Complex solitons in integrable systems with real energies, nonlocal gauge equivalence and BPS solutions from duality

Andreas Fring

The 11th DYNAMICS DAY Asia-Pacific,
Singapore, 16-20 November 2020

based on work with
Julia Cen (Los Alamos National Laboratory, USA)
Francisco Correa (Universidad Austral de Chile, Chile)
Takanobu Taira (City, University of London, UK)

J. Cen, A. Fring, J. Phys. A: Math. Theor. 49 (2016) 365202
F. Correa, A. Fring, J. High Energ. Phys. (2016) 2016: 8
J. Cen, F. Correa, A. Fring, J. Math. Phys. 58 (2017) 032901
J. Cen, F. Correa, A. Fring, J. Phys. A 50 (2017) 435201
J. Cen, A. Fring, Physica D: Nonlin Phen. 397 (2019) 17
J. Cen, F. Correa, A. Fring, J. Math. Phys. 60 (2019) 081508
J. Cen, A. Fring, J. of Nonlin Math. Phys. 27 (2020) 17
J. Cen, F. Correa, A. Fring, J. Phys. A 53 (2020) 195201
A. Fring, T.Taira, J. Phys. A: Math. Theor. 53 (2020) 455701

Outline

- Why study complex PT-symmetric solitons?

Outline

- Why study complex PT-symmetric solitons?
  real energies, degeneracy problem, new features

Outline

- Why study complex PT-symmetric solitons?
  real energies, degeneracy problem, new features
- Nonlocality via zero curvature condition

Outline

- Why study complex PT-symmetric solitons?
  real energies, degeneracy problem, new features
- Nonlocality via zero curvature condition
- Nonlocality in solution procedures (Hirota, Darboux)

Outline

- Why study complex PT-symmetric solitons?
  real energies, degeneracy problem, new features
- Nonlocality via zero curvature condition
- Nonlocality in solution procedures (Hirota, Darboux)
- Nonlocal gauge equivalence, extendent continuous limit of Heisenberg spin chain, extendent Landau-Lifschitz equation

Outline

- Why study complex PT-symmetric solitons?
  real energies, degeneracy problem, new features
- Nonlocality via zero curvature condition
- Nonlocality in solution procedures (Hirota, Darboux)
- Nonlocal gauge equivalence, extendent continuous limit of Heisenberg spin chain, extendent Landau-Lifschitz equation
- BPS solutions from duality

Outline

- Why study complex PT-symmetric solitons?
  real energies, degeneracy problem, new features
- Nonlocality via zero curvature condition
- Nonlocality in solution procedures (Hirota, Darboux)
- Nonlocal gauge equivalence, extendent continuous limit of Heisenberg spin chain, extendent Landau-Lifschitz equation
- BPS solutions from duality
- Conclusions and Outlook

Reality of N-Soliton charges

The complex KdV equation equals two coupled real equations

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Leftrightarrow \quad \begin{cases} p_t + 6pp_x + p_{xxx} - 6qq_x = 0 \\ q_t + 6\,(pq)_x + q_{xxx} = 0 \end{cases}$$

with $u(x,t) = p(x,t) + iq(x,t)$, $p(x,t)$, $q(x,t) \in \mathbb{R}$

Reality of N-Soliton charges

The complex KdV equation equals two coupled real equations

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Leftrightarrow \quad \begin{cases} p_t + 6pp_x + p_{xxx} - 6qq_x = 0 \\ q_t + 6\,(pq)_x + q_{xxx} = 0 \end{cases}$$

with $u(x,t) = p(x,t) + iq(x,t)$, $p(x,t)$, $q(x,t) \in \mathbb{R}$

- Unifies some know special cases:
  - for $(pq)_x \to pq_x$: complex KdV $\Rightarrow$ Hirota-Satsuma equations
  - for $q_{xxx} \to 0$ complex KdV $\Rightarrow$ Ito equations

Reality of N-Soliton charges

The complex KdV equation equals two coupled real equations

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Leftrightarrow \quad \begin{cases} p_t + 6pp_x + p_{xxx} - 6qq_x = 0 \\ q_t + 6\left(pq\right)_x + q_{xxx} = 0 \end{cases}$$

with $u(x,t) = p(x,t) + iq(x,t)$, $p(x,t)$, $q(x,t) \in \mathbb{R}$

- Unifies some know special cases:
  - for $(pq)_x \to pq_x$: complex KdV $\Rightarrow$ Hirota-Satsuma equations
  - for $q_{xxx} \to 0$ complex KdV $\Rightarrow$ Ito equations
- $\mathcal{PT}$-symmetry:
  $x \to -x$, $t \to -t$, $i \to -i$, $u \to u$, $p \to p$, $q \to -q$

Reality of N-Soliton charges

The complex KdV equation equals two coupled real equations

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Leftrightarrow \quad \begin{cases} p_t + 6pp_x + p_{xxx} - 6qq_x = 0 \\ q_t + 6(pq)_x + q_{xxx} = 0 \end{cases}$$

with $u(x,t) = p(x,t) + iq(x,t)$, $p(x,t)$, $q(x,t) \in \mathbb{R}$

- Unifies some know special cases:
  - for $(pq)_x \to pq_x$: complex KdV $\Rightarrow$ Hirota-Satsuma equations
  - for $q_{xxx} \to 0$ complex KdV $\Rightarrow$ Ito equations
- $\mathcal{PT}$-symmetry:
  $x \to -x$, $t \to -t$, $i \to -i$, $u \to u$, $p \to p$, $q \to -q$
- Integrability:
  Lax pair:

$$L_t = [M, L] \quad L = \partial_x^2 + \frac{1}{6}u, \;\; M = 4\partial_x^3 + u\partial_x + \frac{1}{2}u_x$$

Solutions from Hirota's direct method

Convert KdV equation into Hirota's bilinear form

$$\left(D_x^4 + D_x D_t\right) \tau \cdot \tau = 0$$

with $u = 2(\ln \tau)_{xx}$. (D_x, D_t are Hirota derivatives)

Solutions from Hirota's direct method

Convert KdV equation into Hirota's bilinear form

$$\left(D_x^4 + D_x D_t\right) \tau \cdot \tau = 0$$

with $u = 2(\ln \tau)_{xx}$. (D_x, D_t are Hirota derivatives)
Expanding $\tau = \sum_{k=0}^{\infty} \lambda^k \tau^k$ gives multi-soliton solutions

$$\begin{aligned}
\tau_{\mu;\alpha}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} \\
\tau_{\mu,\nu;\alpha,\beta}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} + e^{\eta_{\nu;\beta}} + \varkappa(\alpha,\beta) e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}} \\
\tau_{\mu,\nu,\rho;\alpha,\beta,\gamma}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} + e^{\eta_{\nu;\beta}} + e^{\eta_{\rho;\gamma}} + \varkappa(\alpha,\beta) e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}} \\
&\quad + \varkappa(\alpha,\gamma) e^{\eta_{\mu;\alpha}+\eta_{\rho;\gamma}} + \varkappa(\beta,\gamma) e^{\eta_{\nu;\beta}+\eta_{\rho;\gamma}} \\
&\quad + \varkappa(\alpha,\beta)\varkappa(\alpha,\gamma)\varkappa(\beta,\gamma) e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}+\eta_{\rho;\gamma}}
\end{aligned}$$

with $\eta_{\mu;\alpha} := \alpha x - \alpha^3 t + \mu$, $\varkappa(\alpha,\beta) := (\alpha-\beta)^2/(\alpha+\beta)^2$
$\mu, \nu, \rho \in \mathbb{C}$, $\alpha, \beta, \gamma \in \mathbb{R}$

One-soliton solution

We find

$$u_{i\theta;\alpha}(x,t) = \frac{\alpha^2 + \alpha^2 \cos\theta \cosh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2} - i \frac{\alpha^2 \sin\theta \sinh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2}$$

One-soliton solution

We find

$$u_{i\theta;\alpha}(x,t) = \frac{\alpha^2 + \alpha^2 \cos\theta \cosh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2} - i \frac{\alpha^2 \sin\theta \sinh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2}$$

The solution found by Khare and Saxena is the special case

$$u_{\pm i\frac{\pi}{2};\alpha}(x,t) = \alpha^2 \operatorname{sech}^2\left(\alpha x - \alpha^3 t\right) \mp i\alpha^2 \tanh\left(\alpha x - \alpha^3 t\right) \operatorname{sech}\left(\alpha x - \alpha^3 t\right)$$

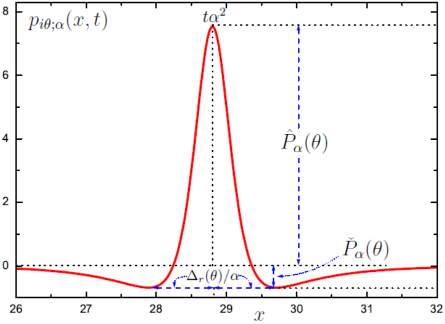
One-soliton solution

We find

$$u_{i\theta;\alpha}(x,t) = \frac{\alpha^2 + \alpha^2 \cos\theta \cosh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2} - i \frac{\alpha^2 \sin\theta \sinh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2}$$

The solution found by Khare and Saxena is the special case

$$u_{\pm i\frac{\pi}{2};\alpha}(x,t) = \alpha^2 \operatorname{sech}^2(\alpha x - \alpha^3 t) \mp i\alpha^2 \tanh(\alpha x - \alpha^3 t) \operatorname{sech}(\alpha x - \alpha^3 t)$$



$$\begin{aligned} \hat{P}_\alpha(\theta) &= \frac{\alpha^2}{2} \sec^2\left(\frac{\theta}{2}\right) \\ \check{P}_\alpha(\theta) &= \frac{\alpha^2}{4} \cot^2(\theta) \\ \Delta_r(\theta) &= \operatorname{arccosh}(\cos\theta - 2\sec\theta) \end{aligned}$$

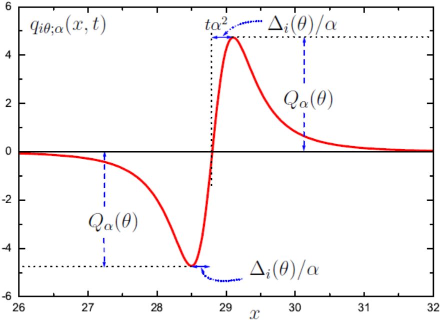
One-soliton solution

We find

$$u_{i\theta;\alpha}(x,t) = \frac{\alpha^2 + \alpha^2\cos\theta\cosh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2} - i\frac{\alpha^2\sin\theta\sinh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2}$$

The solution found by Khare and Saxena is the special case

$$u_{\pm i\frac{\pi}{2};\alpha}(x,t) = \alpha^2\mathrm{sech}^2\left(\alpha x - \alpha^3 t\right) \mp i\alpha^2\tanh\left(\alpha x - \alpha^3 t\right)\mathrm{sech}\left(\alpha x - \alpha^3 t\right)$$



$$Q_\alpha(\theta) = \frac{8\alpha^2\sqrt{5+\cos(2\theta)+\cos\theta A}}{[6\cos\theta + A]^2 / \sin\theta}$$

$$\Delta_i(\theta) = \operatorname{arccosh}\left[\frac{1}{2}\cos\theta + \frac{1}{4}A\right]$$

$$A = \sqrt{2}\sqrt{17+\cos(2\theta)}$$

Real charges from one-soliton solution

$$\begin{aligned}
\text{Mass}: m_\alpha &= \int_{-\infty}^{\infty} u_{i\theta;\alpha}(x,t)dx = 2\alpha \\
\text{Momentum}: p_\alpha &= \int_{-\infty}^{\infty} u_{i\theta;\alpha}^2 dx = \frac{2}{3}\alpha^3 \\
\text{Energy}: E_\alpha &= \int_{-\infty}^{\infty} \left[2u_{i\theta;\alpha}^3 - (u_{i\theta;\alpha})_x^2\right] dx = \frac{2}{5}\alpha^5
\end{aligned}$$

Real charges from one-soliton solution

$$\text{Mass}: m_\alpha = \int_{-\infty}^{\infty} u_{i\theta;\alpha}(x,t)dx = 2\alpha$$

$$\text{Momentum}: p_\alpha = \int_{-\infty}^{\infty} u^2_{i\theta;\alpha} dx = \frac{2}{3}\alpha^3$$

$$\text{Energy}: E_\alpha = \int_{-\infty}^{\infty} \left[2u^3_{i\theta;\alpha} - (u_{i\theta;\alpha})^2_x\right] dx = \frac{2}{5}\alpha^5$$

Generic: $I_n = \int_{-\infty}^{\infty} w_{2n-2}(x,t)dx = \frac{2}{2n-1}\alpha^{2n-1}$

Reality follows immediately from $\mathcal{PT}$-symmetry

Real charges from one-soliton solution

$$\text{Mass}: m_\alpha = \int_{-\infty}^{\infty} u_{i\theta;\alpha}(x,t)dx = 2\alpha$$

$$\text{Momentum}: p_\alpha = \int_{-\infty}^{\infty} u_{i\theta;\alpha}^2 dx = \frac{2}{3}\alpha^3$$

$$\text{Energy}: E_\alpha = \int_{-\infty}^{\infty} \left[2u_{i\theta;\alpha}^3 - (u_{i\theta;\alpha})_x^2\right] dx = \frac{2}{5}\alpha^5$$

Generic: $I_n = \int_{-\infty}^{\infty} w_{2n-2}(x,t)dx = \frac{2}{2n-1}\alpha^{2n-1}$

Reality follows immediately from $\mathcal{PT}$-symmetry

$\mathcal{PT}$-broken solutions $(\mu = \kappa + i\theta) \Rightarrow \mathcal{PT}$-symmetric I_n:

Real charges from one-soliton solution

$$\text{Mass}: m_\alpha = \int_{-\infty}^{\infty} u_{i\theta;\alpha}(x,t)dx = 2\alpha$$

$$\text{Momentum}: p_\alpha = \int_{-\infty}^{\infty} u_{i\theta;\alpha}^2 dx = \frac{2}{3}\alpha^3$$

$$\text{Energy}: E_\alpha = \int_{-\infty}^{\infty} \left[2u_{i\theta;\alpha}^3 - (u_{i\theta;\alpha})_x^2\right] dx = \frac{2}{5}\alpha^5$$

Generic: $I_n = \int_{-\infty}^{\infty} w_{2n-2}(x,t)dx = \frac{2}{2n-1}\alpha^{2n-1}$

Reality follows immediately from $\mathcal{PT}$-symmetry

$\mathcal{PT}$-broken solutions ($\mu = \kappa + i\theta$) $\Rightarrow$ $\mathcal{PT}$-symmetric I_n:

$u_{\kappa+i\theta;\alpha}(x,t) = u_{i\theta;\alpha}(x+\kappa/\alpha, t)$ then absorb in integral limits

Real charges from one-soliton solution

$$\begin{aligned}
\text{Mass}: m_\alpha &= \int_{-\infty}^{\infty} u_{i\theta;\alpha}(x,t)dx = 2\alpha \\
\text{Momentum}: p_\alpha &= \int_{-\infty}^{\infty} u_{i\theta;\alpha}^2 dx = \frac{2}{3}\alpha^3 \\
\text{Energy}: E_\alpha &= \int_{-\infty}^{\infty} \left[2u_{i\theta;\alpha}^3 - (u_{i\theta;\alpha})_x^2\right] dx = \frac{2}{5}\alpha^5
\end{aligned}$$

Generic: $I_n = \int_{-\infty}^{\infty} w_{2n-2}(x,t)dx = \frac{2}{2n-1}\alpha^{2n-1}$

Reality follows immediately from $\mathcal{PT}$-symmetry

$\mathcal{PT}$-broken solutions $(\mu = \kappa + i\theta) \Rightarrow \mathcal{PT}$-symmetric I_n:

$u_{\kappa+i\theta;\alpha}(x,t) = u_{i\theta;\alpha}(x+\kappa/\alpha, t)$ then absorb in integral limits

$u_{\kappa+i\theta;\alpha}(x,t) = u_{i\theta;\alpha}(x, t-\kappa/\alpha^3)$ then use charges are conserved

Real charges from one-soliton solution

$$\text{Mass}: m_\alpha = \int_{-\infty}^{\infty} u_{i\theta;\alpha}(x,t)dx = 2\alpha$$

$$\text{Momentum}: p_\alpha = \int_{-\infty}^{\infty} u_{i\theta;\alpha}^2 dx = \frac{2}{3}\alpha^3$$

$$\text{Energy}: E_\alpha = \int_{-\infty}^{\infty} \left[2u_{i\theta;\alpha}^3 - (u_{i\theta;\alpha})_x^2\right] dx = \frac{2}{5}\alpha^5$$

Generic: $I_n = \int_{-\infty}^{\infty} w_{2n-2}(x,t)dx = \frac{2}{2n-1}\alpha^{2n-1}$

Reality follows immediately from $\mathcal{PT}$-symmetry

$\mathcal{PT}$-broken solutions $(\mu = \kappa + i\theta) \Rightarrow \mathcal{PT}$-symmetric I_n:

$u_{\kappa+i\theta;\alpha}(x,t) = u_{i\theta;\alpha}(x+\kappa/\alpha, t)$ then absorb in integral limits

$u_{\kappa+i\theta;\alpha}(x,t) = u_{i\theta;\alpha}(x, t-\kappa/\alpha^3)$ then use charges are conserved

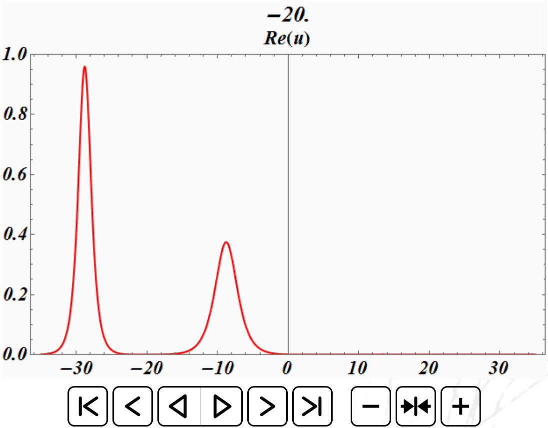
This is not possible for N-soliton solutions with $N > 2$.

Nondegenerate two-soliton solutions

$$u_{\mu,\nu;\alpha,\beta} = 2\left[ln\left(\tau_{\mu,\nu;\alpha,\beta}(x,t)\right)\right]_{xx}$$

Nondegenerate two-soliton solutions

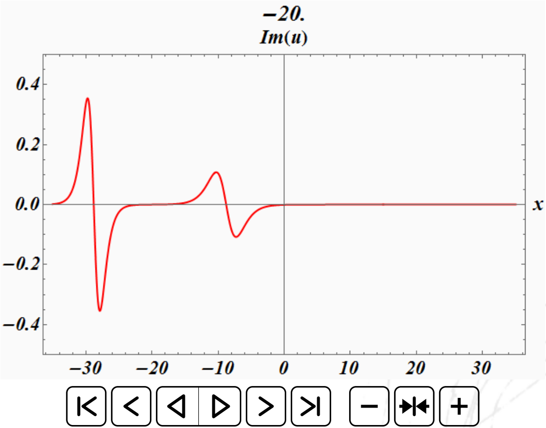
$$u_{\mu,\nu;\alpha,\beta} = 2\left[ln\left(\tau_{\mu,\nu;\alpha,\beta}(x,t)\right)\right]_{xx}$$



$\alpha = 6/5,\ \beta = 4/5,\ \mu = i\pi/3,\ \nu = i\pi/4$

Nondegenerate two-soliton solutions

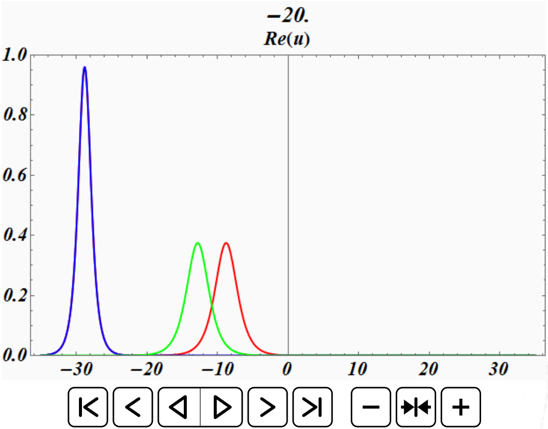
$$u_{\mu,\nu;\alpha,\beta} = 2\left[ln\left(\tau_{\mu,\nu;\alpha,\beta}(x,t)\right)\right]_{xx}$$



$\alpha = 6/5$, $\beta = 4/5$, $\mu = i\pi/3$, $\nu = i\pi/4$

Nondegenerate two-soliton solutions

$$u_{\mu,\nu;\alpha,\beta} = 2\left[\ln\left(\tau_{\mu,\nu;\alpha,\beta}(x,t)\right)\right]_{xx}$$



$p_{\mu,\nu;\alpha,\beta}(x,t)$

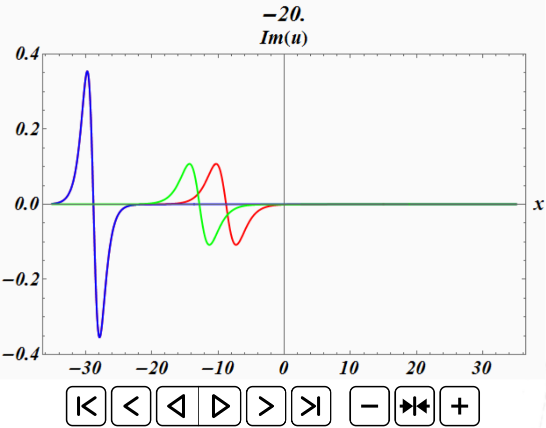
$p_{\mu;\alpha}(x,t)$

$p_{\nu;\beta}(x,t)$

$$\alpha = 6/5,\ \beta = 4/5,\ \mu = i\pi/3,\ \nu = i\pi/4$$

Nondegenerate two-soliton solutions

$$u_{\mu,\nu;\alpha,\beta} = 2\left[ln\left(\tau_{\mu,\nu;\alpha,\beta}(x,t)\right)\right]_{xx}$$



$q_{\mu,\nu;\alpha,\beta}(x,t)$
$q_{\mu;\alpha}(x,t)$
$q_{\nu;\beta}(x,t)$

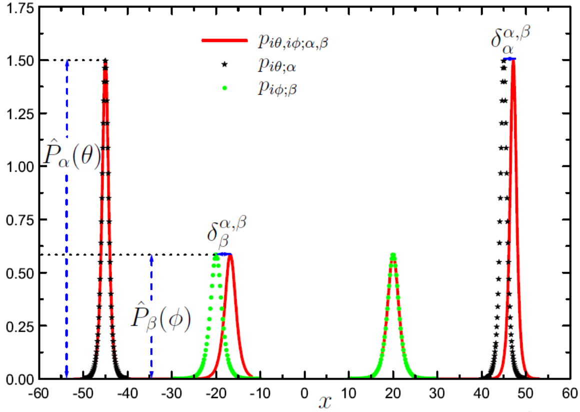
$\alpha = 6/5,\ \beta = 4/5,\ \mu = i\pi/3,\ \nu = i\pi/4$

Time-delays and lateral displacements

Comparing trajectories in the asymptotic past and future

Time-delays and lateral displacements

Comparing trajectories in the asymptotic past and future



$$\delta_x^{y,z} := \frac{2}{x} \ln \left(\frac{y+z}{y-z} \right)$$

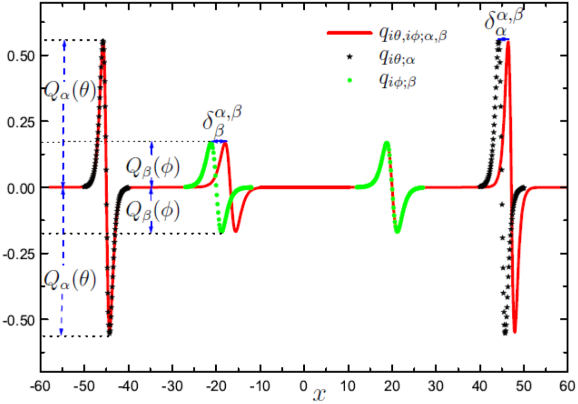
Displacements: $(\Delta_x)_\alpha = \delta_\alpha^{\alpha,\beta}$ and $(\Delta_x)_\beta = -\delta_\beta^{\alpha,\beta}$

Time-delays: $(\Delta_t)_\alpha = -\frac{1}{\alpha^2}\delta_\alpha^{\alpha,\beta}$ and $(\Delta_t)_\beta = \frac{1}{\beta^2}\delta_\beta^{\alpha,\beta}$

Consistency relations: $\sum_k m_k(\Delta_x)_k = 0$ and $\sum_k p_k(\Delta_t)_k = 0$

Time-delays and lateral displacements

Comparing trajectories in the asymptotic past and future



$$\delta_x^{y,z} := \frac{2}{x} \ln\left(\frac{y+z}{y-z}\right)$$

Displacements: $(\Delta_x)_\alpha = \delta_\alpha^{\alpha,\beta}$ and $(\Delta_x)_\beta = -\delta_\beta^{\alpha,\beta}$

Time-delays: $(\Delta_t)_\alpha = -\frac{1}{\alpha^2}\delta_\alpha^{\alpha,\beta}$ and $(\Delta_t)_\beta = \frac{1}{\beta^2}\delta_\beta^{\alpha,\beta}$

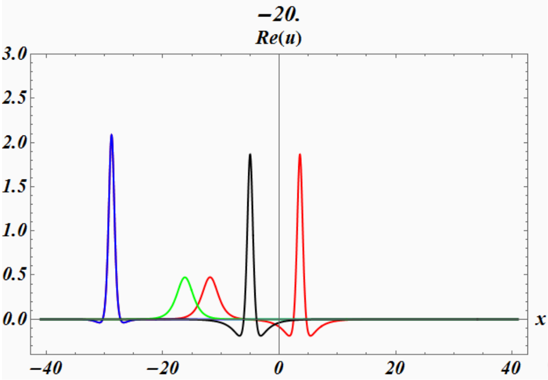
Consistency relations: $\sum_k m_k(\Delta_x)_k = 0$ and $\sum_k p_k(\Delta_t)_k = 0$

Nondegenerate three-soliton solutions

$$u_{\mu,\nu,\rho;\alpha,\beta,\gamma} = 2\left[ln\left(\tau_{\mu,\nu,\rho;\alpha,\beta,\gamma}(x,t)\right)\right]_{xx}$$

Nondegenerate three-soliton solutions

$$u_{\mu,\nu,\rho;\alpha,\beta,\gamma} = 2\left[ln\left(\tau_{\mu,\nu,\rho;\alpha,\beta,\gamma}(x,t)\right)\right]_{xx}$$

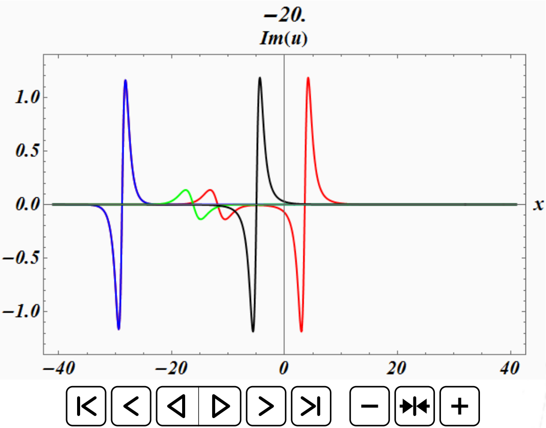


$p_{\mu,\nu,\rho;\alpha,\beta,\gamma}(x,t)$
$p_{\mu;\alpha}(x,t)$
$p_{\nu;\beta}(x,t)$
$p_{\rho;\gamma}(x,t)$

$$\alpha = 6/5,\ \beta = 9/10,\ \gamma = 1/2,\ \mu = i7/5\pi,\ \nu = i1/4\pi,\ \rho = i7/6\pi$$

Nondegenerate three-soliton solutions

$$u_{\mu,\nu,\rho;\alpha,\beta,\gamma} = 2\left[ln\left(\tau_{\mu,\nu,\rho;\alpha,\beta,\gamma}(x,t)\right)\right]_{xx}$$



$q_{\mu,\nu,\rho;\alpha,\beta,\gamma}(x,t)$
$q_{\mu;\alpha}(x,t)$
$q_{\nu;\beta}(x,t)$
$q_{\rho;\gamma}(x,t)$

$\alpha = 6/5,\ \beta = 9/10,\ \gamma = 1/2,\ \mu = i7/5\pi,\ \nu = i1/4\pi,\ \rho = i7/6\pi$

Factorized Scattering

Displacements:

$$
\begin{aligned}
(\Delta_x)_\alpha &= \delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma} \\
(\Delta_x)_\beta &= \delta_\beta^{\beta,\gamma} - \delta_\beta^{\alpha,\beta} \\
(\Delta_x)_\gamma &= -\delta_\gamma^{\alpha,\gamma} - \delta_\gamma^{\beta,\gamma}
\end{aligned}
$$

Factorized Scattering

Displacements:

$$
\begin{aligned}
(\Delta_x)_\alpha &= \delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma} \\
(\Delta_x)_\beta &= \delta_\beta^{\beta,\gamma} - \delta_\beta^{\alpha,\beta} \\
(\Delta_x)_\gamma &= -\delta_\gamma^{\alpha,\gamma} - \delta_\gamma^{\beta,\gamma}
\end{aligned}
$$

Time-delays:

$$
\begin{aligned}
(\Delta_t)_\alpha &= -\frac{1}{\alpha^2}\left(\delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma}\right) \\
(\Delta_t)_\beta &= \frac{1}{\beta^2}\left(\delta_\beta^{\alpha,\beta} - \delta_\beta^{\beta,\gamma}\right) \\
(\Delta_t)_\gamma &= \frac{1}{\gamma^2}\left(\delta_\gamma^{\alpha,\gamma} + \delta_\gamma^{\beta,\gamma}\right)
\end{aligned}
$$

Factorized Scattering

Displacements:

$$
\begin{aligned}
(\Delta_x)_\alpha &= \delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma} \\
(\Delta_x)_\beta &= \delta_\beta^{\beta,\gamma} - \delta_\beta^{\alpha,\beta} \\
(\Delta_x)_\gamma &= -\delta_\gamma^{\alpha,\gamma} - \delta_\gamma^{\beta,\gamma}
\end{aligned}
$$

Time-delays:

$$
\begin{aligned}
(\Delta_t)_\alpha &= -\frac{1}{\alpha^2}\left(\delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma}\right) \\
(\Delta_t)_\beta &= \frac{1}{\beta^2}\left(\delta_\beta^{\alpha,\beta} - \delta_\beta^{\beta,\gamma}\right) \\
(\Delta_t)_\gamma &= \frac{1}{\gamma^2}\left(\delta_\gamma^{\alpha,\gamma} + \delta_\gamma^{\beta,\gamma}\right)
\end{aligned}
$$

Consistency relations: $\sum_k m_k (\Delta_x)_k = 0$ and $\sum_k p_k (\Delta_t)_k = 0$

Factorized Scattering

Displacements:

$$
\begin{aligned}
(\Delta_x)_\alpha &= \delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma} \\
(\Delta_x)_\beta &= \delta_\beta^{\beta,\gamma} - \delta_\beta^{\alpha,\beta} \\
(\Delta_x)_\gamma &= -\delta_\gamma^{\alpha,\gamma} - \delta_\gamma^{\beta,\gamma}
\end{aligned}
$$

Time-delays:

$$
\begin{aligned}
(\Delta_t)_\alpha &= -\frac{1}{\alpha^2}\left(\delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma}\right) \\
(\Delta_t)_\beta &= \frac{1}{\beta^2}\left(\delta_\beta^{\alpha,\beta} - \delta_\beta^{\beta,\gamma}\right) \\
(\Delta_t)_\gamma &= \frac{1}{\gamma^2}\left(\delta_\gamma^{\alpha,\gamma} + \delta_\gamma^{\beta,\gamma}\right)
\end{aligned}
$$

Consistency relations: $\sum_k m_k(\Delta_x)_k = 0$ and $\sum_k p_k(\Delta_t)_k = 0$

N-soliton time-delays $\equiv$ sum of two-soliton time-delays

Factorized Scattering

Displacements:

$$
\begin{aligned}
(\Delta_x)_\alpha &= \delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma} \\
(\Delta_x)_\beta &= \delta_\beta^{\beta,\gamma} - \delta_\beta^{\alpha,\beta} \\
(\Delta_x)_\gamma &= -\delta_\gamma^{\alpha,\gamma} - \delta_\gamma^{\beta,\gamma}
\end{aligned}
$$

Time-delays:

$$
\begin{aligned}
(\Delta_t)_\alpha &= -\frac{1}{\alpha^2}\left(\delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma}\right) \\
(\Delta_t)_\beta &= \frac{1}{\beta^2}\left(\delta_\beta^{\alpha,\beta} - \delta_\beta^{\beta,\gamma}\right) \\
(\Delta_t)_\gamma &= \frac{1}{\gamma^2}\left(\delta_\gamma^{\alpha,\gamma} + \delta_\gamma^{\beta,\gamma}\right)
\end{aligned}
$$

Consistency relations: $\sum_k m_k(\Delta_x)_k = 0$ and $\sum_k p_k(\Delta_t)_k = 0$

N-soliton time-delays $\equiv$ sum of two-soliton time-delays

Classical factorization

This corresponds to the factorization of the quantum S-matrix described by the Yang-Baxter and bootstrap equation.

Reality of complex N-soliton charges

Asymptotically complex N-solitons factor into N one-solitons

Charges based on one-solitons solutions are real by $\mathcal{PT}$-symmetry

Reality of complex N-soliton charges

Asymptotically complex N-solitons factor into N one-solitons

Charges based on one-solitons solutions are real by $\mathcal{PT}$-symmetry

Therefore

Reality condition

$\mathcal{PT}$-symmetry and integrability ensure the reality of all charges.

Regularization of degenerate multi-solitons

- In general for real solutions:

  The limit $E_\alpha \to E_\beta$ gives $\lim_{\alpha\to\beta} u_{\alpha,\beta,\gamma,\ldots}(x,t) \to \infty$

  The best scenario still has cusps.

Regularization of degenerate multi-solitons

- In general for real solutions:

  The limit $E_\alpha \to E_\beta$ gives $\lim_{\alpha\to\beta} u_{\alpha,\beta,\gamma,\ldots}(x,t) \to \infty$

  The best scenario still has cusps.

- In the complex case the limits become finite.

  Technically we use Wronskians as τ-functions involving solutions of the Schrödinger equation and Jordan states obtained from Darboux-Crum transformations.

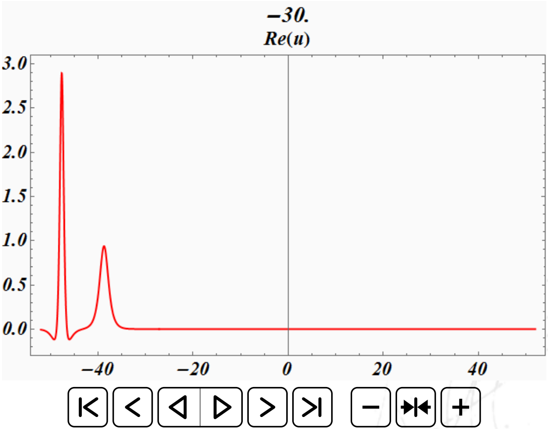
  A link to Hirota's direct method and solutions obtained from a superpositon principle based on Bäcklund transformations is also established.

Degenerate two-soliton solutions

$$u_{i\theta,i\phi;\alpha,\alpha}(x,t) = \frac{2\alpha^2\left[\left(\alpha x - 3\alpha^3 t + i\phi\right)\sinh\left(\eta_{i\theta;\alpha}\right) - 2\cosh\left(\eta_{i\theta;\alpha}\right) - 2\right]}{\left[\alpha x - 3\alpha^3 t + i\phi + \sinh\left(\eta_{i\theta;\alpha}\right)\right]^2}$$

Degenerate two-soliton solutions

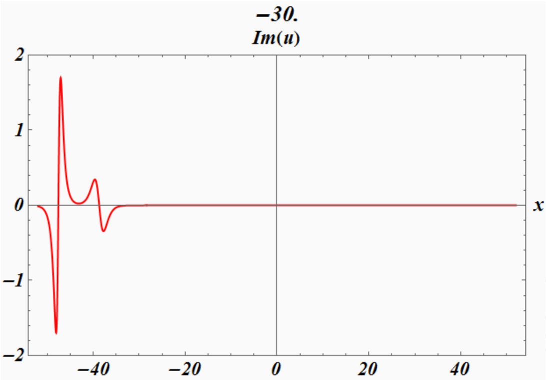
$$u_{i\theta,i\phi;\alpha,\alpha}(x,t)=\frac{2\alpha^2\left[\left(\alpha x-3\alpha^3 t+i\phi\right)\sinh\left(\eta_{i\theta;\alpha}\right)-2\cosh\left(\eta_{i\theta;\alpha}\right)-2\right]}{\left[\alpha x-3\alpha^3 t+i\phi+\sinh\left(\eta_{i\theta;\alpha}\right)\right]^2}$$



$\alpha=6/5$, $\theta=\pi/3$, $\phi=\pi/4$

Degenerate two-soliton solutions

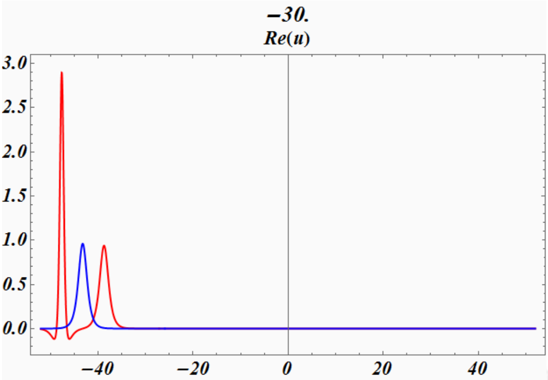
$$u_{i\theta,i\phi;\alpha,\alpha}(x,t) = \frac{2\alpha^2\left[\left(\alpha x - 3\alpha^3 t + i\phi\right)\sinh\left(\eta_{i\theta;\alpha}\right) - 2\cosh\left(\eta_{i\theta;\alpha}\right) - 2\right]}{\left[\alpha x - 3\alpha^3 t + i\phi + \sinh\left(\eta_{i\theta;\alpha}\right)\right]^2}$$



$\alpha = 6/5$, $\theta = \pi/3$, $\phi = \pi/4$

Degenerate two-soliton solutions

$$u_{i\theta,i\phi;\alpha,\alpha}(x,t)=\frac{2\alpha^2\left[\left(\alpha x-3\alpha^3 t+i\phi\right)\sinh\left(\eta_{i\theta;\alpha}\right)-2\cosh\left(\eta_{i\theta;\alpha}\right)-2\right]}{\left[\alpha x-3\alpha^3 t+i\phi+\sinh\left(\eta_{i\theta;\alpha}\right)\right]^2}$$



$p_{i\theta,i\phi;\alpha,\alpha}(x,t)$

$p_{i\theta;\alpha}(x,t)$

Relative displacement:

$\Delta(t)=\frac{1}{\alpha}\ln\left(4\alpha^3|t|\right)$

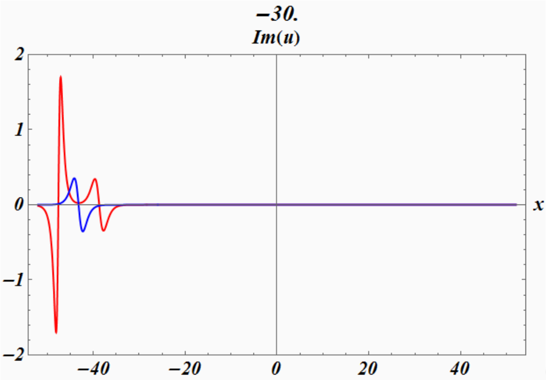
Total displacement:

$\pm 2\Delta(t)$

$\alpha=6/5,\ \theta=\pi/3,\ \phi=\pi/4$

Degenerate two-soliton solutions

$$u_{i\theta,i\phi;\alpha,\alpha}(x,t) = \frac{2\alpha^2\left[\left(\alpha x - 3\alpha^3 t + i\phi\right)\sinh\left(\eta_{i\theta;\alpha}\right) - 2\cosh\left(\eta_{i\theta;\alpha}\right) - 2\right]}{\left[\alpha x - 3\alpha^3 t + i\phi + \sinh\left(\eta_{i\theta;\alpha}\right)\right]^2}$$



$q_{i\theta,i\phi;\alpha,\alpha}(x,t)$

$q_{i\theta;\alpha}(x,t)$

Relative displacement:

$\Delta(t) = \frac{1}{\alpha}\ln\left(4\alpha^3|t|\right)$

Total displacement:

$\pm 2\Delta(t)$

$\alpha = 6/5,\ \theta = \pi/3,\ \phi = \pi/4$

Degenerate N-soliton solutions ($\alpha_1 = \alpha_2 = \ldots \alpha_N$)

Notation:

$\lim_{\alpha_2,\ldots,\alpha_N \to \alpha_1 = \alpha} u_{i\theta_1 = i\theta,\ldots,i\theta_N;\alpha_1,\ldots,\alpha_N} = p_{i\theta,\ldots,i\theta_N;N\alpha} + i q_{i\theta,\ldots,i\theta_N;N\alpha}$

Asymptotic limits:

$$\lim_{t\to\sigma\infty} p_{i\theta,\ldots,i\theta_{2n};2n\alpha}\left[t\alpha^2 + \sigma\Delta_{n,\ell,1}(t), t\right] = \hat{P}_\alpha\left(\theta + \frac{1-(-1)^{n+\ell+1}}{2}\pi\right)$$

$$\lim_{t\to\sigma\infty} p_{i\theta,\ldots,i\theta_{2n};2n\alpha}\left[t\alpha^2 - \sigma\Delta_{n,\ell,1}(t), t\right] = \hat{P}_\alpha\left(\theta + \frac{1-(-1)^{n+\ell}}{2}\pi\right)$$

for $n = 1, 2, \ldots,\ \ell = 1, 2, \ldots, n,\ \sigma = \pm 1$

$$\lim_{t\to\sigma\infty} p_{i\theta,\ldots,i\theta_{2n+1};(2n+1)\alpha}\left[t\alpha^2 \pm \Delta_{n,\ell,0}(t), t\right] = \hat{P}_\alpha\left(\theta + \frac{1-(-1)^{n+\ell}}{2}\pi\right)$$

for $n = 0, 1, 2, \ldots,\ \ell = 0, 1, 2, \ldots, n$

Time-dependent displacements:

$$\Delta_{n,\ell,\kappa}(t) = \frac{1}{\alpha}\ln\left[\frac{(n-\ell)!}{(n+\ell-\kappa)!}\left(4\,|t|\,\alpha^3\right)^{2\ell-\kappa}\right]$$

Nonlocality

Consider higher order nonlinear Schrödinger equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q \qquad = 0$$

Nonlocality

Consider higher order nonlinear Schrödinger equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon\left[\alpha q_{xxx} + \beta |q|^2 q_x + \gamma q |q|_x^2\right] = 0$$

$\mathcal{PT}$-symmetry: $\mathcal{PT}: x \to -x,\ t \to -t,\ i \to -i,\ q \to q$

Nonlocality

Consider higher order nonlinear Schrödinger equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon\left[\alpha q_{xxx} + \beta |q|^2 q_x + \gamma q |q|_x^2\right] = 0$$

$\mathcal{PT}$-symmetry: $\mathcal{PT} : x \to -x,\ t \to -t,\ i \to -i,\ q \to q$

Integrable cases:

$\varepsilon = 0 \equiv$nonlinear Schrödinger equation (NLSE)

$\alpha : \beta : \gamma = 0 : 1 : 1 \equiv$ derivative NLSE of type I

$\alpha : \beta : \gamma = 0 : 1 : 0 \equiv$ derivative NLSE of type II

$\alpha : \beta : \gamma = 1 : 6 : 3 \equiv$ Sasa-Satsuma equation

Nonlocality

Consider higher order nonlinear Schrödinger equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon \left[\alpha q_{xxx} + \beta |q|^2 q_x + \gamma q |q|^2_x\right] = 0$$

$\mathcal{PT}$-symmetry: $\mathcal{PT} : x \to -x,\ t \to -t,\ i \to -i,\ q \to q$

Integrable cases:

$\varepsilon = 0 \equiv$ nonlinear Schrödinger equation (NLSE)

$\alpha : \beta : \gamma = 0 : 1 : 1 \equiv$ derivative NLSE of type I

$\alpha : \beta : \gamma = 0 : 1 : 0 \equiv$ derivative NLSE of type II

$\alpha : \beta : \gamma = 1 : 6 : 3 \equiv$ Sasa-Satsuma equation

$\alpha : \beta : \gamma = 1 : 6 : 0 \equiv$ Hirota equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon \left[q_{xxx} + 6 |q|^2 q_x\right] = 0$$

Zero curvature condition

$$\partial_t U - \partial_x V + [U, V] = 0$$

Zero curvature condition

$$\partial_t U - \partial_x V + [U, V] = 0$$

$$U = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}, \qquad V = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix}$$

Zero curvature condition

$$\partial_t U - \partial_x V + [U, V] = 0$$

$$U = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}, \qquad V = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix}$$

$$\begin{aligned} A_x(x,t) &= q(x,t)C(x,t) - r(x,t)B(x,t) \\ B_x(x,t) &= q_t(x,t) - 2q(x,t)A(x,t) - 2i\lambda B(x,t) \\ C_x(x,t) &= r_t(x,t) + 2r(x,t)A(x,t) + 2i\lambda C(x,t) \end{aligned}$$

$$\begin{aligned} A(x,t) &= -i\alpha qr - 2i\alpha\lambda^2 + \beta\left(rq_x - qr_x - 4i\lambda^3 - 2i\lambda qr\right) \\ B(x,t) &= i\alpha q_x + 2\alpha\lambda q + \beta\left(2q^2 r - q_{xx} + 2i\lambda q_x + 4\lambda^2 q\right) \\ C(x,t) &= -i\alpha r_x + 2\alpha\lambda r + \beta\left(2qr^2 - r_{xx} - 2i\lambda r_x + 4\lambda^2 r\right) \end{aligned}$$

Zero curvature condition

$$\partial_t U - \partial_x V + [U, V] = 0$$

$$U = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}, \qquad V = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix}$$

$$\begin{aligned} A_x(x,t) &= q(x,t)C(x,t) - r(x,t)B(x,t) \\ B_x(x,t) &= q_t(x,t) - 2q(x,t)A(x,t) - 2i\lambda B(x,t) \\ C_x(x,t) &= r_t(x,t) + 2r(x,t)A(x,t) + 2i\lambda C(x,t) \end{aligned}$$

$$\begin{aligned} A(x,t) &= -i\alpha qr - 2i\alpha\lambda^2 + \beta\left(rq_x - qr_x - 4i\lambda^3 - 2i\lambda qr\right) \\ B(x,t) &= i\alpha q_x + 2\alpha\lambda q + \beta\left(2q^2 r - q_{xx} + 2i\lambda q_x + 4\lambda^2 q\right) \\ C(x,t) &= -i\alpha r_x + 2\alpha\lambda r + \beta\left(2qr^2 - r_{xx} - 2i\lambda r_x + 4\lambda^2 r\right) \end{aligned}$$

$$\begin{aligned} q_t - i\alpha q_{xx} + 2i\alpha q^2 r + \beta\left[q_{xxx} - 6qrq_x\right] &= 0 \\ r_t + i\alpha r_{xx} - 2i\alpha qr^2 + \beta\left(r_{xxx} - 6qrr_x\right) &= 0 \end{aligned}$$

Nonlocality from zero curvature condition

Complex conjugate pair: $r(x,t) = \kappa q^*(x,t)$ (Hirota equation)

$$
\begin{aligned}
iq_t &= -\alpha\left(q_{xx} - 2\kappa\left|q\right|^2 q\right) - i\beta\left(q_{xxx} - 6\kappa\left|q\right|^2 q_x\right)\\
-iq_t^* &= -\alpha\left(q_{xx}^* - 2\kappa\left|q\right|^2 q^*\right) + i\beta\left(q_{xxx}^* - 6\kappa\left|q\right|^2 q_x^*\right)
\end{aligned}
$$

Nonlocality from zero curvature condition

Complex conjugate pair: $r(x,t) = \kappa q^*(x,t)$ (Hirota equation)

$$\begin{aligned} iq_t &= -\alpha\left(q_{xx} - 2\kappa\left|q\right|^2 q\right) - i\beta\left(q_{xxx} - 6\kappa\left|q\right|^2 q_x\right) \\ -iq_t^* &= -\alpha\left(q_{xx}^* - 2\kappa\left|q\right|^2 q^*\right) + i\beta\left(q_{xxx}^* - 6\kappa\left|q\right|^2 q_x^*\right) \end{aligned}$$

$\mathcal{P}$ conjugate pair: $r(x,t) = \kappa q^*(-x,t)$ (Nonlocal Hirota equ^n)

$$\begin{aligned} iq_t &= -\alpha\left[q_{xx} - 2\kappa\tilde{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q\tilde{q}^* q_x] \\ -i\tilde{q}_t^* &= -\alpha\left[\tilde{q}_{xx}^* - 2\kappa q(\tilde{q}^*)^2\right] - \delta(\tilde{q}_{xxx}^* - 6\kappa\tilde{q}^* q\tilde{q}_x^*) \end{aligned}$$

$$\beta = i\delta,\ \alpha,\delta\in\mathbb{R},\ q := q(x,t);\ \tilde{q} := q(-x,t)$$

Nonlocality from zero curvature condition

Complex conjugate pair: $r(x,t) = \kappa q^*(x,t)$ (Hirota equation)

$$\begin{aligned} iq_t &= -\alpha\left(q_{xx} - 2\kappa\left|q\right|^2 q\right) - i\beta\left(q_{xxx} - 6\kappa\left|q\right|^2 q_x\right) \\ -iq_t^* &= -\alpha\left(q_{xx}^* - 2\kappa\left|q\right|^2 q^*\right) + i\beta\left(q_{xxx}^* - 6\kappa\left|q\right|^2 q_x^*\right) \end{aligned}$$

$\mathcal{P}$ conjugate pair: $r(x,t) = \kappa q^*(-x,t)$ (Nonlocal Hirota equ^n)

$$\begin{aligned} iq_t &= -\alpha\left[q_{xx} - 2\kappa\tilde{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q\tilde{q}^* q_x] \\ -i\tilde{q}_t^* &= -\alpha\left[\tilde{q}_{xx}^* - 2\kappa q(\tilde{q}^*)^2\right] - \delta(\tilde{q}_{xxx}^* - 6\kappa\tilde{q}^* q\tilde{q}_x^*) \end{aligned}$$

$$\beta = i\delta,\ \alpha,\delta \in \mathbb{R},\ q := q(x,t);\ \tilde{q} := q(-x,t)$$

$\mathcal{T}$ conjugate pair: $r(x,t) = \kappa q^*(x,-t)$

$$\begin{aligned} iq_t &= -i\hat{\delta}\left[q_{xx} - 2\kappa\hat{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q\hat{q}^* q_x] \\ i\hat{q}_t^* &= i\hat{\delta}\left[\hat{q}_{xx}^* - 2\kappa q(\hat{q}^*)^2\right] + \delta(\hat{q}_{xxx}^* - 6\kappa\hat{q}^* q\hat{q}_x^*) \end{aligned}$$

$$\alpha = i\hat{\delta},\ \beta = i\delta,\ \hat{\delta},\delta \in \mathbb{R};\ \hat{q} := q(x,-t)$$

$\mathcal{PT}$-conjugate pair: $r(x,t) = \kappa q^*(-x,-t)$

$$
\begin{aligned}
q_t &= -\check{\delta}\left[q_{xx} - 2\kappa \check{q}^* q^2\right] - \beta[q_{xxx} - 6\kappa q \check{q}^* q_x] \\
-\check{q}_t^* &= -\check{\delta}\left[\check{q}_{xx}^* - 2\kappa q (\check{q}^*)^2\right] + \beta(\check{q}_{xxx}^* - 6\kappa \check{q}^* q \check{q}_x^*)
\end{aligned}
$$

$\alpha = i\check{\delta}$; $\check{\delta}, \beta \in \mathbb{R}$; $\check{q} := q(-x,-t)$

$\mathcal{PT}$-conjugate pair: $r(x,t) = \kappa q^*(-x,-t)$

$$
\begin{aligned}
q_t &= -\check{\delta}\left[q_{xx} - 2\kappa \check{q}^* q^2\right] - \beta[q_{xxx} - 6\kappa q \check{q}^* q_x] \\
-\check{q}_t^* &= -\check{\delta}\left[\check{q}_{xx}^* - 2\kappa q (\check{q}^*)^2\right] + \beta(\check{q}_{xxx}^* - 6\kappa \check{q}^* q \check{q}_x^*)
\end{aligned}
$$

$\alpha = i\check{\delta}$; $\check{\delta}, \beta \in \mathbb{R}$; $\check{q} := q(-x,-t)$

$\mathcal{P}$ transformed pair: $r(x,t) = \kappa q(-x,t)$:

$$
\begin{aligned}
iq_t &= -\alpha\left[q_{xx} - 2\kappa \tilde{q} q^2\right] + \delta[q_{xxx} - 6\kappa q \tilde{q} q_x] \\
-i\tilde{q}_t &= -\alpha\left[\tilde{q}_{xx} - 2\kappa q \tilde{q}^2\right] - \delta(\tilde{q}_{xxx} - 6\kappa \tilde{q} q \tilde{q}_x)
\end{aligned}
$$

$\beta = i\delta$; $\alpha, \delta \in \mathbb{R}$

$\mathcal{PT}$-conjugate pair: $r(x,t) = \kappa q^*(-x,-t)$

$$\begin{aligned} q_t &= -\check{\delta}\left[q_{xx} - 2\kappa \check{q}^* q^2\right] - \beta[q_{xxx} - 6\kappa q \check{q}^* q_x] \\ -\check{q}_t^* &= -\check{\delta}\left[\check{q}_{xx}^* - 2\kappa q (\check{q}^*)^2\right] + \beta(\check{q}_{xxx}^* - 6\kappa \check{q}^* q \check{q}_x^*) \end{aligned}$$

$$\alpha = i\check{\delta};\ \check{\delta}, \beta \in \mathbb{R}\ ;\ \check{q} := q(-x,-t)$$

$\mathcal{P}$ transformed pair: $r(x,t) = \kappa q(-x,t)$:

$$\begin{aligned} iq_t &= -\alpha\left[q_{xx} - 2\kappa \tilde{q} q^2\right] + \delta[q_{xxx} - 6\kappa q \tilde{q} q_x] \\ -i\tilde{q}_t &= -\alpha\left[\tilde{q}_{xx} - 2\kappa q \tilde{q}^2\right] - \delta(\tilde{q}_{xxx} - 6\kappa \tilde{q} q \tilde{q}_x) \end{aligned}$$

$$\beta = i\delta;\ \alpha, \delta \in \mathbb{R}$$

$\mathcal{T}$ transformed pair: $r(x,t) = \kappa q(x,-t)$

$$\begin{aligned} iq_t &= -i\hat{\delta}\left[q_{xx} - 2\kappa \hat{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q \hat{q}^* q_x] \\ i\hat{q}_t^* &= i\hat{\delta}\left[\hat{q}_{xx}^* - 2\kappa q (\hat{q}^*)^2\right] + \delta(\hat{q}_{xxx}^* - 6\kappa \hat{q}^* q \hat{q}_x^*) \end{aligned}$$

$$\alpha = i\hat{\delta};\ \beta = i\delta; \hat{\delta}, \delta \in \mathbb{R}$$

Nonlocality in Hirota's direct method

Bilinearisation of the local Hirota equation ($q = g/f$)

$$f^3 \left[iq_t + \alpha q_{xx} - 2\kappa\alpha \left|q\right|^2 q + i\beta\left(q_{xxx} - 6\kappa \left|q\right|^2 q_x \right) \right] =$$

$$f \left[iD_t g \cdot f + \alpha D_x^2 g \cdot f + i\beta D_x^3 g \cdot f \right] + \left[3i\beta \left(\frac{g}{f} f_x - g_x \right) - \alpha g \right]$$

$$\times \left[D_x^2 f \cdot f + 2\kappa \left|g\right|^2 \right]$$

$$D_x^n f \cdot g = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k \frac{\partial^{n-k}}{\partial x^{n-k}} f(x) \frac{\partial^k}{\partial x^k} g(x)$$

Nonlocality in Hirota's direct method

Bilinearisation of the local Hirota equation ($q = g/f$)

$$f^3 \left[iq_t + \alpha q_{xx} - 2\kappa\alpha \left|q\right|^2 q + i\beta \Big(q_{xxx} - 6\kappa \left|q\right|^2 q_x \Big) \right] =$$

$$f \left[iD_t g \cdot f + \alpha D_x^2 g \cdot f + i\beta D_x^3 g \cdot f \right] + \left[3i\beta \left(\frac{g}{f} f_x - g_x \right) - \alpha g \right]$$

$$\times \left[D_x^2 f \cdot f + 2\kappa \left|g\right|^2 \right]$$

$$D_x^n f \cdot g = \sum\nolimits_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\partial^{n-k}}{\partial x^{n-k}} f(x) \frac{\partial^k}{\partial x^k} g(x)$$

$$iD_t g \cdot f + \alpha D_x^2 g \cdot f + i\beta D_x^3 g \cdot f = 0, \qquad D_x^2 f \cdot f = -2\kappa \left|g\right|^2$$

Nonlocality in Hirota's direct method

Bilinearisation of the local Hirota equation $(q = g/f)$

$$f^3 \left[iq_t + \alpha q_{xx} - 2\kappa\alpha \left|q\right|^2 q + i\beta\left(q_{xxx} - 6\kappa \left|q\right|^2 q_x \right) \right] =$$
$$f \left[iD_t g \cdot f + \alpha D_x^2 g \cdot f + i\beta D_x^3 g \cdot f \right] + \left[3i\beta \left(\frac{g}{f} f_x - g_x \right) - \alpha g \right]$$
$$\times \left[D_x^2 f \cdot f + 2\kappa \left|g\right|^2 \right]$$

$$D_x^n f \cdot g = \sum\nolimits_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (-1)^k \frac{\partial^{n-k}}{\partial x^{n-k}} f(x) \frac{\partial^k}{\partial x^k} g(x)$$

$$iD_t g \cdot f + \alpha D_x^2 g \cdot f + i\beta D_x^3 g \cdot f = 0, \qquad D_x^2 f \cdot f = -2\kappa \left|g\right|^2$$

Solve by formal power series that becomes **exact**

$$f(x,t) = \sum_{k=0}^{\infty} \varepsilon^{2k} f_{2k}(x,t), \quad \text{and} \quad g(x,t) = \sum_{k=1}^{\infty} \varepsilon^{2k-1} g_{2k-1}(x,t)$$

Bilinearisation of the nonlocal Hirota equation

$$f^3\tilde{f}^*\left[iq_t+\alpha q_{xx}+2\alpha\tilde{q}^*q^2-\delta(q_{xxx}+6q\tilde{q}^*q_x)\right]=$$
$$f\tilde{f}^*\left[iD_tg\cdot f+\alpha D_x^2g\cdot f-\delta D_x^3g\cdot f\right]+\left(\frac{3\delta}{f}D_xg\cdot f-\alpha g\right)$$
$$\times\left(\tilde{f}^*D_x^2f\cdot f-2fg\tilde{g}^*\right)$$

not bilinear yet

$$iD_tg\cdot f+\alpha D_x^2g\cdot f-\delta D_x^3g\cdot f=0,\qquad \tilde{f}^*D_x^2f\cdot f=2fg\tilde{g}^*$$

Bilinearisation of the nonlocal Hirota equation

$$f^3\tilde{f}^*\left[iq_t+\alpha q_{xx}+2\alpha\tilde{q}^*q^2-\delta(q_{xxx}+6q\tilde{q}^*q_x)\right]=$$
$$f\tilde{f}^*\left[iD_tg\cdot f+\alpha D_x^2g\cdot f-\delta D_x^3g\cdot f\right]+\left(\frac{3\delta}{f}D_xg\cdot f-\alpha g\right)$$
$$\times\left(\tilde{f}^*D_x^2f\cdot f-2fg\tilde{g}^*\right)$$

not bilinear yet

$$iD_tg\cdot f+\alpha D_x^2g\cdot f-\delta D_x^3g\cdot f=0,\qquad \tilde{f}^*D_x^2f\cdot f=2fg\tilde{g}^*$$

introduce additional auxiliary function

$$D_x^2f\cdot f=hg,\qquad \text{and}\qquad 2f\tilde{g}^*=h\tilde{f}^*$$

Solve again formal power series that becomes **exact**

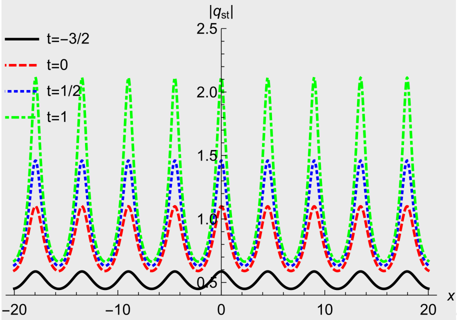
$$h(x,t)=\sum\nolimits_k\varepsilon^k h_k(x,t).$$

Two-types of nonlocal solutions (one-soliton)

Truncated expansions: $f = 1 + \varepsilon^2 f_2, \quad g = \varepsilon g_1, \quad h = \varepsilon h_1$

$$
\begin{aligned}
0 &= \varepsilon\left[i\,(g_1)_t + \alpha\,(g_1)_{xx} - \delta (g_1)_{xxx}\right] \\
&\quad + \varepsilon^3\left[2\,(f_2)_x\,(g_1)_x - g_1\left[(f_2)_{xx} + i\,(f_2)_t\right] + i f_2\left[(g_1)_t + i\,(g_1)_{xx}\right]\right] \\
0 &= \varepsilon^2\left[2(f_2)_{xx} - g_1 h_1\right] + \varepsilon^4\left[2 f_2 (f_2)_{xx} - 2(f_2)_x^2\right] \\
0 &= \varepsilon\left[2\tilde{g}_1^* - h_1\right] + \varepsilon^3\left[2 f_2 \tilde{g}_1^* - \tilde{f}_2^* h_1\right]
\end{aligned}
$$

Standard solution, solve six equations independently, then $\varepsilon \to 1$



$$q_{\text{st}}^{(1)} = \frac{\lambda(\mu - \mu^*)^2 \tau_{\mu,\gamma}}{(\mu - \mu^*)^2 + |\lambda|^2 \tau_{\mu,\gamma} \tilde{\tau}^*_{\mu,\gamma}}$$

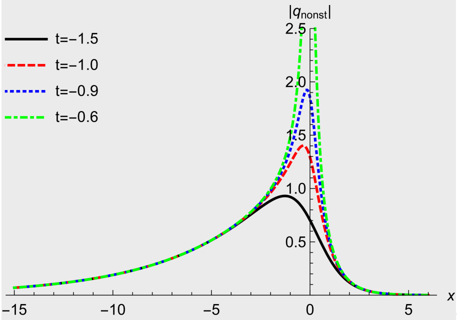
$$\tau_{\mu,\gamma}(x,t) := e^{\mu x + \mu^2 (i\alpha - \beta\mu) t + \gamma}$$

Two-types of nonlocal solutions (one-soliton)

Truncated expansions: $f = 1 + \varepsilon^2 f_2, \quad g = \varepsilon g_1, \quad h = \varepsilon h_1$

$$
\begin{aligned}
0 &= \varepsilon[i\,(g_1)_t + \alpha\,(g_1)_{xx} - \delta(g_1)_{xxx}] \\
&\quad + \varepsilon^3[2\,(f_2)_x\,(g_1)_x - g_1\,[(f_2)_{xx} + i\,(f_2)_t] + if_2\,[(g_1)_t + i\,(g_1)_{xx}]] \\
0 &= \varepsilon^2[2(f_2)_{xx} - g_1 h_1] + \varepsilon^4\left[2f_2(f_2)_{xx} - 2(f_2)_x^2\right] \\
0 &= [2\tilde{g}_1^* - h_1] + \left[2f_2\tilde{g}_1^* - \tilde{f}_2^* h_1\right]
\end{aligned}
$$

Nonstandard solution, solve five equations, last one for $\varepsilon = 1$



$$
q_{\text{nonst}}^{(1)} = \frac{(\mu+\nu)\tau_{\mu,i\gamma}}{1+\tau_{\mu,i\gamma}\tilde{\tau}^*_{-\nu,-i\theta}}.
$$

$$
\tau_{\mu,\gamma}(x,t) := e^{\mu x + \mu^2(i\alpha - \beta\mu)t + \gamma}
$$

Two-soliton solution

Truncated expansions:

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4, \quad g = \varepsilon g_1 + \varepsilon^3 g_3, \quad h = \varepsilon h_1 + \varepsilon^3 h_3$$

$$q_{\mathrm{nl}}^{(2)}(x,t) = \frac{g_1(x,t) + g_3(x,t)}{1 + f_2(x,t) + f_4(x,t)}$$

$$\begin{aligned}
g_1 &= \tau_{\mu,\gamma} + \tau_{\nu,\delta} \\
g_3 &= \frac{(\mu-\nu)^2}{(\mu-\mu^*)^2(\nu-\mu^*)^2}\tau_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\mu,\gamma} + \frac{(\mu-\nu)^2}{(\mu-\nu^*)^2(\nu-\nu^*)^2}\tau_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta} \\
f_2 &= \frac{\tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}}{(\mu-\mu^*)^2} + \frac{\tau_{\nu,\delta}\tilde{\tau}^*_{\mu,\gamma}}{(\nu-\mu^*)^2} + \frac{\tau_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}}{(\mu-\nu^*)^2} + \frac{\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta}}{(\nu-\nu^*)^2} \\
f_4 &= \frac{(\mu-\nu)^2(\mu^*-\nu^*)^2}{(\mu-\mu^*)^2(\nu-\mu^*)^2(\mu-\nu^*)^2(\nu-\nu^*)^2}\tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta} \\
h_1 &= 2\tilde{\tau}^*_{\mu,\gamma} + 2\tilde{\tau}^*_{\nu,\delta} \\
h_3 &= \frac{2(\mu^*-\nu^*)^2}{(\mu-\mu^*)^2(\nu^*-\mu)^2}\tilde{\tau}^*_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}\tau_{\mu,\gamma} + \frac{2(\mu^*-\nu^*)^2}{(\mu^*-\nu)^2(\nu-\nu^*)^2}\tilde{\tau}^*_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}\tau_{\nu,\delta}
\end{aligned}$$

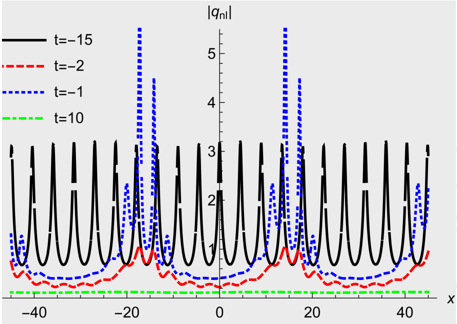
Two-soliton solution

Truncated expansions:

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4, \quad g = \varepsilon g_1 + \varepsilon^3 g_3, \quad h = \varepsilon h_1 + \varepsilon^3 h_3$$

$$q_{\text{nl}}^{(2)}(x,t) = \frac{g_1(x,t) + g_3(x,t)}{1 + f_2(x,t) + f_4(x,t)}$$

Nonlocal regular two-soliton solution



Nonlocality in Darboux-Crum transformations

Quantum mechanical analogue to supersymmetry, intertwining

$$L_n H_{n-1} = H_n L_n$$

Nonlocality in Darboux-Crum transformations

Quantum mechanical analogue to supersymmetry, intertwining

$$L_n H_{n-1} = H_n L_n$$

iteration $\mathcal{L}_n H_0 = H_n \mathcal{L}_n$, $\mathcal{L}_n := L_n L_{n-1} \ldots L_1$, $\Psi_n(\lambda) = \mathcal{L}_n \Psi(\lambda)$

Nonlocality in Darboux-Crum transformations

Quantum mechanical analogue to supersymmetry, intertwining

$$L_n H_{n-1} = H_n L_n$$

iteration $\mathcal{L}_n H_0 = H_n \mathcal{L}_n$, $\mathcal{L}_n := L_n L_{n-1} \ldots L_1$, $\Psi_n(\lambda) = \mathcal{L}_n \Psi(\lambda)$

In Hirota case, Hamiltonian of Dirac type :

$$\Psi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \quad \Psi_x = U\Psi \;\Leftrightarrow\; \begin{array}{c} -i\varphi_x + iq\phi = -\lambda\varphi \\ i\phi_x - ir\varphi = -\lambda\phi \end{array} \;\Leftrightarrow H\Psi(\lambda) = -\lambda\Psi(\lambda)$$

Nonlocality in Darboux-Crum transformations

Quantum mechanical analogue to supersymmetry, intertwining

$$L_n H_{n-1} = H_n L_n$$

iteration $\mathcal{L}_n H_0 = H_n \mathcal{L}_n$, $\mathcal{L}_n := L_n L_{n-1} \ldots L_1$, $\Psi_n(\lambda) = \mathcal{L}_n \Psi(\lambda)$

In Hirota case, Hamiltonian of Dirac type :

$$\Psi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \; \Psi_x = U\Psi \;\Leftrightarrow\; \begin{array}{c} -i\varphi_x + iq\phi = -\lambda\varphi \\ i\phi_x - ir\varphi = -\lambda\phi \end{array} \;\Leftrightarrow H\Psi(\lambda) = -\lambda\Psi(\lambda)$$

with

$$H = \begin{pmatrix} -i\partial_x & iq \\ -ir & i\partial_x \end{pmatrix} = -i\sigma_3\partial_x + V,$$

Nonlocality in Darboux-Crum transformations

Quantum mechanical analogue to supersymmetry, intertwining

$$L_n H_{n-1} = H_n L_n$$

iteration $\mathcal{L}_n H_0 = H_n \mathcal{L}_n$, $\mathcal{L}_n := L_n L_{n-1} \dots L_1$, $\Psi_n(\lambda) = \mathcal{L}_n \Psi(\lambda)$

In Hirota case, Hamiltonian of Dirac type :

$$\Psi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \; \Psi_x = U\Psi \;\Leftrightarrow\; \begin{array}{c} -i\varphi_x + iq\phi = -\lambda\varphi \\ i\phi_x - ir\varphi = -\lambda\phi \end{array} \;\Leftrightarrow H\Psi(\lambda) = -\lambda\Psi(\lambda)$$

with

$$H_n = \begin{pmatrix} -i\partial_x & iq_n \\ -ir_n & i\partial_x \end{pmatrix} = -i\sigma_3\partial_x + V_n,$$

Nonlocality in Darboux-Crum transformations

Quantum mechanical analogue to supersymmetry, intertwining

$$L_n H_{n-1} = H_n L_n$$

iteration $\mathcal{L}_n H_0 = H_n \mathcal{L}_n$, $\mathcal{L}_n := L_n L_{n-1} \dots L_1$, $\Psi_n(\lambda) = \mathcal{L}_n \Psi(\lambda)$

In Hirota case, Hamiltonian of Dirac type :

$$\Psi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \quad \Psi_x = U\Psi \;\Leftrightarrow\; \begin{array}{c} -i\varphi_x + iq\phi = -\lambda\varphi \\ i\phi_x - ir\varphi = -\lambda\phi \end{array} \;\Leftrightarrow H\Psi(\lambda) = -\lambda\Psi(\lambda)$$

with

$$H_n = \begin{pmatrix} -i\partial_x & iq_n \\ -ir_n & i\partial_x \end{pmatrix} = -i\sigma_3\partial_x + V_n,$$

iterated potentials V_n $\Leftrightarrow$ multi-soliton solutions

Solve the "seed" equations for $q = r = 0$:

$$\tilde{\Psi}_1(x,t;\lambda) = \begin{pmatrix} \varphi_1(x,t;\lambda) \\ \phi_1(x,t;\lambda) \end{pmatrix} = \begin{pmatrix} e^{\lambda x + 2i\lambda^2(\alpha - 2\delta\lambda)t + \gamma_1} \\ e^{-\lambda x - 2i\lambda^2(\alpha - 2\delta\lambda)t + \gamma_2} \end{pmatrix}$$

Implement nonlocality in the construction Ψ_2:
Two choices to achieve $r(x,t) = \pm q^*(-x,t)$

$$1\colon\ \varphi_2 = \pm\tilde{\phi}_1^*,\ \phi_2 = \tilde{\varphi}_1^* \qquad 2\colon\ \phi_1 = \tilde{\varphi}_1^*,\ \phi_2 = \pm\tilde{\varphi}_2^*$$

The second choice is not available in the local case.

Solve the "seed" equations for $q = r = 0$:

$$\tilde{\Psi}_1(x,t;\lambda) = \begin{pmatrix} \varphi_1(x,t;\lambda) \\ \phi_1(x,t;\lambda) \end{pmatrix} = \begin{pmatrix} e^{\lambda x + 2i\lambda^2(\alpha - 2\delta\lambda)t + \gamma_1} \\ e^{-\lambda x - 2i\lambda^2(\alpha - 2\delta\lambda)t + \gamma_2} \end{pmatrix}$$

Implement nonlocality in the construction Ψ_2:
Two choices to achieve $r(x,t) = \pm q^*(-x,t)$

$$1:\ \varphi_2 = \pm\tilde{\phi}_1^*,\ \phi_2 = \tilde{\varphi}_1^* \qquad 2:\ \phi_1 = \tilde{\varphi}_1^*,\ \phi_2 = \pm\tilde{\varphi}_2^*$$

The second choice is not available in the local case.
Choice 1:

$$\tilde{\Psi}_2(x,t;\lambda) = \begin{pmatrix} \varphi_2(x,t;\lambda) \\ \phi_2(x,t;\lambda) \end{pmatrix} = \begin{pmatrix} \mp e^{\lambda^* x + 2i(\lambda^*)^2(\alpha - 2\delta\lambda)t + \gamma_2^*} \\ e^{-\lambda^* x - 2i(\lambda^*)^2(\alpha - 2\delta\lambda^*)t + \gamma_1^*} \end{pmatrix}$$

with $\lambda, \gamma_1, \gamma_2 \in \mathbb{C}$

$$q_{\mathrm{st}}^{(1)}(x,t) = \frac{2(\lambda^* - \lambda)e^{2\lambda^* x + 2i(\lambda^*)^2(\alpha - 2\delta\lambda^*)t - \gamma_1^* + \gamma_2^*}}{1 + e^{2(\lambda^* - \lambda)x + 4i\left[\alpha(\lambda^*)^2 - \alpha\lambda^2 + 2\delta\lambda^3 - 2\delta(\lambda^*)^3\right]t - \gamma_1 + \gamma_2 - \gamma_1^* + \gamma_2^*}}.$$

Solve the "seed" equations for $q = r = 0$:

$$\tilde{\Psi}_1(x,t;\lambda) = \begin{pmatrix} \varphi_1(x,t;\lambda) \\ \phi_1(x,t;\lambda) \end{pmatrix} = \begin{pmatrix} e^{\lambda x + 2i\lambda^2(\alpha - 2\delta\lambda)t + \gamma_1} \\ e^{-\lambda x - 2i\lambda^2(\alpha - 2\delta\lambda)t + \gamma_2} \end{pmatrix}$$

Implement nonlocality in the construction Ψ_2:
Two choices to achieve $r(x,t) = \pm q^*(-x,t)$

$$1\colon\ \varphi_2 = \pm\tilde{\phi}_1^*,\ \phi_2 = \tilde{\varphi}_1^* \qquad 2\colon\ \phi_1 = \tilde{\varphi}_1^*,\ \phi_2 = \pm\tilde{\varphi}_2^*$$

The second choice is not available in the local case.
Choice 2:

$$\tilde{\Psi}_2(x,t;\nu) = \begin{pmatrix} \varphi_2(x,t;\nu) \\ \phi_2(x,t;\nu) \end{pmatrix} = \begin{pmatrix} e^{\nu x + 2i\nu^2(\alpha - 2\delta\nu)t + \gamma_3} \\ -e^{-\nu x - 2i\nu^2(\alpha - 2\delta\nu)t + \gamma_3^*} \end{pmatrix}$$

$$q^{(1)}_{\text{nonst}}(x,t) = \frac{2(\nu - \mu)e^{\gamma_1 - \gamma_1^* + 2\mu x + 4i\mu^2(\alpha - 2\delta\mu)t}}{1 + e^{2(\mu-\nu)x + 4i(\alpha\mu^2 - \alpha\nu^2 - 2\delta\mu^3 + 2\delta\nu^3)t + \gamma_1 - \gamma_1^* - \gamma_3 + \gamma_3^*}}.$$

Nonlocal n-soliton solutions: $q_n = q + 2\frac{\det D_n^q}{\det W_n}$, $r_n = r - 2\frac{\det D_n^r}{\det W_n}$

$$W_n = \begin{pmatrix} \varphi_1^{(n-1)} & \varphi_1^{(n-2)} & \cdots & \varphi_1 & \phi_1^{(n-1)} & \cdots & \phi_1' & \phi_1 \\ \varphi_2^{(n-1)} & \varphi_2^{(n-2)} & \cdots & \varphi_2 & \phi_2^{(n-1)} & \cdots & \phi_2' & \phi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{2n}^{(n-1)} & \varphi_{2n}^{(n-2)} & \cdots & \varphi_{2n} & \phi_{2n}^{(n-1)} & \cdots & \phi_{2n}' & \phi_{2n} \end{pmatrix}$$

$$D_n^q = \begin{pmatrix} \phi_1^{(n-2)} & \phi_1^{(n-3)} & \cdots & \phi_1 & \varphi_1^{(n)} & \cdots & \varphi_1' & \varphi_1 \\ \phi_2^{(n-2)} & \phi_2^{(n-3)} & \cdots & \phi_2 & \varphi_2^{(n)} & \cdots & \varphi_2' & \varphi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{2n}^{(n-2)} & \phi_{2n}^{(n-3)} & \cdots & \phi_{2n} & \varphi_{2n}^{(n)} & \cdots & \varphi_{2n}' & \varphi_{2n} \end{pmatrix}$$

$$D_n^r = \begin{pmatrix} \phi_1^{(n)} & \phi_1^{(n-1)} & \cdots & \phi_1 & \varphi_1^{(n-2)} & \cdots & \varphi_1' & \varphi_1 \\ \phi_2^{(n)} & \phi_2^{(n-1)} & \cdots & \phi_2 & \varphi_2^{(n-2)} & \cdots & \varphi_2' & \varphi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{2n}^{(n)} & \phi_{2n}^{(n-1)} & \cdots & \phi_{2n} & \varphi_{2n}^{(n-2)} & \cdots & \varphi_{2n}' & \varphi_{2n} \end{pmatrix}$$

Nonlocal n-soliton solutions: $q_n = q + 2\frac{\det D_n^q}{\det W_n}, \;\; r_n = r - 2\frac{\det D_n^r}{\det W_n}$

$$W_n = \begin{pmatrix} \varphi_1^{(n-1)} & \varphi_1^{(n-2)} & \cdots & \varphi_1 & \phi_1^{(n-1)} & \cdots & \phi_1' & \phi_1 \\ \varphi_2^{(n-1)} & \varphi_2^{(n-2)} & \cdots & \varphi_2 & \phi_2^{(n-1)} & \cdots & \phi_2' & \phi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{2n}^{(n-1)} & \varphi_{2n}^{(n-2)} & \cdots & \varphi_{2n} & \phi_{2n}^{(n-1)} & \cdots & \phi_{2n}' & \phi_{2n} \end{pmatrix}$$

$$D_n^q = \begin{pmatrix} \phi_1^{(n-2)} & \phi_1^{(n-3)} & \cdots & \phi_1 & \varphi_1^{(n)} & \cdots & \varphi_1' & \varphi_1 \\ \phi_2^{(n-2)} & \phi_2^{(n-3)} & \cdots & \phi_2 & \varphi_2^{(n)} & \cdots & \varphi_2' & \varphi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{2n}^{(n-2)} & \phi_{2n}^{(n-3)} & \cdots & \phi_{2n} & \varphi_{2n}^{(n)} & \cdots & \varphi_{2n}' & \varphi_{2n} \end{pmatrix}$$

In the nonlocal case use

$$\tilde{S}_{2n}^{\text{st}} = \left\{ \tilde{\Psi}_1(x,t;\lambda_1), \tilde{\Psi}_2(x,t;\lambda_1), \tilde{\Psi}_1(x,t;\lambda_2), \tilde{\Psi}_2(x,t;\lambda_2), \ldots \right\}$$

or

$$\tilde{S}_{2n}^{\text{nonst}} = \left\{ \tilde{\Psi}_1(x,t;\mu_1), \tilde{\Psi}_2(x,t;\nu_1), \tilde{\Psi}_1(x,t;\mu_2), \tilde{\Psi}_2(x,t;\nu_2), \ldots \right\}$$

Nonlocal gauge equivalence

Two zero curvature conditions are related as

$$\partial_t U_i - \partial_x V_i + [U_i, V_i] = 0 \quad \Leftrightarrow \quad \Psi_{i,t} = V_i \Psi_i, \; \Psi_{i,x} = U_i \Psi_i \quad i = 1, 2$$

$$U_1 = G U_2 G^{-1} + G_x G^{-1} \qquad V_1 = G V_2 G^{-1} + G_t G^{-1}$$

when the auxiliary fields are related by a gauge field $\Psi_1 = G \Psi_2$

Nonlocal gauge equivalence

Two zero curvature conditions are related as

$$\partial_t U_i - \partial_x V_i + [U_i, V_i] = 0 \quad \Leftrightarrow \quad \Psi_{i,t} = V_i \Psi_i, \ \Psi_{i,x} = U_i \Psi_i \quad i = 1, 2$$

$$U_1 = G U_2 G^{-1} + G_x G^{-1} \qquad V_1 = G V_2 G^{-1} + G_t G^{-1}$$

when the auxiliary fields are related by a gauge field $\Psi_1 = G \Psi_2$

System 1 $\equiv$ (nonlocal) Hirota equation $\Rightarrow$ system 2:

$$U_2 = -i\lambda G^{-1} \sigma_3 G, \qquad V_2 = \lambda G^{-1} B_1 G + \lambda^2 G^{-1} B_2 G + \lambda^3 G^{-1} B_3 G$$

Nonlocal gauge equivalence

Two zero curvature conditions are related as

$$\partial_t U_i - \partial_x V_i + [U_i, V_i] = 0 \quad \Leftrightarrow \quad \Psi_{i,t} = V_i \Psi_i, \ \Psi_{i,x} = U_i \Psi_i \quad i = 1, 2$$

$$U_1 = G U_2 G^{-1} + G_x G^{-1} \qquad V_1 = G V_2 G^{-1} + G_t G^{-1}$$

when the auxiliary fields are related by a gauge field $\Psi_1 = G\Psi_2$

System 1 $\equiv$ (nonlocal) Hirota equation $\Rightarrow$ system 2:

$$U_2 = -i\lambda G^{-1}\sigma_3 G, \qquad V_2 = \lambda G^{-1} B_1 G + \lambda^2 G^{-1} B_2 G + \lambda^3 G^{-1} B_3 G$$

Extended continuous limit of the Heisenberg spin chain

For $\quad S := G^{-1}\sigma_3 G \Rightarrow S_t = i\alpha\left(S_x^2 + S S_{xx}\right) - \beta\left[\frac{3}{2}\left(S S_x^2\right)_x + S_{xxx}\right]$

Nonlocal gauge equivalence

Two zero curvature conditions are related as

$$\partial_t U_i - \partial_x V_i + [U_i, V_i] = 0 \quad \Leftrightarrow \quad \Psi_{i,t} = V_i \Psi_i,\ \Psi_{i,x} = U_i \Psi_i \quad i = 1, 2$$

$$U_1 = G U_2 G^{-1} + G_x G^{-1} \qquad V_1 = G V_2 G^{-1} + G_t G^{-1}$$

when the auxiliary fields are related by a gauge field $\Psi_1 = G\Psi_2$

System 1 $\equiv$ (nonlocal) Hirota equation $\Rightarrow$ system 2:

$$U_2 = -i\lambda G^{-1}\sigma_3 G, \qquad V_2 = \lambda G^{-1} B_1 G + \lambda^2 G^{-1} B_2 G + \lambda^3 G^{-1} B_3 G$$

Extended continuous limit of the Heisenberg spin chain

For $\quad S := G^{-1}\sigma_3 G \Rightarrow S_t = i\alpha \left(S_x^2 + S S_{xx} \right) - \beta \left[\frac{3}{2} \left(S S_x^2 \right)_x + S_{xxx} \right]$

Extended version of the Landau-Lifschitz equation

For $\quad S = \mathbf{s} \cdot \boldsymbol{\sigma} \Rightarrow \mathbf{s}_t = -\alpha \mathbf{s} \times \mathbf{s}_{xx} - \frac{3}{2}\beta \left(\mathbf{s}_x \cdot \mathbf{s}_x \right) \mathbf{s}_x + \beta \mathbf{s} \times \left(\mathbf{s} \times \mathbf{s}_{xxx} \right)$

Parameterise

$$S = \begin{pmatrix} -\omega & u \\ v & \omega \end{pmatrix} \qquad \omega^2 + uv = 1$$

components of the matrix equation give

$$\begin{aligned} u_t &= i\alpha(u\omega_x - \omega u_x)_x - \beta \left[u_{xx} + 3/2u(u_x v_x + \omega_x^2)\right]_x \\ v_t &= -i\alpha(v\omega_x - \omega v_x)_x - \beta \left[v_{xx} + 3/2v(v_x u_x + \omega_x^2)\right]_x \end{aligned}$$

Parameterise

$$S = \begin{pmatrix} -\omega & u \\ v & \omega \end{pmatrix} \qquad \omega^2 + uv = 1$$

components of the matrix equation give

$$\begin{aligned} u_t &= i\alpha(u\omega_x - \omega u_x)_x - \beta\left[u_{xx} + 3/2u(u_x v_x + \omega_x^2)\right]_x \\ v_t &= -i\alpha(v\omega_x - \omega v_x)_x - \beta\left[v_{xx} + 3/2v(v_x u_x + \omega_x^2)\right]_x \end{aligned}$$

Nonlocality via $u(x,t) = \kappa v^*(-x,t), \omega(x,t) = \omega^*(-x,t)$

Parameterise

$$S = \begin{pmatrix} -\omega & u \\ v & \omega \end{pmatrix} \qquad \omega^2 + uv = 1$$

components of the matrix equation give

$$\begin{aligned} u_t &= i\alpha(u\omega_x - \omega u_x)_x - \beta\left[u_{xx} + 3/2u(u_x v_x + \omega_x^2)\right]_x \\ v_t &= -i\alpha(v\omega_x - \omega v_x)_x - \beta\left[v_{xx} + 3/2v(v_x u_x + \omega_x^2)\right]_x \end{aligned}$$

Nonlocality via $u(x,t) = \kappa v^*(-x,t), \omega(x,t) = \omega^*(-x,t)$
Solutions from auto-gauge transformation $\equiv$ DC transformation

$$\begin{aligned} u_1 &= \frac{2\varphi_1\varphi_2(\lambda_2\varphi_1\phi_2 - \lambda_1\varphi_2\phi_1)(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2(\varphi_2\phi_1 - \varphi_1\phi_2)^2} \\ v_1 &= \frac{2\phi_1\phi_2(\lambda_1\varphi_1\phi_2 - \lambda_2\varphi_2\phi_1)(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2(\varphi_2\phi_1 - \varphi_1\phi_2)^2} \\ \omega_1 &= 1 - \frac{2\varphi_1\varphi_2\phi_1\phi_2(\lambda_1 - \lambda_2)^2}{\lambda_1\lambda_2(\varphi_2\phi_1 - \varphi_1\phi_2)^2} \end{aligned}$$

Back to Hirota equation:
solve $S = G^{-1}\sigma_3 G$ for G and use $G_x = A_0 G,\ G_t = B_0 G$

$$A_0 = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix}$$

$$B_0 = i\alpha\left[\sigma_3 (A_0)_x - \sigma_3 A_0^2\right] + \beta\left[2A_0^3 + (A_0)_x A_0 - A_0 (A_0)_x - (A_0)_{xx}\right]$$

$$q(x,t) = \frac{\mu(t)}{2}\left(\frac{v_x}{v} + \frac{\omega v_x - \omega_x v}{v}\right)\exp\int^x \frac{\omega(s,t)v_s(s,t) - \omega_s(s,t)v(s,t)}{v(s,t)}ds$$

$$r(x,t) = \frac{1}{2\mu(t)}\left(\frac{v_x}{v} - \frac{\omega v_x - \omega_x v}{v}\right)\exp\int^x \frac{\omega_s(s,t)v(s,t) - \omega(s,t)v_s(s,t)}{v(s,t)}ds$$

Back to Hirota equation:
solve $S = G^{-1}\sigma_3 G$ for G and use $G_x = A_0 G,\ G_t = B_0 G$

$$A_0 = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix}$$

$$B_0 = i\alpha\left[\sigma_3 (A_0)_x - \sigma_3 A_0^2\right] + \beta\left[2A_0^3 + (A_0)_x A_0 - A_0 (A_0)_x - (A_0)_{xx}\right]$$

$$q(x,t) = \frac{\mu(t)}{2}\left(\frac{v_x}{v} + \frac{\omega v_x - \omega_x v}{v}\right)\exp\int^x \frac{\omega(s,t)v_s(s,t) - \omega_s(s,t)v(s,t)}{v(s,t)}ds$$

$$r(x,t) = \frac{1}{2\mu(t)}\left(\frac{v_x}{v} - \frac{\omega v_x - \omega_x v}{v}\right)\exp\int^x \frac{\omega_s(s,t)v(s,t) - \omega(s,t)v_s(s,t)}{v(s,t)}ds$$

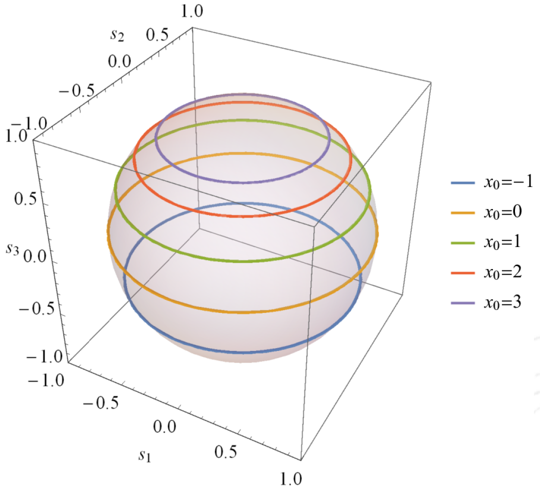
$$q_n(x,t) = \frac{\mu_n}{\prod_{k=1}^{2n}\lambda_k}\left(\frac{(\det\mathcal{V}_n)_x}{\det\Upsilon_n} - \frac{(\det\mathcal{W}_n)_x \det\mathcal{V}_n}{\det\mathcal{W}_n \det\Upsilon_n}\right)$$

$$r_n(x,t) = \frac{\prod_{k=1}^{2n}\lambda_k}{\mu_n}\left(\frac{(\det\Upsilon_n)_x}{\det\mathcal{V}_n} - \frac{(\det\mathcal{W}_n)_x \det\Upsilon_n}{\det\mathcal{W}_n \det\mathcal{V}_n}\right)$$

Nonlocality: $\quad r_n(x,t) = \kappa\prod_{i=1}^{n}\left|\lambda_{2i-1}\right|^4/\mu_n^2 q_n^*(-x,t)$

Trajectories of local extended version of the LL equation

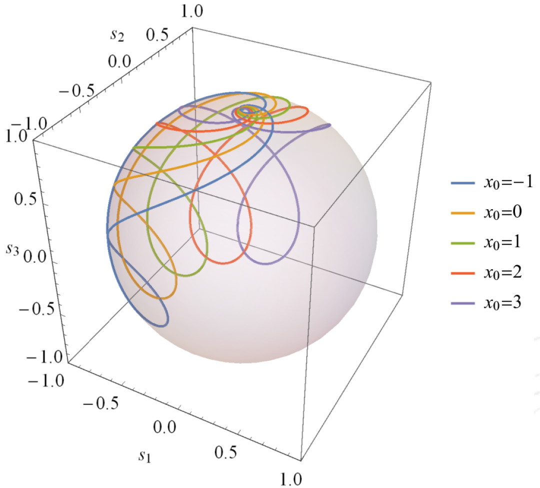
Gauge equivalent version of nonlinear Schrödinger equation $\beta = 0$



$|\mathbf{s}|^2 = 1$, $\mathbf{s}$ is real, real shift parameter

Trajectories of local extended version of the LL equation

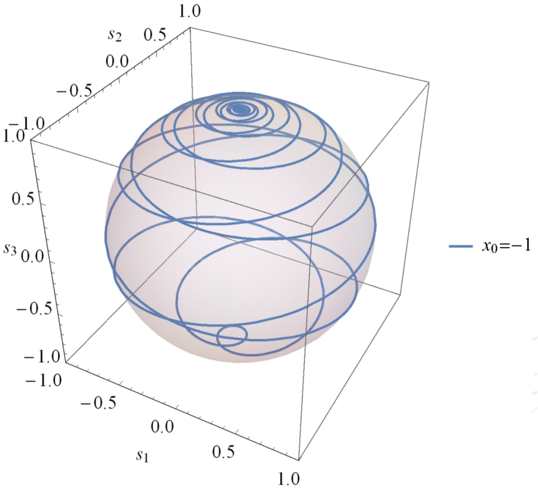
Gauge equivalent version of nonlinear Schrödinger equation $\beta = 0$



$|\mathbf{s}|^2 = 1$, $\mathbf{s}$ is real, complex shift parameter

Trajectories of local extended version of the LL equation

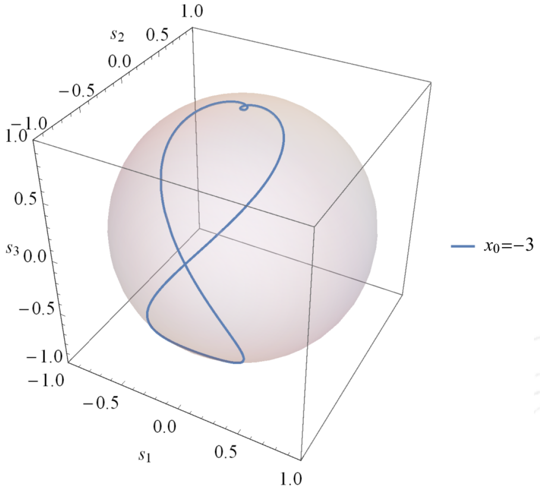
Gauge equivalent version of Hirota equation $\beta \neq 0$



$|\mathbf{s}|^2 = 1$, $\mathbf{s}$ is real, real shift parameter

Trajectories of local extended version of the LL equation

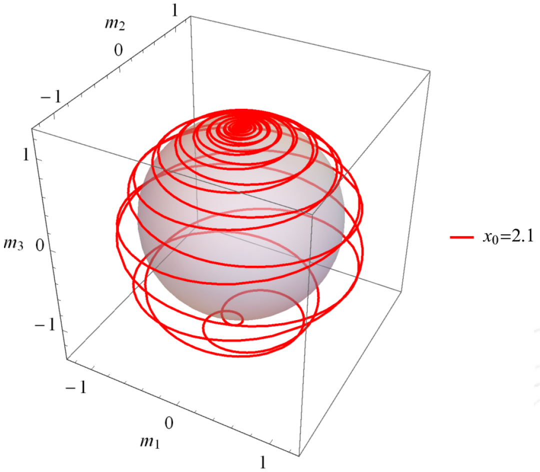
Gauge equivalent version of Hirota equation $\beta \neq 0$



$|\mathbf{s}|^2 = 1$, $\mathbf{s}$ is real, complex shift parameter

Trajectories of nonlocal extended version of the LL equation

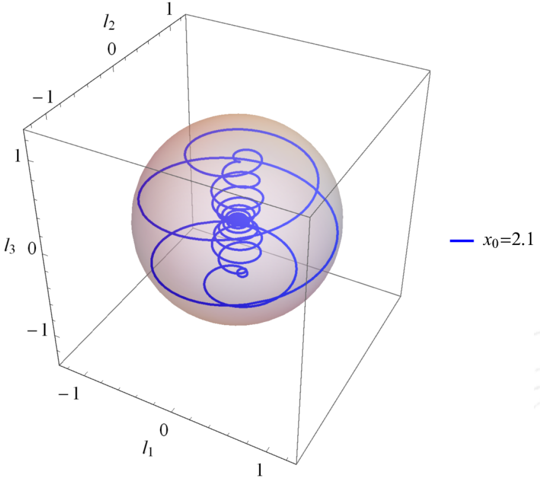
Gauge equivalent version of nonlocal Hirota equation $\beta \neq 0$



$|\mathbf{s}|^2 = 1$, $\mathbf{s} = \mathbf{m} + i\mathbf{l}$ is complex

Trajectories of nonlocal extended version of the LL equation

Gauge equivalent version of of nonlocal Hirota equation $\beta \neq 0$



$|\mathbf{s}|^2 = 1$, $\mathbf{s} = \mathbf{m} + i\mathbf{l}$ is complex

Bogomolny-Prasad-Sommerfield (BPS) solitons

Consider complex scalar field theory

$$\mathcal{L} = \frac{1}{2}\eta_{ab}\partial_\mu\phi_a\partial^\mu\phi_b - \mathcal{V}(\phi)$$

Taking the energy functional and topological charge of the form

$$E = \frac{1}{2}\int d^2x \left(A_\alpha^2 + \tilde{A}_\alpha^2\right) \qquad Q = \int d^2x A_\alpha \tilde{A}_\alpha,$$

Bogomolny-Prasad-Sommerfield (BPS) solitons

Consider complex scalar field theory

$$\mathcal{L} = \frac{1}{2}\eta_{ab}\partial_\mu\phi_a\partial^\mu\phi_b - \mathcal{V}(\phi)$$

Taking the energy functional and topological charge of the form

$$E = \frac{1}{2}\int d^2x \left(A_\alpha^2 + \tilde{A}_\alpha^2\right) \qquad Q = \int d^2x A_\alpha \tilde{A}_\alpha,$$

the Bogomolny bound for E is saturated ($E = |Q|$) by Q if

$$A_\alpha = \pm\tilde{A}_\alpha$$

Bogomolny-Prasad-Sommerfield (BPS) solitons

Consider complex scalar field theory

$$\mathcal{L} = \frac{1}{2}\eta_{ab}\partial_\mu\phi_a\partial^\mu\phi_b - \mathcal{V}(\phi)$$

Taking the energy functional and topological charge of the form

$$E = \frac{1}{2}\int d^2x \left(A_\alpha^2 + \tilde{A}_\alpha^2\right) \qquad Q = \int d^2x A_\alpha \tilde{A}_\alpha,$$

the Bogomolny bound for E is saturated ($E = |Q|$) by Q if

$$A_\alpha = \pm\tilde{A}_\alpha$$

These two equations result as a compatibility equation between the Euler-Lagrange equations and $\delta Q = 0$.

non-Hermitian sine-Gordon model

$$\mathcal{V} = \frac{1}{2(1+\lambda^2)} \left[(\sin\phi_1 - \mu)^2 + 2i\lambda \left(\sin\phi_1 - \mu\right) \sin\phi_2 + \sin^2\phi_2 \right]$$

non-Hermitian sine-Gordon model

$$\mathcal{V} = \frac{1}{2(1+\lambda^2)}\left[(\sin\phi_1 - \mu)^2 + 2i\lambda(\sin\phi_1 - \mu)\sin\phi_2 + \sin^2\phi_2\right]$$

static BPS equations

$$BPS_1^{\pm} \quad : \qquad \partial_x\phi_1 = \pm\frac{1}{1+\lambda^2}(\sin\phi_1 - \mu + i\lambda\sin\phi_2) =: G_1^{\pm}$$

$$BPS_2^{\pm} \quad : \qquad \partial_x\phi_2 = \pm\frac{1}{1+\lambda^2}[i\lambda(\sin\phi_1 - \mu) + \sin\phi_2] =: G_2^{\pm}$$

non-Hermitian sine-Gordon model

$$\mathcal{V} = \frac{1}{2(1+\lambda^2)}\left[(\sin\phi_1 - \mu)^2 + 2i\lambda\,(\sin\phi_1 - \mu)\sin\phi_2 + \sin^2\phi_2\right]$$

static BPS equations

$$BPS_1^{\pm} \quad : \qquad \partial_x\phi_1 = \pm\frac{1}{1+\lambda^2}\left(\sin\phi_1 - \mu + i\lambda\sin\phi_2\right) =: G_1^{\pm}$$

$$BPS_2^{\pm} \quad : \qquad \partial_x\phi_2 = \pm\frac{1}{1+\lambda^2}\left[i\lambda\,(\sin\phi_1 - \mu) + \sin\phi_2\right] =: G_2^{\pm}$$

with modified $\mathcal{CPT}$-symmetry

$$\phi_1(x) \to [\phi_1(-x)]^\dagger\,, \phi_2(x) \to -[\phi_2(-x)]^\dagger\,, \Leftrightarrow BPS_i^{\pm} \to \left(BPS_i^{\mp}\right)^*$$

non-Hermitian sine-Gordon model

$$\mathcal{V} = \frac{1}{2(1+\lambda^2)} \left[(\sin\phi_1 - \mu)^2 + 2i\lambda \left(\sin\phi_1 - \mu\right) \sin\phi_2 + \sin^2\phi_2 \right]$$

static BPS equations

$$BPS_1^{\pm} \quad : \qquad \partial_x \phi_1 = \pm \frac{1}{1+\lambda^2} \left(\sin\phi_1 - \mu + i\lambda \sin\phi_2\right) =: G_1^{\pm}$$

$$BPS_2^{\pm} \quad : \qquad \partial_x \phi_2 = \pm \frac{1}{1+\lambda^2} \left[i\lambda \left(\sin\phi_1 - \mu\right) + \sin\phi_2\right] =: G_2^{\pm}$$

with modified $\mathcal{CPT}$-symmetry

$$\phi_1(x) \to [\phi_1(-x)]^\dagger, \phi_2(x) \to -[\phi_2(-x)]^\dagger, \Leftrightarrow BPS_i^{\pm} \to \left(BPS_i^{\mp}\right)^*$$

Thus we have

$$\mathcal{V}\left[\phi_{\pm}(x)\right] = \mathcal{V}^\dagger \left[\phi_{\mp}(-x)\right].$$

which guarantees the reality of the energy.

Solutions and energy

Hermitian limit $\lambda = 0$:

$$\phi_1^{\pm(n)} = 2\arctan\left[\frac{1}{\mu} + \frac{\sqrt{(1-\mu^2)}}{\mu}\tanh\left[\frac{1}{2}\sqrt{(1-\mu^2)}(\kappa_1 \pm x)\right]\right] + 2\pi n$$

$$\phi_2^{\pm(n)} = 2\arctan\left(e^{\pm x+\kappa_2}\right) + 2\pi n$$

asymptotic limits:

$$\lim_{x\to\infty}\phi_1^{+(n)}(x) = \lim_{x\to-\infty}\phi_1^{-(n)}(x) = 2n\pi + \mathrm{sign}(\mu)\pi - \arcsin(\mu)$$

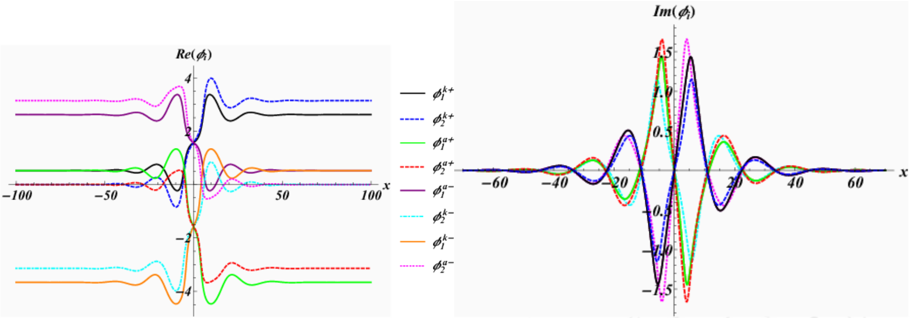
$$\lim_{x\to-\infty}\phi_1^{+(n)}(x) = \lim_{x\to\infty}\phi_1^{-(n)}(x) = 2n\pi + \mathrm{sign}(\mu)\arcsin(\mu)$$

$$\lim_{x\to\pm\infty}\phi_2^{+(n)}(x) = \lim_{x\to\mp\infty}\phi_2^{-(n)}(x) = 2n\pi + \frac{\pi\pm\pi}{2}$$

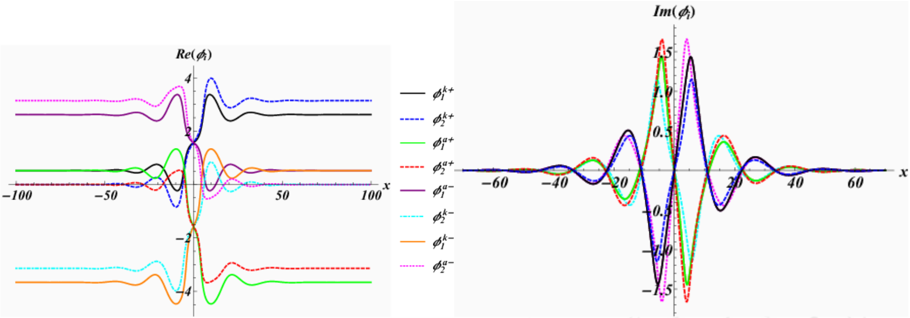
real energy for $|\mu| \leq 1$

$$E^{\pm}(\mu) = 2\left[1 + \sqrt{1-\mu^2} - \mu\arctan\left(\frac{\sqrt{1-\mu^2}}{\mu}\right)\right]$$

non-Hermitian case $\lambda \neq 0$ (numerical solution)



non-Hermitian case $\lambda \neq 0$ (numerical solution)



asymptotic limits are the same $\Rightarrow$ energies are the same

vacua:

$$v_1^{(n,m)} = (\arcsin \mu + 2\pi n, m\pi), \quad v_2^{(n,m)} = (\pi - \arcsin \mu + 2n\pi, m\pi)$$

vacua:

$$v_1^{(n,m)} = (\arcsin\mu + 2\pi n, m\pi), \quad v_2^{(n,m)} = (\pi - \arcsin\mu + 2n\pi, m\pi)$$

nature of the fixed points from eigenvalues of the Jacobian

$$J = \left. \begin{pmatrix} \partial_{\phi_1} G_1^{\pm} & \partial_{\phi_2} G_1^{\pm} \\ \partial_{\phi_1} G_2^{\pm} & \partial_{\phi_2} G_2^{\pm} \end{pmatrix} \right|_{v_j^{(n,m)}}$$

vacua:

$$v_1^{(n,m)} = (\arcsin\mu + 2\pi n, m\pi), \quad v_2^{(n,m)} = (\pi - \arcsin\mu + 2n\pi, m\pi)$$

nature of the fixed points from eigenvalues of the Jacobian

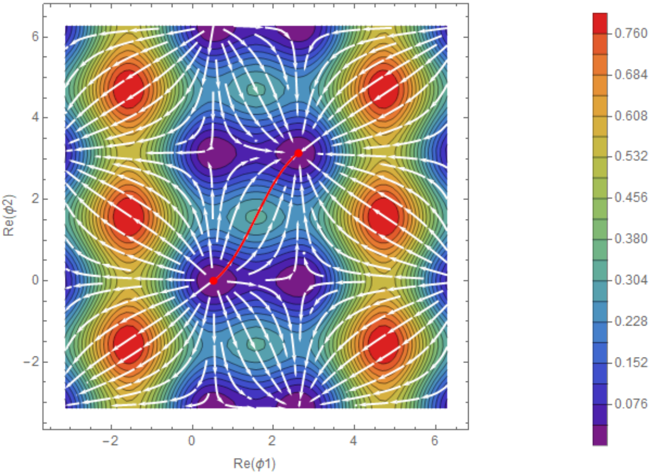
$$J = \left. \begin{pmatrix} \partial_{\phi_1} G_1^{\pm} & \partial_{\phi_2} G_1^{\pm} \\ \partial_{\phi_1} G_2^{\pm} & \partial_{\phi_2} G_2^{\pm} \end{pmatrix} \right|_{v_j^{(n,m)}}$$

solutions interpolate between different vacua as

$$v_1^{(0,0)} \xrightarrow{\phi_1^{k+}\phi_2^{k+}} v_2^{(0,1)}, \quad v_1^{(0,0)} \xrightarrow{\phi_1^{a+}\phi_2^{a+}} v_2^{(-1,1)}$$

$$v_1^{(0,0)} \xrightarrow{\phi_1^{a-}\phi_2^{k-}} v_2^{(0,-1)}, \quad v_1^{(0,0)} \xrightarrow{\phi_1^{k-}\phi_2^{a-}} v_2^{(-1,1)}$$

gradient flow superimposed on the coupled sine-Gordon potential



Conclusions and Outlook

- $\mathcal{PT}$-symmetric complex solitons have real energies and new types of behaviour.

Conclusions and Outlook

- $\mathcal{PT}$-symmetric complex solitons have real energies and new types of behaviour.
- Using $\mathcal{PT}$ conjugations we find new integrable versions of the Hirota equation, with different types of qualitative behaviour.

Conclusions and Outlook

- $\mathcal{PT}$-symmetric complex solitons have real energies and new types of behaviour.
- Using $\mathcal{PT}$ conjugations we find new integrable versions of the Hirota equation, with different types of qualitative behaviour.
- The nonlocality can be systematically implemented into solution procedures, such as Hirota's method and Darboux transformations.

Conclusions and Outlook

- $\mathcal{PT}$-symmetric complex solitons have real energies and new types of behaviour.
- Using $\mathcal{PT}$ conjugations we find new integrable versions of the Hirota equation, with different types of qualitative behaviour.
- The nonlocality can be systematically implemented into solution procedures, such as Hirota's method and Darboux transformations.
- Nonlocality is inherited in gauge equivalent systems.

Conclusions and Outlook

- $\mathcal{PT}$-symmetric complex solitons have real energies and new types of behaviour.
- Using $\mathcal{PT}$ conjugations we find new integrable versions of the Hirota equation, with different types of qualitative behaviour.
- The nonlocality can be systematically implemented into solution procedures, such as Hirota's method and Darboux transformations.
- Nonlocality is inherited in gauge equivalent systems.
- Complex BPS solitons have real energies.

Thank you for your attention

In case you are interested, register for the virtual seminar series on
Pseudo-Hermitian Hamiltonians in Quantum Physics
https://vphhqp.com