Degenrate and nonlocal soliton solutions from PT-symmetry

Andreas Fring

SIG IX (Solitons @ work workshop)
virtual, 28 of June - 02 of July

based on work with
Julia Cen (Los Alamos National Laboratory, USA)
Francisco Correa (Universidad Austral de Chile, Chile)
Takanobu Taira (City, University of London, UK)

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Outline

- Why study complex PT-symmetric solitons?
  real energies, degeneracy problem, new models, ...

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PHYSICAL REVIEW D **103,** 025024 (2021)

Kink moduli spaces: Collective coordinates reconsidered

N. S. Manton,[1,*] K. Oleś,[2,†] T. Romańczukiewicz,[2,‡] and A. Wereszczyński[2,§]
[1]*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*
[2]*Institute of Theoretical Physics, Jagiellonian University, Lojasiewicza 11, 30-348 Kraków, Poland*

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Moduli spaces—finite-dimensional, collective coordinate manifolds—for kinks and antikinks in ϕ^4 theory and sine-Gordon theory are reconsidered. The field theory Lagrangian restricted to moduli space defines a reduced Lagrangian, combining a potential with a kinetic term that can be interpreted as a Riemannian metric on moduli space. Moduli spaces should be metrically complete, or have an infinite potential on their boundary. Examples are constructed for both kink-antikink and kink-antikink-kink configurations. The naive position coordinates of the kinks and antikinks sometimes need to be extended from real to imaginary values, although the field remains real. The previously discussed null-vector problem for the shape modes of ϕ^4 kinks is resolved by a better coordinate choice. In sine-Gordon theory, moduli spaces can be constructed using exact solutions at the critical energy separating scattering and breather (or wobble) solutions; here, energy conservation relates the metric and potential. The reduced dynamics on these moduli spaces accurately reproduces properties of the exact solutions over a range of energies.

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- Conclusions

Reality of N-Soliton charges

The complex KdV equation equals two coupled real equations

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Leftrightarrow \quad \begin{cases} p_t + 6pp_x + p_{xxx} - 6qq_x = 0 \\ q_t + 6\,(pq)_x + q_{xxx} = 0 \end{cases}$$

with $u(x,t) = p(x,t) + iq(x,t)$, $p(x,t)$, $q(x,t) \in \mathbb{R}$

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- Unifies some known special cases:
  - for $(pq)_x \to pq_x$: complex KdV $\Rightarrow$ Hirota-Satsuma equations
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- $\mathcal{PT}$-symmetry:
  $x \to -x$, $t \to -t$, $i \to -i$, $u \to u$, $p \to p$, $q \to -q$

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- $\mathcal{PT}$-symmetry:
  $x \to -x$, $t \to -t$, $i \to -i$, $u \to u$, $p \to p$, $q \to -q$
- Integrability:
  Lax pair:

$$L_t = [M, L] \quad L = \partial_x^2 + \frac{1}{6}u, \;\; M = 4\partial_x^3 + u\partial_x + \frac{1}{2}u_x$$

Solutions from Hirota's direct method

Convert KdV equation into Hirota's bilinear form

$$\left(D_x^4 + D_x D_t\right) \tau \cdot \tau = 0$$

with $u = 2(\ln \tau)_{xx}$. (D_x, D_t are Hirota derivatives)

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Expanding $\tau = \sum_{k=0}^{\infty} \lambda^k \tau^k$ gives multi-soliton solutions

$$\begin{aligned}
\tau_{\mu;\alpha}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} \\
\tau_{\mu,\nu;\alpha,\beta}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} + e^{\eta_{\nu;\beta}} + \varkappa(\alpha,\beta)e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}} \\
\tau_{\mu,\nu,\rho;\alpha,\beta,\gamma}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} + e^{\eta_{\nu;\beta}} + e^{\eta_{\rho;\gamma}} + \varkappa(\alpha,\beta)e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}} \\
&\quad + \varkappa(\alpha,\gamma)e^{\eta_{\mu;\alpha}+\eta_{\rho;\gamma}} + \varkappa(\beta,\gamma)e^{\eta_{\nu;\beta}+\eta_{\rho;\gamma}} \\
&\quad + \varkappa(\alpha,\beta)\varkappa(\alpha,\gamma)\varkappa(\beta,\gamma)e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}+\eta_{\rho;\gamma}}
\end{aligned}$$

with $\eta_{\mu;\alpha} := \alpha x - \alpha^3 t + \mu$, $\varkappa(\alpha,\beta) := (\alpha-\beta)^2/(\alpha+\beta)^2$
$\mu,\nu,\rho \in \mathbb{C}$, $\alpha,\beta,\gamma \in \mathbb{R}$

One-soliton solution

We find

$$u_{i\theta;\alpha}(x,t) = \frac{\alpha^2 + \alpha^2 \cos\theta \cosh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2} - i \frac{\alpha^2 \sin\theta \sinh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2}$$

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The solution found by Khare and Saxena is the special case

$$u_{\pm i\frac{\pi}{2};\alpha}(x,t) = \alpha^2 \mathrm{sech}^2\left(\alpha x - \alpha^3 t\right) \mp i\alpha^2 \tanh\left(\alpha x - \alpha^3 t\right) \mathrm{sech}\left(\alpha x - \alpha^3 t\right)$$

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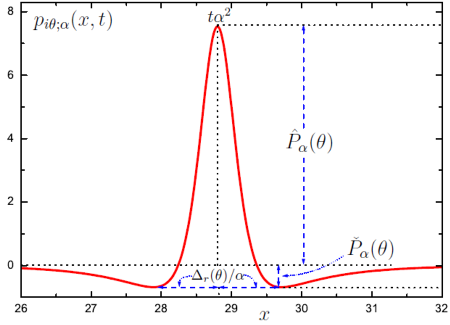

$$\begin{aligned}
\hat{P}_\alpha(\theta) &= \frac{\alpha^2}{2} \sec^2\left(\frac{\theta}{2}\right) \\
\check{P}_\alpha(\theta) &= \frac{\alpha^2}{4} \cot^2(\theta) \\
\Delta_r(\theta) &= \mathrm{arccosh}(\cos\theta - 2\sec\theta)
\end{aligned}$$

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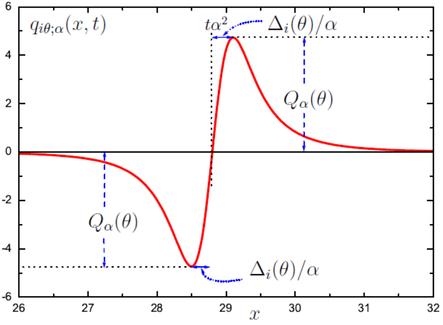

$$Q_\alpha(\theta) = \frac{8\alpha^2 \sqrt{5 + \cos(2\theta) + \cos\theta A}}{[6\cos\theta + A]^2 / \sin\theta}$$

$$\Delta_i(\theta) = \operatorname{arccosh}\left[\frac{1}{2}\cos\theta + \frac{1}{4}A\right]$$

$$A = \sqrt{2}\sqrt{17 + \cos(2\theta)}$$

Real charges from one-soliton solution

$$\begin{aligned}
\text{Mass}: m_\alpha &= \int_{-\infty}^{\infty} u_{i\theta;\alpha}(x,t)dx = 2\alpha \\
\text{Momentum}: p_\alpha &= \int_{-\infty}^{\infty} u_{i\theta;\alpha}^2 dx = \frac{2}{3}\alpha^3 \\
\text{Energy}: E_\alpha &= \int_{-\infty}^{\infty} \left[2u_{i\theta;\alpha}^3 - (u_{i\theta;\alpha})_x^2\right] dx = \frac{2}{5}\alpha^5
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Reality follows immediately from $\mathcal{PT}$-symmetry

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E = \int_{-\infty}^{\infty} dx \mathcal{H}[\phi(x)] = -\int_{\infty}^{-\infty} dx \mathcal{H}[\phi(-x)] = \int_{-\infty}^{\infty} dx \mathcal{H}^\dagger[\phi(x)] = E^*
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This is not possible for N-soliton solutions with $N > 2$.

Nondegenerate two-soliton solutions

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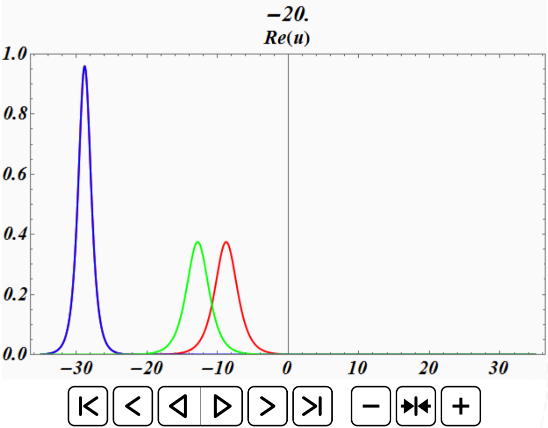

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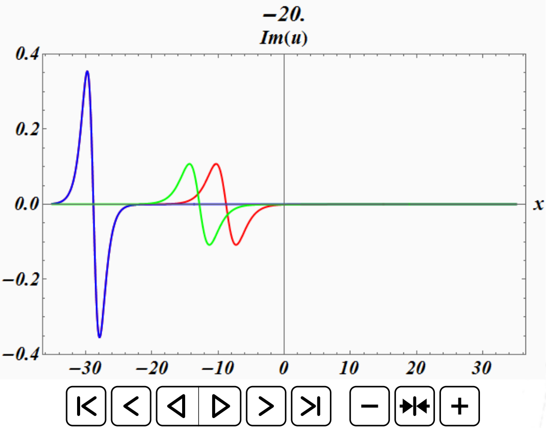

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Time-delays and lateral displacements

Comparing trajectories in the asymptotic past and future

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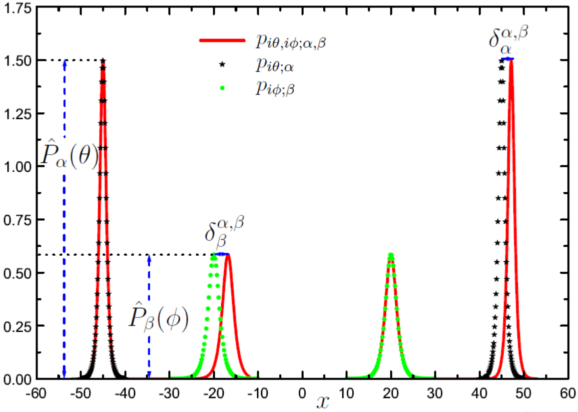

$$\delta_x^{y,z} := \frac{2}{x} \ln\left(\frac{y+z}{y-z}\right)$$

Displacements: $(\Delta_x)_\alpha = \delta_\alpha^{\alpha,\beta}$ and $(\Delta_x)_\beta = -\delta_\beta^{\alpha,\beta}$

Time-delays: $(\Delta_t)_\alpha = -\frac{1}{\alpha^2}\delta_\alpha^{\alpha,\beta}$ and $(\Delta_t)_\beta = \frac{1}{\beta^2}\delta_\beta^{\alpha,\beta}$

Consistency relations: $\sum_k m_k(\Delta_x)_k = 0$ and $\sum_k p_k(\Delta_t)_k = 0$

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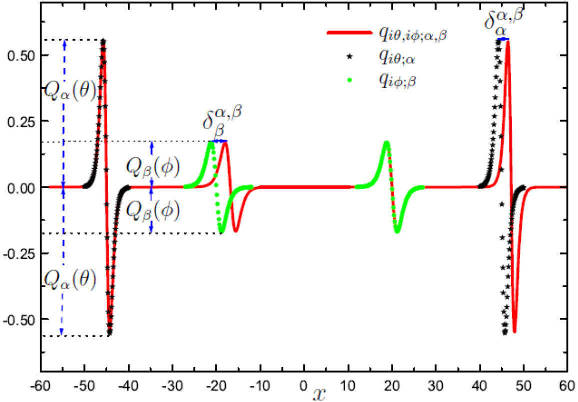

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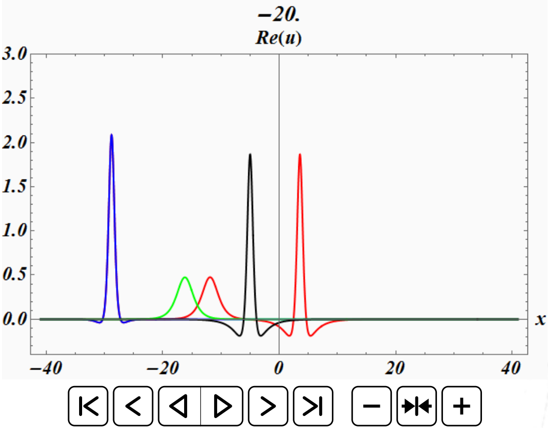

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$$\alpha = 6/5,\ \beta = 9/10,\ \gamma = 1/2,\ \mu = i7/5\pi,\ \nu = i1/4\pi,\ \rho = i7/6\pi$$

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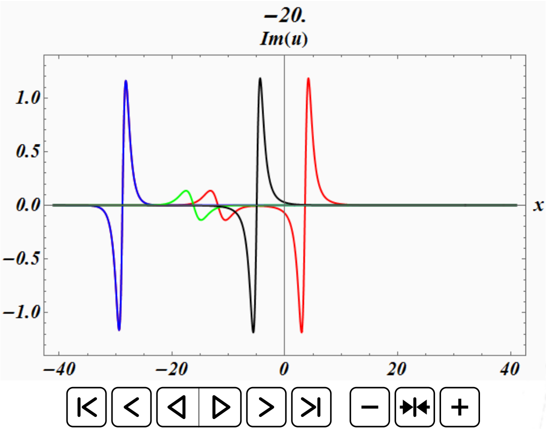

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Factorized Scattering

Displacements:

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\begin{aligned}
(\Delta_x)_\alpha &= \delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma} \\
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N-soliton time-delays $\equiv$ sum of two-soliton time-delays

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$$\begin{aligned}(\Delta_x)_\alpha &= \delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma}\\ (\Delta_x)_\beta &= \delta_\beta^{\beta,\gamma} - \delta_\beta^{\alpha,\beta}\\ (\Delta_x)_\gamma &= -\delta_\gamma^{\alpha,\gamma} - \delta_\gamma^{\beta,\gamma}\end{aligned}$$

Time-delays:

$$\begin{aligned}(\Delta_t)_\alpha &= -\frac{1}{\alpha^2}\left(\delta_\alpha^{\alpha,\beta} + \delta_\alpha^{\alpha,\gamma}\right)\\ (\Delta_t)_\beta &= \frac{1}{\beta^2}\left(\delta_\beta^{\alpha,\beta} - \delta_\beta^{\beta,\gamma}\right)\\ (\Delta_t)_\gamma &= \frac{1}{\gamma^2}\left(\delta_\gamma^{\alpha,\gamma} + \delta_\gamma^{\beta,\gamma}\right)\end{aligned}$$

Consistency relations: $\sum_k m_k(\Delta_x)_k = 0$ and $\sum_k p_k(\Delta_t)_k = 0$

N-soliton time-delays $\equiv$ sum of two-soliton time-delays

Classical factorization

This corresponds to the factorization of the quantum S-matrix described by the Yang-Baxter and bootstrap equation.

Reality of complex N-soliton charges

Asymptotically complex N-solitons factor into N one-solitons

Charges based on one-solitons solutions are real by $\mathcal{PT}$-symmetry

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Asymptotically complex N-solitons factor into N one-solitons

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Therefore

Reality condition

$\mathcal{PT}$-symmetry and integrability ensure the reality of all charges.

Regularization of degenerate multi-solitons

- In general for real solutions:

  The limit $E_\alpha \to E_\beta$ gives $\lim_{\alpha\to\beta} u_{\alpha,\beta,\gamma,\ldots}(x,t) \to \infty$

  The best scenario still has cusps.

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- In the complex case the limits become finite.

  Technically we use Wronskians as τ-functions involving solutions of the Schrödinger equation and Jordan states obtained from Darboux-Crum transformations.

  A link to Hirota's direct method and solutions obtained from a superpositon principle based on Bäcklund transformations is also established.

Degenerate two-soliton solutions

$$u_{i\theta,i\phi;\alpha,\alpha}(x,t)=\frac{2\alpha^2\left[\left(\alpha x-3\alpha^3 t+i\phi\right)\sinh\left(\eta_{i\theta;\alpha}\right)-2\cosh\left(\eta_{i\theta;\alpha}\right)-2\right]}{\left[\alpha x-3\alpha^3 t+i\phi+\sinh\left(\eta_{i\theta;\alpha}\right)\right]^2}$$

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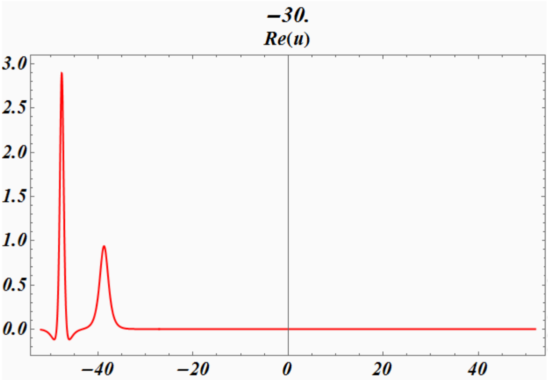

$\alpha = 6/5,\ \theta = \pi/3,\ \phi = \pi/4$

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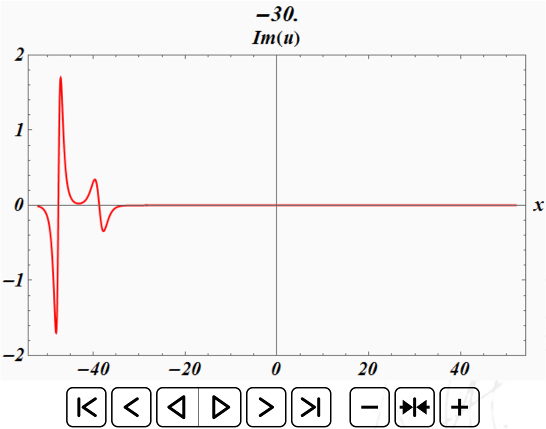

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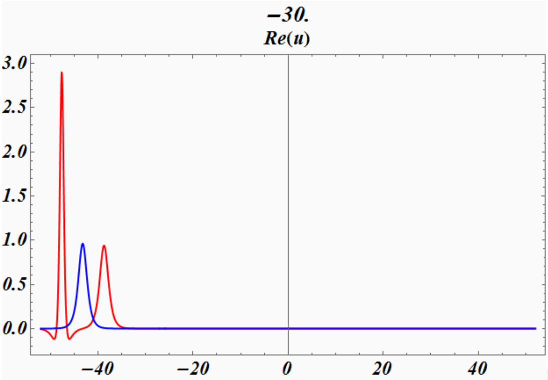

$p_{i\theta,i\phi;\alpha,\alpha}(x,t)$

$p_{i\theta;\alpha}(x,t)$

Relative displacement:

$\Delta(t) = \frac{1}{\alpha}\ln\left(4\alpha^3|t|\right)$

Total displacement:

$\pm 2\Delta(t)$

$\alpha = 6/5,\ \theta = \pi/3,\ \phi = \pi/4$

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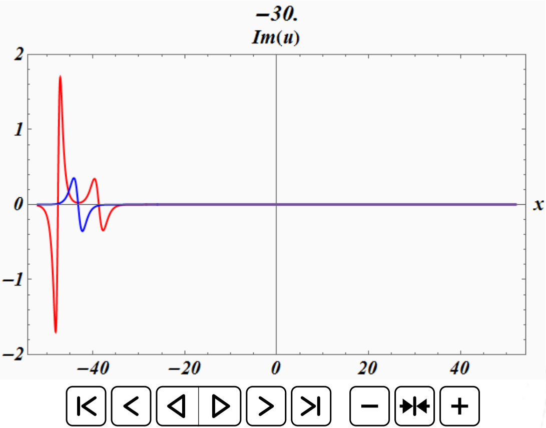

$\alpha=6/5,\ \theta=\pi/3,\ \phi=\pi/4$

$q_{i\theta,i\phi;\alpha,\alpha}(x,t)$
$q_{i\theta;\alpha}(x,t)$

Relative displacement:
$\Delta(t)=\frac{1}{\alpha}\ln\left(4\alpha^3|t|\right)$
Total displacement:
$\pm2\Delta(t)$

The equivalent solution in the sine-Gordon model reduces for $\alpha \to 1$ to the solution with energy that separates scattering solutions from breather solutions on moduli spaces.
see [J. Cen, F. Correa, A. Fring, Degenerate multi-solitons in the sine-Gordon equation, Journal of Physics A: 50 (2017): 435201.]

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$$\pm(t-t_0) = \int \sqrt{\frac{g(a)}{2(E-V(a))}}\, da, \qquad (25)$$

and can be calculated numerically.

Before presenting the solutions, let us recall the analogous exact solutions of sine-Gordon theory. The breather solution

$$\phi(x,t) = -4\arctan\left(\frac{\sqrt{1-\omega^2}\sin(\omega t)}{\omega\cosh(\sqrt{1-\omega^2}x)}\right) \qquad (26)$$

has frequency ω in the range (0, 1) and energy

$$E = 16\sqrt{1-\omega^2}. \qquad (27)$$

This solution exactly matches (19) if we ignore the shape-changing factor $\sqrt{1-\omega^2}$ in $\cosh(\sqrt{1-\omega^2}x)$ and say that the breather has modulus dynamics given by

$$\sinh(a(t)) = -\frac{\sqrt{1-\omega^2}\sin(\omega t)}{\omega}. \qquad (28)$$

The kink-antikink scattering solution is the analytic continuation of the breather, when the frequency becomes imaginary. Setting $\omega = iq$ in (26), with q real, we obtain the solution

Particularly relevant is the exact sG solution with the critical energy $E = 16$,

$\alpha \to 1 \Rightarrow$
$$\phi(x,t) = -4\arctan\left(\frac{t}{\cosh(x)}\right), \qquad (33)$$

which can be regarded either as a scattering solution where the initial incoming velocities have decreased to zero, or as a breather of infinite period, where the kink and antikink reach spatial infinity with zero velocity. It evolves precisely through the configurations in our moduli space, with

$$\sinh(a(t)) = -t. \qquad (34)$$

Equation (34) also solves the equation of motion for $a(t)$ on the moduli space, because a solution of the field equation, being a stationary point of the action for unconstrained field variations, is automatically a stationary point of the action for a smoothly embedded set of constrained fields (the fields in the moduli space), provided the Lagrangian of the constrained problem is the restriction of the Lagrangian of the unconstrained problem, as here. This has an interesting consequence. Differentiating (34), we see that $\cosh(a(t))\dot{a} = -1$, and substituting this into the energy conservation equation (24) we derive the relation (23) between the metric and potential on moduli space.

Let us now investigate the accuracy of the moduli space

Degenerate N-soliton solutions ($\alpha_1 = \alpha_2 = \ldots \alpha_N$)

Notation:

$\lim_{\alpha_2,\ldots,\alpha_N \to \alpha_1 = \alpha} u_{i\theta_1 = i\theta,\ldots,i\theta_N;\alpha_1,\ldots,\alpha_N} = p_{i\theta,\ldots,i\theta_N;N\alpha} + iq_{i\theta,\ldots,i\theta_N;N\alpha}$

Asymptotic limits:

$$\lim_{t\to\sigma\infty} p_{i\theta,\ldots,i\theta_{2n};2n\alpha}\left[t\alpha^2 + \sigma\Delta_{n,\ell,1}(t), t\right] = \hat{P}_\alpha\left(\theta + \frac{1-(-1)^{n+\ell+1}}{2}\pi\right)$$

$$\lim_{t\to\sigma\infty} p_{i\theta,\ldots,i\theta_{2n};2n\alpha}\left[t\alpha^2 - \sigma\Delta_{n,\ell,1}(t), t\right] = \hat{P}_\alpha\left(\theta + \frac{1-(-1)^{n+\ell}}{2}\pi\right)$$

for $n = 1, 2, \ldots,$ $\ell = 1, 2, \ldots, n,$ $\sigma = \pm 1$

$$\lim_{t\to\sigma\infty} p_{i\theta,\ldots,i\theta_{2n+1};(2n+1)\alpha}\left[t\alpha^2 \pm \Delta_{n,\ell,0}(t), t\right] = \hat{P}_\alpha\left(\theta + \frac{1-(-1)^{n+\ell}}{2}\pi\right)$$

for $n = 0, 1, 2, \ldots,$ $\ell = 0, 1, 2, \ldots, n$

Time-dependent displacements:

$$\Delta_{n,\ell,\kappa}(t) = \frac{1}{\alpha}\ln\left[\frac{(n-\ell)!}{(n+\ell-\kappa)!}\left(4\left|t\right|\alpha^3\right)^{2\ell-\kappa}\right]$$

Nonlocality

Consider higher order nonlinear Schrödinger equation

$$iq_t + \frac{1}{2} q_{xx} + |q|^2 q \qquad = 0$$

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$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon\left[\alpha q_{xxx} + \beta |q|^2 q_x + \gamma q |q|^2_x\right] = 0$$

$\mathcal{PT}$-symmetry: $\mathcal{PT}: x \to -x,\ t \to -t,\ i \to -i,\ q \to q$

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Integrable cases:

$\varepsilon = 0 \equiv$nonlinear Schrödinger equation (NLSE)

$\alpha : \beta : \gamma = 0 : 1 : 1 \equiv$ derivative NLSE of type I

$\alpha : \beta : \gamma = 0 : 1 : 0 \equiv$ derivative NLSE of type II

$\alpha : \beta : \gamma = 1 : 6 : 3 \equiv$ Sasa-Satsuma equation

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Consider higher order nonlinear Schrödinger equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon\left[\alpha q_{xxx} + \beta |q|^2 q_x + \gamma q |q|_x^2\right] = 0$$

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Integrable cases:

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$\alpha : \beta : \gamma = 0 : 1 : 0 \equiv$ derivative NLSE of type II
$\alpha : \beta : \gamma = 1 : 6 : 3 \equiv$ Sasa-Satsuma equation
$\alpha : \beta : \gamma = 1 : 6 : 0 \equiv$ Hirota equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon\left[q_{xxx} + 6|q|^2 q_x\right] = 0$$

Zero curvature condition

$$\partial_t U - \partial_x V + [U, V] = 0$$

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$$\begin{aligned} A_x(x,t) &= q(x,t)C(x,t) - r(x,t)B(x,t) \\ B_x(x,t) &= q_t(x,t) - 2q(x,t)A(x,t) - 2i\lambda B(x,t) \\ C_x(x,t) &= r_t(x,t) + 2r(x,t)A(x,t) + 2i\lambda C(x,t) \end{aligned}$$

$$\begin{aligned} A(x,t) &= -i\alpha qr - 2i\alpha\lambda^2 + \beta\left(rq_x - qr_x - 4i\lambda^3 - 2i\lambda qr\right) \\ B(x,t) &= i\alpha q_x + 2\alpha\lambda q + \beta\left(2q^2 r - q_{xx} + 2i\lambda q_x + 4\lambda^2 q\right) \\ C(x,t) &= -i\alpha r_x + 2\alpha\lambda r + \beta\left(2qr^2 - r_{xx} - 2i\lambda r_x + 4\lambda^2 r\right) \end{aligned}$$

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$$\begin{aligned} q_t - i\alpha q_{xx} + 2i\alpha q^2 r + \beta\left[q_{xxx} - 6qrq_x\right] &= 0 \\ r_t + i\alpha r_{xx} - 2i\alpha qr^2 + \beta\left(r_{xxx} - 6qrr_x\right) &= 0 \end{aligned}$$

Nonlocality from zero curvature condition

Complex conjugate pair: $r(x,t) = \kappa q^*(x,t)$ (Hirota equation)

$$\begin{aligned} iq_t &= -\alpha\left(q_{xx} - 2\kappa\left|q\right|^2 q\right) - i\beta\left(q_{xxx} - 6\kappa\left|q\right|^2 q_x\right) \\ -iq_t^* &= -\alpha\left(q_{xx}^* - 2\kappa\left|q\right|^2 q^*\right) + i\beta\left(q_{xxx}^* - 6\kappa\left|q\right|^2 q_x^*\right) \end{aligned}$$

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$\mathcal{P}$ conjugate pair: $r(x,t) = \kappa q^*(-x,t)$ (Nonlocal Hirota equ^n)

$$\begin{aligned} iq_t &= -\alpha\left[q_{xx} - 2\kappa\tilde{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q\tilde{q}^* q_x] \\ -i\tilde{q}_t^* &= -\alpha\left[\tilde{q}_{xx}^* - 2\kappa q(\tilde{q}^*)^2\right] - \delta(\tilde{q}_{xxx}^* - 6\kappa\tilde{q}^* q\tilde{q}_x^*) \end{aligned}$$

$$\beta = i\delta,\ \alpha,\delta \in \mathbb{R},\ q := q(x,t);\ \tilde{q} := q(-x,t)$$

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\end{aligned}
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$\mathcal{T}$ conjugate pair: $r(x,t) = \kappa q^*(x,-t)$

$$
\begin{aligned}
iq_t &= -i\hat{\delta}\left[q_{xx} - 2\kappa\hat{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q\hat{q}^* q_x] \\
i\hat{q}_t^* &= i\hat{\delta}\left[\hat{q}_{xx}^* - 2\kappa q(\hat{q}^*)^2\right] + \delta(\hat{q}_{xxx}^* - 6\kappa\hat{q}^* q\hat{q}_x^*)
\end{aligned}
$$

$$\alpha = i\hat{\delta},\ \beta = i\delta,\ \hat{\delta},\delta \in \mathbb{R};\ \hat{q} := q(x,-t)$$

$\mathcal{PT}$-conjugate pair: $r(x,t) = \kappa q^*(-x,-t)$

$$
\begin{aligned}
q_t &= -\check{\delta}\left[q_{xx} - 2\kappa \check{q}^* q^2\right] - \beta[q_{xxx} - 6\kappa q \check{q}^* q_x] \\
-\check{q}_t^* &= -\check{\delta}\left[\check{q}_{xx}^* - 2\kappa q (\check{q}^*)^2\right] + \beta(\check{q}_{xxx}^* - 6\kappa \check{q}^* q \check{q}_x^*)
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$$
\begin{aligned}
iq_t &= -\alpha\left[q_{xx} - 2\kappa \tilde{q} q^2\right] + \delta[q_{xxx} - 6\kappa q \tilde{q} q_x] \\
-i\tilde{q}_t &= -\alpha\left[\tilde{q}_{xx} - 2\kappa q \tilde{q}^2\right] - \delta(\tilde{q}_{xxx} - 6\kappa \tilde{q} q \tilde{q}_x)
\end{aligned}
$$

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$$
\begin{aligned}
q_t &= -\check{\delta}\left[q_{xx} - 2\kappa \check{q}^* q^2\right] - \beta[q_{xxx} - 6\kappa q \check{q}^* q_x] \\
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\end{aligned}
$$

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$\mathcal{T}$ transformed pair: $r(x,t) = \kappa q(x,-t)$

$$
\begin{aligned}
iq_t &= -i\hat{\delta}\left[q_{xx} - 2\kappa \hat{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q \hat{q}^* q_x] \\
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\end{aligned}
$$

$\alpha = i\hat{\delta}$; $\beta = i\delta; \hat{\delta}, \delta \in \mathbb{R}$

Nonlocality in Hirota's direct method

Bilinearisation of the local Hirota equation $(q = g/f)$

$$f^3 \left[iq_t + \alpha q_{xx} - 2\kappa\alpha \left| q \right|^2 q + i\beta \left(q_{xxx} - 6\kappa \left| q \right|^2 q_x \right) \right] =$$

$$f \left[iD_t g \cdot f + \alpha D_x^2 g \cdot f + i\beta D_x^3 g \cdot f \right] + \left[3i\beta \left(\frac{g}{f} f_x - g_x \right) - \alpha g \right]$$

$$\times \left[D_x^2 f \cdot f + 2\kappa \left| g \right|^2 \right]$$

$$D_x^n f \cdot g = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k \frac{\partial^{n-k}}{\partial x^{n-k}} f(x) \frac{\partial^k}{\partial x^k} g(x)$$

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$$\times \left[D_x^2 f \cdot f + 2\kappa \left| g \right|^2 \right]$$

$$D_x^n f \cdot g = \sum\nolimits_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (-1)^k \frac{\partial^{n-k}}{\partial x^{n-k}} f(x) \frac{\partial^k}{\partial x^k} g(x)$$

$$iD_t g \cdot f + \alpha D_x^2 g \cdot f + i\beta D_x^3 g \cdot f = 0, \qquad D_x^2 f \cdot f = -2\kappa \left| g \right|^2$$

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$$f\left[iD_t g\cdot f + \alpha D_x^2 g\cdot f + i\beta D_x^3 g\cdot f\right] + \left[3i\beta\left(\frac{g}{f}f_x - g_x\right) - \alpha g\right]$$
$$\times\left[D_x^2 f\cdot f + 2\kappa\left|g\right|^2\right]$$

$$D_x^n f\cdot g = \sum\nolimits_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}(-1)^k\frac{\partial^{n-k}}{\partial x^{n-k}}f(x)\frac{\partial^k}{\partial x^k}g(x)$$

$$iD_t g\cdot f + \alpha D_x^2 g\cdot f + i\beta D_x^3 g\cdot f = 0, \qquad D_x^2 f\cdot f = -2\kappa\left|g\right|^2$$

Solve by formal power series that becomes **exact**

$$f(x,t) = \sum_{k=0}^{\infty}\varepsilon^{2k}f_{2k}(x,t), \quad \text{and} \quad g(x,t) = \sum_{k=1}^{\infty}\varepsilon^{2k-1}g_{2k-1}(x,t)$$

Bilinearisation of the nonlocal Hirota equation

$$f^3\tilde{f}^*\left[iq_t+\alpha q_{xx}+2\alpha\tilde{q}^*q^2-\delta(q_{xxx}+6q\tilde{q}^*q_x)\right]=$$
$$f\tilde{f}^*\left[iD_tg\cdot f+\alpha D_x^2g\cdot f-\delta D_x^3g\cdot f\right]+\left(\frac{3\delta}{f}D_xg\cdot f-\alpha g\right)$$
$$\times\left(\tilde{f}^*D_x^2f\cdot f-2fg\tilde{g}^*\right)$$

not bilinear yet

$$iD_tg\cdot f+\alpha D_x^2g\cdot f-\delta D_x^3g\cdot f=0,\qquad \tilde{f}^*D_x^2f\cdot f=2fg\tilde{g}^*$$

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$$iD_tg\cdot f+\alpha D_x^2g\cdot f-\delta D_x^3g\cdot f=0,\qquad \tilde{f}^*D_x^2f\cdot f=2fg\tilde{g}^*$$

introduce additional auxiliary function

$$D_x^2f\cdot f=hg,\qquad \text{and} \qquad 2f\tilde{g}^*=h\tilde{f}^*$$

Solve again formal power series that becomes **exact**

$$h(x,t)=\sum\nolimits_k\varepsilon^kh_k(x,t).$$

Two-types of nonlocal solutions (one-soliton)

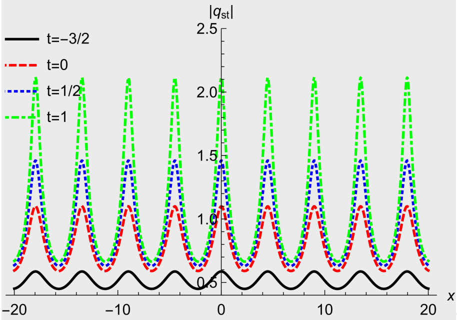
Truncated expansions: $f = 1 + \varepsilon^2 f_2, \quad g = \varepsilon g_1, \quad h = \varepsilon h_1$

$$0 = \varepsilon[i\,(g_1)_t + \alpha\,(g_1)_{xx} - \delta(g_1)_{xxx}] \\ +\varepsilon^3[2\,(f_2)_x\,(g_1)_x - g_1\,[(f_2)_{xx} + i\,(f_2)_t] + if_2\,[(g_1)_t + i\,(g_1)_{xx}]]$$

$$0 = \varepsilon^2[2(f_2)_{xx} - g_1 h_1] + \varepsilon^4\left[2f_2(f_2)_{xx} - 2(f_2)_x^2\right]$$

$$0 = \varepsilon[2\tilde{g}_1^* - h_1] + \varepsilon^3\left[2f_2\tilde{g}_1^* - \tilde{f}_2^* h_1\right]$$

Standard solution, solve six equations independently, then $\varepsilon \to 1$



$$q_{st}^{(1)} = \frac{\lambda(\mu - \mu^*)^2 \tau_{\mu,\gamma}}{(\mu - \mu^*)^2 + |\lambda|^2\,\tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}}$$

$$\tau_{\mu,\gamma}(x,t) := e^{\mu x + \mu^2(i\alpha - \beta\mu)t + \gamma}$$

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$$
\begin{aligned}
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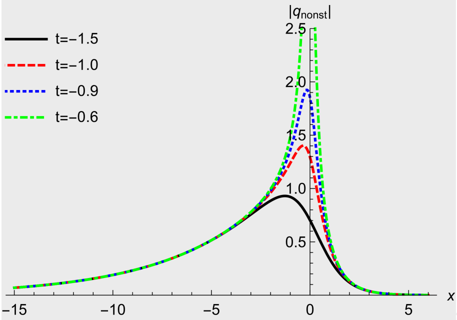
&\quad + \varepsilon^3[2\,(f_2)_x\,(g_1)_x - g_1\,[(f_2)_{xx} + i\,(f_2)_t] + i f_2\,[(g_1)_t + i\,(g_1)_{xx}]] \\
0 &= \varepsilon^2[2(f_2)_{xx} - g_1 h_1] + \varepsilon^4\left[2 f_2 (f_2)_{xx} - 2(f_2)_x^2\right] \\
0 &= [2\tilde{g}_1^* - h_1] + \left[2 f_2 \tilde{g}_1^* - \tilde{f}_2^* h_1\right]
\end{aligned}
$$

Nonstandard solution, solve five equations, last one for $\varepsilon = 1$



$$
q_{\text{nonst}}^{(1)} = \frac{(\mu + \nu)\tau_{\mu,i\gamma}}{1 + \tau_{\mu,i\gamma}\tilde{\tau}^*_{-\nu,-i\theta}}.
$$

$$
\tau_{\mu,\gamma}(x,t) := e^{\mu x + \mu^2(i\alpha - \beta\mu)t + \gamma}
$$

Two-soliton solution

Truncated expansions:

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4, \quad g = \varepsilon g_1 + \varepsilon^3 g_3, \quad h = \varepsilon h_1 + \varepsilon^3 h_3$$

$$q_{\mathrm{nl}}^{(2)}(x,t) = \frac{g_1(x,t) + g_3(x,t)}{1 + f_2(x,t) + f_4(x,t)}$$

$$\begin{aligned}
g_1 &= \tau_{\mu,\gamma} + \tau_{\nu,\delta} \\
g_3 &= \frac{(\mu-\nu)^2}{(\mu-\mu^*)^2(\nu-\mu^*)^2}\tau_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\mu,\gamma} + \frac{(\mu-\nu)^2}{(\mu-\nu^*)^2(\nu-\nu^*)^2}\tau_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta} \\
f_2 &= \frac{\tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}}{(\mu-\mu^*)^2} + \frac{\tau_{\nu,\delta}\tilde{\tau}^*_{\mu,\gamma}}{(\nu-\mu^*)^2} + \frac{\tau_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}}{(\mu-\nu^*)^2} + \frac{\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta}}{(\nu-\nu^*)^2} \\
f_4 &= \frac{(\mu-\nu)^2(\mu^*-\nu^*)^2}{(\mu-\mu^*)^2(\nu-\mu^*)^2(\mu-\nu^*)^2(\nu-\nu^*)^2}\tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta} \\
h_1 &= 2\tilde{\tau}^*_{\mu,\gamma} + 2\tilde{\tau}^*_{\nu,\delta} \\
h_3 &= \frac{2(\mu^*-\nu^*)^2}{(\mu-\mu^*)^2(\nu^*-\mu)^2}\tilde{\tau}^*_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}\tau_{\mu,\gamma} + \frac{2(\mu^*-\nu^*)^2}{(\mu^*-\nu)^2(\nu-\nu^*)^2}\tilde{\tau}^*_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}\tau_{\nu,\delta}
\end{aligned}$$

Two-soliton solution

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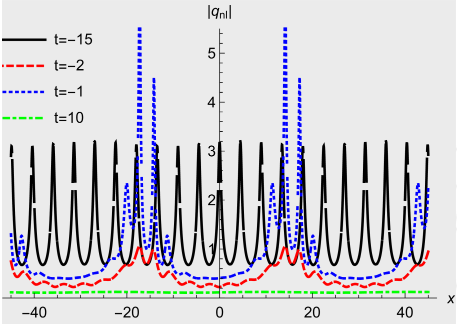
Nonlocal regular two-soliton solution



Nonlocality in Darboux-Crum transformations

Quantum mechanical analogue to supersymmetry, intertwining

$$L_n H_{n-1} = H_n L_n$$

Nonlocality in Darboux-Crum transformations

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iteration $\mathcal{L}_n H_0 = H_n \mathcal{L}_n$, $\mathcal{L}_n := L_n L_{n-1} \ldots L_1$, $\Psi_n(\lambda) = \mathcal{L}_n \Psi(\lambda)$

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In Hirota case, Hamiltonian of Dirac type :

$$\Psi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \ \Psi_x = U\Psi \ \Leftrightarrow \ \begin{array}{c} -i\varphi_x + iq\phi = -\lambda\varphi \\ i\phi_x - ir\varphi = -\lambda\phi \end{array} \ \Leftrightarrow H\Psi(\lambda) = -\lambda\Psi(\lambda)$$

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with

$$H = \begin{pmatrix} -i\partial_x & iq \\ -ir & i\partial_x \end{pmatrix} = -i\sigma_3\partial_x + V,$$

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iterated potentials V_n $\Leftrightarrow$ multi-soliton solutions

Solve the "seed" equations for $q = r = 0$:

$$\tilde{\Psi}_1(x,t;\lambda) = \begin{pmatrix} \varphi_1(x,t;\lambda) \\ \phi_1(x,t;\lambda) \end{pmatrix} = \begin{pmatrix} e^{\lambda x + 2i\lambda^2(\alpha - 2\delta\lambda)t + \gamma_1} \\ e^{-\lambda x - 2i\lambda^2(\alpha - 2\delta\lambda)t + \gamma_2} \end{pmatrix}$$

Implement nonlocality in the construction Ψ_2:
Two choices to achieve $r(x,t) = \pm q^*(-x,t)$

$$1\colon\ \varphi_2 = \pm\tilde{\phi}_1^*,\ \phi_2 = \tilde{\varphi}_1^* \qquad 2\colon\ \phi_1 = \tilde{\varphi}_1^*,\ \phi_2 = \pm\tilde{\varphi}_2^*$$

The second choice is not available in the local case.

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Choice 1:

$$\tilde{\Psi}_2(x,t;\lambda) = \begin{pmatrix} \varphi_2(x,t;\lambda) \\ \phi_2(x,t;\lambda) \end{pmatrix} = \begin{pmatrix} \mp e^{\lambda^* x + 2i(\lambda^*)^2(\alpha - 2\delta\lambda)t + \gamma_2^*} \\ e^{-\lambda^* x - 2i(\lambda^*)^2(\alpha - 2\delta\lambda^*)t + \gamma_1^*} \end{pmatrix}$$

with $\lambda, \gamma_1, \gamma_2 \in \mathbb{C}$

$$q_{\text{st}}^{(1)}(x,t) = \frac{2(\lambda^* - \lambda)e^{2\lambda^* x + 2i(\lambda^*)^2(\alpha - 2\delta\lambda^*)t - \gamma_1^* + \gamma_2^*}}{1 + e^{2(\lambda^* - \lambda)x + 4i\left[\alpha(\lambda^*)^2 - \alpha\lambda^2 + 2\delta\lambda^3 - 2\delta(\lambda^*)^3\right]t - \gamma_1 + \gamma_2 - \gamma_1^* + \gamma_2^*}}.$$

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The second choice is not available in the local case.
Choice 2:

$$\tilde{\Psi}_2(x,t;\nu) = \begin{pmatrix} \varphi_2(x,t;\nu) \\ \phi_2(x,t;\nu) \end{pmatrix} = \begin{pmatrix} e^{\nu x + 2i\nu^2(\alpha - 2\delta\nu)t + \gamma_3} \\ -e^{-\nu x - 2i\nu^2(\alpha - 2\delta\nu)t + \gamma_3^*} \end{pmatrix}$$

$$q^{(1)}_{\text{nonst}}(x,t) = \frac{2(\nu - \mu)e^{\gamma_1 - \gamma_1^* + 2\mu x + 4i\mu^2(\alpha - 2\delta\mu)t}}{1 + e^{2(\mu-\nu)x + 4i(\alpha\mu^2 - \alpha\nu^2 - 2\delta\mu^3 + 2\delta\nu^3)t + \gamma_1 - \gamma_1^* - \gamma_3 + \gamma_3^*}}.$$

Nonlocal n-soliton solutions: $q_n = q + 2\frac{\det D_n^q}{\det W_n}$, $r_n = r - 2\frac{\det D_n^r}{\det W_n}$

$$W_n = \begin{pmatrix} \varphi_1^{(n-1)} & \varphi_1^{(n-2)} & \cdots & \varphi_1 & \phi_1^{(n-1)} & \cdots & \phi_1' & \phi_1 \\ \varphi_2^{(n-1)} & \varphi_2^{(n-2)} & \cdots & \varphi_2 & \phi_2^{(n-1)} & \cdots & \phi_2' & \phi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{2n}^{(n-1)} & \varphi_{2n}^{(n-2)} & \cdots & \varphi_{2n} & \phi_{2n}^{(n-1)} & \cdots & \phi_{2n}' & \phi_{2n} \end{pmatrix}$$

$$D_n^q = \begin{pmatrix} \phi_1^{(n-2)} & \phi_1^{(n-3)} & \cdots & \phi_1 & \varphi_1^{(n)} & \cdots & \varphi_1' & \varphi_1 \\ \phi_2^{(n-2)} & \phi_2^{(n-3)} & \cdots & \phi_2 & \varphi_2^{(n)} & \cdots & \varphi_2' & \varphi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{2n}^{(n-2)} & \phi_{2n}^{(n-3)} & \cdots & \phi_{2n} & \varphi_{2n}^{(n)} & \cdots & \varphi_{2n}' & \varphi_{2n} \end{pmatrix}$$

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In the nonlocal case use

$$\tilde{S}_{2n}^{\text{st}} = \left\{ \tilde{\Psi}_1(x,t;\lambda_1), \tilde{\Psi}_2(x,t;\lambda_1), \tilde{\Psi}_1(x,t;\lambda_2), \tilde{\Psi}_2(x,t;\lambda_2), \ldots \right\}$$

or

$$\tilde{S}_{2n}^{\text{nonst}} = \left\{ \tilde{\Psi}_1(x,t;\mu_1), \tilde{\Psi}_2(x,t;\nu_1), \tilde{\Psi}_1(x,t;\mu_2), \tilde{\Psi}_2(x,t;\nu_2), \ldots \right\}$$

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- We can systematically construct degenerate multi-soliton solutions
- Using $\mathcal{PT}$ conjugations we find new integrable versions of the Hirota equation, with different types of qualitative behaviour.
- The nonlocality can be systematically implemented into solution procedures, such as Hirota's method and Darboux transformations.

Thank you for your attention

In case you are interested, register for the virtual seminar series on
Pseudo-Hermitian Hamiltonians in Quantum Physics
https://vphhqp.com