangle\,, \quad \text{with} \quad D(\alpha, t) := e^{\alpha \hat{a}^\dagger(t) - \alpha^* \hat{a}(t)}$$

yield

$$\Delta X|^2_{|\alpha,t\rangle} = \Delta Y|^2_{|\alpha,t\rangle} = \Delta X|^2_{\psi_{0,0}} \quad \Delta P_x|^2_{|\alpha,t\rangle} = \Delta P_y|^2_{|\alpha,t\rangle} = \Delta P_x|^2_{\psi_{0,0}}$$

Squeezed coherent states

$$|\alpha, \beta, t\rangle := S(\beta, t) D(\alpha, t)\,|0,0\rangle\,, \quad \text{with} \quad S(\beta, t) := e^{\frac{\beta}{2}[\hat{a}^2(t) - \hat{a}^{\dagger 2}(t)]}$$

yield for instance

$$\Delta X|^2_{|\alpha,\beta,t\rangle} =$$

$$\frac{\hbar}{2}\left[\sigma^2 e^\beta + \frac{\theta^2(t)}{4\hbar^2}\left(\frac{1}{\sigma^2} e^\beta + \frac{\dot{\sigma}^2}{a^2} e^{-\beta}\right)\right] \cosh\beta + \frac{\theta(t)}{4}(1 - e^{2\beta})$$

Use β to minimise uncertainties:

- Possible for generic t for $\Delta x \, \Delta p_x|_{|\alpha,\beta,t\rangle}$:

$$\beta(t) = \beta_{\min}(t) = 1/2 \ln \left[\left(a\sqrt{a^2 + 8\sigma^2\dot{\sigma}^2} - a^2 \right) / (4\sigma^2\dot{\sigma}^2) \right]$$

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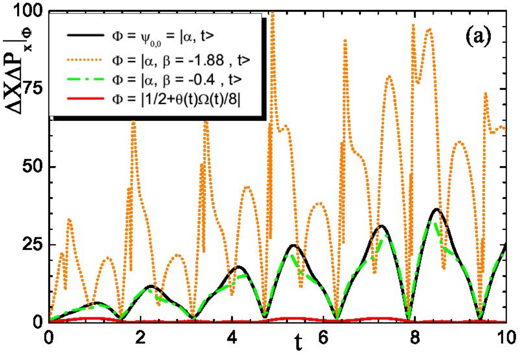
- Possible for specific t for $\Delta X\, \Delta P_x|_{|\alpha,\beta,t\rangle}$, e.g. $t = 4$:

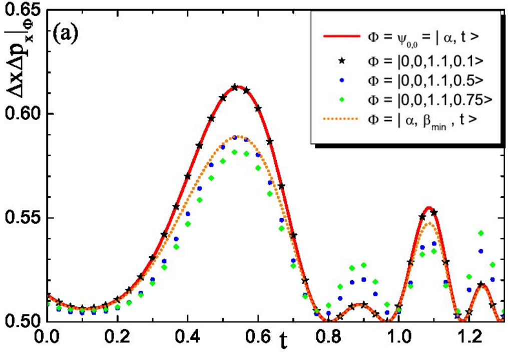


for background fields $\theta(t) = \alpha \sin(\gamma t)$, $\Omega(t) = \beta \sin(\gamma t/2)$

Glauber versus Gaussian Klauder coherent states

$$|n, m_0, \phi_0, s\rangle := \frac{1}{\sqrt{N(m_0)}} \sum_{m=0}^{\infty} \exp\left[-\frac{(m-m_0)^2}{4s^2}\right] e^{im\phi_0} |n, m-n\rangle$$

Quality depends on background fields:



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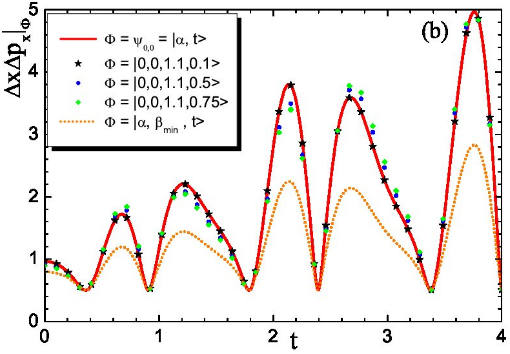

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Thank you for your attention