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Introduction to PT-quantum mechanics, deformations of integrable models

Andreas Fring

Universität Stuttgart, 16th of April 2019



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based on: A. Fring,
PT-Symmetric Deformations of Nonlinear Integrable Systems,
ch. 9 in "PT Symmetry: In Quantum and Classical Physics",
World Scientific Publishing Co., Singapore, 2018

Outline

- 1 Introduction to PT-quantum mechanics
- 2 PT-deformed quantum spin chains
- 3 PT-deformed Calogero-Moser-Sutherland models
- 4 PT-deformed KdV/Ito systems
- 5 Conclusions



Hermiticity is good to have for two reasons, but

Why is Hermiticity a good property to have?

- Hermiticity ensures real energies

Schrödinger equation $H\psi = E\psi$

$$\left. \begin{aligned} \langle \psi | H | \psi \rangle &= E \langle \psi | \psi \rangle \\ \langle \psi | H^\dagger | \psi \rangle &= E^* \langle \psi | \psi \rangle \end{aligned} \right\} \Rightarrow 0 = (E - E^*) \langle \psi | \psi \rangle$$

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- Hermiticity ensures conservation of probability densities

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$$

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- Thus when $H \neq H^\dagger$ one usually thinks of dissipation.
- However, these systems are usually open and do not possess a self-consistent description.

Hermiticity is not essential

- Operators \mathcal{O} which are left invariant under an antilinear involution \mathcal{I} and whose eigenfunctions Φ also respect this symmetry,

$$[\mathcal{O}, \mathcal{I}] = 0 \quad \wedge \quad \mathcal{I}\Phi = \Phi$$

have a real eigenvalue spectrum.

[E. Wigner, *J. Math. Phys.* 1 (1960) 409]

- By defining a new metric also a consistent quantum mechanical framework has been developed for theories involving such operators.

[F. Scholtz, H. Geyer, F. Hahne, *Ann. Phys.* 213 (1992) 74,
C. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243,
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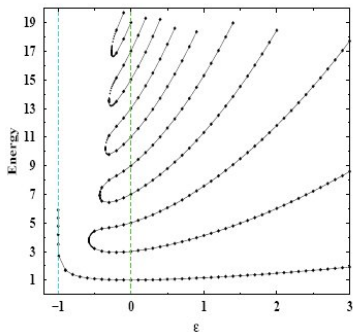
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In particular this also holds for \mathcal{O} being non-Hermitian.

There are plenty of well studied examples of non-Hermitian systems in the literature

"Recent" classical example

$$\mathcal{H} = \frac{1}{2}p^2 + x^2(ix)^\varepsilon \quad \text{for } \varepsilon \geq 0$$



[C.M. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243]

A more classical example

- Lattice Reggeon field theory:

$$\mathcal{H} = \sum_{\vec{i}} \left[\Delta a_{\vec{i}}^{\dagger} a_{\vec{i}} + i g a_{\vec{i}}^{\dagger} (a_{\vec{i}} + a_{\vec{i}}^{\dagger}) a_{\vec{i}} + \tilde{g} \sum_{\vec{j}} (a_{\vec{i}+\vec{j}}^{\dagger} - a_{\vec{i}}^{\dagger}) (a_{\vec{i}+\vec{j}} - a_{\vec{i}}) \right]$$

- $a_{\vec{i}}^{\dagger}$, $a_{\vec{i}}$ are creation and annihilation operators, $\Delta, g, \tilde{g} \in \mathbb{R}$

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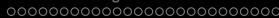
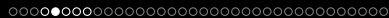
[J.L. Cardy, R. Sugar, *Phys. Rev. D*12 (1975) 2514]

- for one site this is almost $i\hat{x}^3$

$$\begin{aligned} \mathcal{H} &= \Delta a^{\dagger} a + i g a^{\dagger} (a + a^{\dagger}) a \\ &= \frac{1}{2} (\hat{p}^2 + \hat{x}^2 - 1) + i \frac{g}{\sqrt{2}} (\hat{x}^3 + \hat{p}^2 \hat{x} - 2\hat{x} + i\hat{p}) \end{aligned}$$

with $a = (\omega \hat{x} + i\hat{p})/\sqrt{2\omega}$, $a^{\dagger} = (\omega \hat{x} - i\hat{p})/\sqrt{2\omega}$

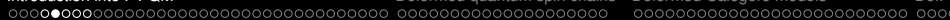
[P. Assis and A.F., *J. Phys.* A41 (2008) 244001]



- quantum spin chains: ($c=-22/5$ CFT)

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^N \sigma_i^x + \lambda \sigma_i^z \sigma_{i+1}^z + ih \sigma_i^z \quad \lambda, h \in \mathbb{R}$$

[G. von Gehlen, J. Phys. A24 (1991) 5371]



Ubiquitous non-Hermitian Hamiltonians (examples from the literature)

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- Toda field theory:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{\beta^2} \sum_{k=\mathbf{a}}^\ell n_k \exp(\beta \alpha_k \cdot \phi)$$

$\mathbf{a} = 1 \equiv$ conformal field theory (Lie algebras)



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$\beta \in i\mathbb{R} \equiv$ backscattering (Yang-Baxter, quantum groups)

- deformed space-time structure
 - deformed Heisenberg canonical commutation relations

$$aa^\dagger - q^2 a^\dagger a = q^{g(N)}, \quad \text{with } N = a^\dagger a$$

$$X = \alpha a^\dagger + \beta a, \quad P = i\gamma a^\dagger - i\delta a, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

$$\begin{aligned}
 [X, P] &= i\hbar q^{g(N)}(\alpha\delta + \beta\gamma) \\
 &+ \frac{i\hbar(q^2 - 1)}{\alpha\delta + \beta\gamma} \left(\delta\gamma X^2 + \alpha\beta P^2 + i\alpha\delta XP - i\beta\gamma PX \right)
 \end{aligned}$$

- limit: $\beta \rightarrow \alpha, \delta \rightarrow \gamma, g(N) \rightarrow 0, q \rightarrow e^{2\tau\gamma^2}, \gamma \rightarrow 0$

$$[X, P] = i\hbar \left(1 + \tau P^2 \right)$$



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$$[X, P] = i\hbar (1 + \tau P^2)$$

- representation: $X = (1 + \tau p_0^2)x_0$, $P = p_0$, $[x_0, p_0] = i\hbar$



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- with the standard inner product X is not Hermitian

$$X^\dagger = X + 2\tau i\hbar P \quad \text{and} \quad P^\dagger = P$$



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- example harmonic oscillator:

$$\begin{aligned} H_{ho} &= \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2} (1 + \tau p_0^2)x_0(1 + \tau p_0^2)x_0, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2} \left[(1 + \tau p_0^2)^2 x_0^2 + 2i\hbar\tau p_0(1 + \tau p_0^2)x_0 \right]. \end{aligned}$$

[B. Bagchi and A.F., Phys. Lett. A373 (2009) 4307]

- "dynamical" noncommutative space-time

Replace

$$\begin{aligned} [x_0, y_0] &= i\theta, & [x_0, p_{x_0}] &= i\hbar, & [y_0, p_{y_0}] &= i\hbar, \\ [p_{x_0}, p_{y_0}] &= 0, & [x_0, p_{y_0}] &= 0, & [y_0, p_{x_0}] &= 0, \end{aligned}$$

with $\theta \in \mathbb{R}$, by

$$\begin{aligned} [X, Y] &= i\theta(1 + \tau Y^2) & [X, P_x] &= i\hbar(1 + \tau Y^2) \\ [Y, P_y] &= i\hbar(1 + \tau Y^2) & [X, P_y] &= 2i\tau Y(\theta P_y + \hbar X) \\ [P_x, P_y] &= 0 & [Y, P_x] &= 0 \end{aligned}$$

\Rightarrow Non-Hermitian representation

$$X = (1 + \tau y_0^2)x_0 \quad Y = y_0 \quad P_x = p_{x_0} \quad P_y = (1 + \tau y_0^2)p_{y_0}$$

$$X^\dagger = X + 2i\tau\theta Y \quad Y^\dagger = Y \quad P_y^\dagger = P_y - 2i\tau\hbar Y \quad P_x^\dagger = P_x$$

[A.F., L. Gouba, F. Scholtz, J.Phys. A43 (2010) 345401]

[A.F., L. Gouba, B. Bagchi, J.Phys. A43 (2010) 425202]



Pseudo/Quasi-Hermiticity

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \Leftrightarrow H^\dagger \rho = \rho H \quad \rho = \eta^\dagger \eta \quad (*)$$

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positivity of ρ	✓	✓	×
ρ Hermitian	✓	✓	✓
ρ invertible	✓	×	✓
terminology	(*)	quasi-Herm.	pseudo-Herm.
spectrum of H	real	could be real	real
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- quasi-Hermiticity: [J. Dieudonné, Proc. Int. Symp. (1961) 115]
[F. Scholtz, H. Geyer, F. Hahne, Ann. Phys. 213 (1992) 74]

Supersymmetry (Darboux transformation)

Decompose Hamiltonian \mathcal{H} as:

$$\mathcal{H} = H_+ \oplus H_- = Q\tilde{Q} \oplus \tilde{Q}Q$$

- intertwining operators: $QH_- = H_+Q$ and $\tilde{Q}H_+ = H_-\tilde{Q}$

$$\Rightarrow [\mathcal{H}, Q] = [\mathcal{H}, \tilde{Q}] = 0$$

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 \Rightarrow isospectral Hamiltonians

$$H_{\pm}^m = -\Delta + V_{\pm}^m + E_m \quad H_{\pm}^m \phi_n^{\pm} = E_n \phi_n^{\pm} \quad \text{for } n > m$$

Unbroken \mathcal{PT} -symmetry guarantees real eigenvalues (QM)

- \mathcal{PT} -symmetry: $\mathcal{PT} : x \rightarrow -x \quad p \rightarrow p \quad i \rightarrow -i$
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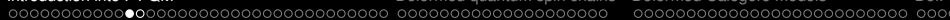
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- \mathcal{PT} -symmetry: $\mathcal{PT} : x \rightarrow -x \quad p \rightarrow p \quad i \rightarrow -i$
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- \mathcal{PT} is an anti-linear operator:

$$\mathcal{PT}(\lambda\Phi + \mu\Psi) = \lambda^*\mathcal{PT}\Phi + \mu^*\mathcal{PT}\Psi \quad \lambda, \mu \in \mathbb{C}$$

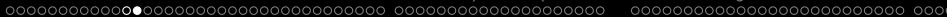
- Real eigenvalues from unbroken \mathcal{PT} -symmetry:

$$[\mathcal{H}, \mathcal{PT}] = 0 \quad \wedge \quad \mathcal{PT}\Phi = \Phi \quad \Rightarrow \quad \varepsilon = \varepsilon^* \quad \text{for } \mathcal{H}\Phi = \varepsilon\Phi$$

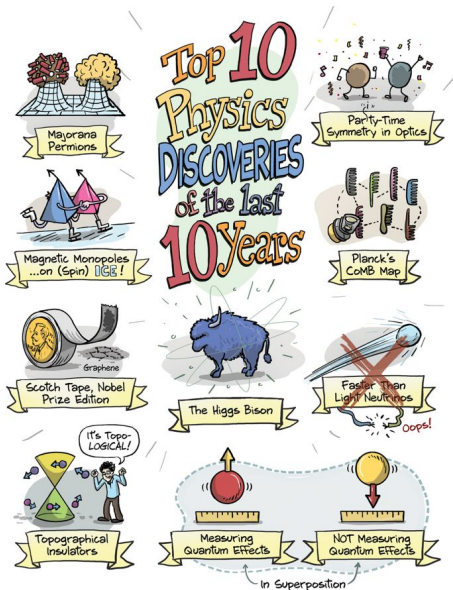
- *Proof:*

$$\varepsilon\Phi = \mathcal{H}\Phi = \mathcal{H}\mathcal{PT}\Phi = \mathcal{PT}\mathcal{H}\Phi = \mathcal{PT}\varepsilon\Phi = \varepsilon^*\mathcal{PT}\Phi = \varepsilon^*\Phi$$

\mathcal{PT} -symmetry is only an example of an antilinear involution



Nature Physics volume 11, page 799 (2015)



How to formulate a quantum mechanical framework?

- 1 orthogonality
- 2 observables
- 3 uniqueness
- 4 technicalities (new metric etc)

Orthogonality

- Take h to be a Hermitian and diagonalisable Hamiltonian:

$$\langle \phi_n | h \phi_m \rangle = \langle h \phi_n | \phi_m \rangle$$

$$\left. \begin{aligned} \langle \phi_n | h \phi_m \rangle &= \varepsilon_m \langle \phi_n | \phi_m \rangle \\ \langle h \phi_n | \phi_m \rangle &= \varepsilon_n^* \langle \phi_n | \phi_m \rangle \end{aligned} \right\} \Rightarrow \mathbf{0} = (\varepsilon_m - \varepsilon_n^*) \langle \phi_n | \phi_m \rangle$$

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$$\Rightarrow \quad n = m : \varepsilon_n = \varepsilon_n^* \quad n \neq m : \langle \phi_n | \phi_m \rangle = 0$$

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$$\Rightarrow \quad n = m : \varepsilon_n = \varepsilon_n^* \quad n \neq m : \langle \phi_n | \phi_m \rangle = 0$$

- Take H to be a non-Hermitian Hamiltonian:

$$H |\Phi_n\rangle = \varepsilon_n |\Phi_n\rangle$$

- reality and orthogonality no longer guaranteed. Define

$$\langle \Phi_n | \Phi_m \rangle_\eta := \langle \Phi_n | \eta^2 \Phi_m \rangle$$

- when $\langle \Phi_n | H \Phi_m \rangle_\eta = \langle H \Phi_n | \Phi_m \rangle_\eta \Rightarrow \langle \Phi_n | \Phi_m \rangle_\eta = \delta_{n,m}$

H is Hermitian with respect to new metric

- Assume pseudo-Hermiticity:

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \Leftrightarrow H^\dagger \eta^\dagger \eta = \eta^\dagger \eta H$$

$$\Phi = \eta^{-1} \phi \quad \eta^\dagger = \eta$$

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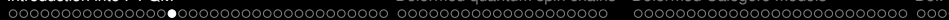
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\Rightarrow Eigenvalues of H are real, eigenstates are orthogonal



Observables

- Observables are Hermitian with respect to the new metric

$$\langle \Phi_n | \mathcal{O} \Phi_m \rangle_\eta = \langle \mathcal{O} \Phi_n | \Phi_m \rangle_\eta$$

$$\mathcal{O} = \eta^{-1} o \eta \quad \Leftrightarrow \quad \mathcal{O}^\dagger = \rho \mathcal{O} \rho^{-1}$$

- o is an observable in the Hermitian system
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- o is an observable in the Hermitian system
- \mathcal{O} is an observable in the non-Hermitian system
- Ambiguities:
 - Given H the metric is not uniquely defined for unknown h .
 - ⇒ Given only H the observables are not uniquely defined.
 - This is different in the Hermitian case.
 - Fixing one more observable achieves uniqueness.

[Scholtz, Geyer, Hahne, , *Ann. Phys.* 213 (1992) 74]

General technique:

- Given H

either	solve	$\eta H \eta^{-1} = h$	for	$\eta \Rightarrow \rho = \eta^\dagger \eta$
or	solve	$H^\dagger = \rho H \rho^{-1}$	for	$\rho \Rightarrow \eta = \sqrt{\rho}$

General technique:

- Given H $\left\{ \begin{array}{l} \text{either solve } \eta H \eta^{-1} = h \text{ for } \eta \Rightarrow \rho = \eta^\dagger \eta \\ \text{or solve } H^\dagger = \rho H \rho^{-1} \text{ for } \rho \Rightarrow \eta = \sqrt{\rho} \end{array} \right.$
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- involves complicated commutation relations
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Note:

- Thus, this is not re-inventing or disputing the validity of quantum mechanics.
- We only give up the restrictive requirement that Hamiltonians have to be Hermitian.

[A. Mostafazadeh, Int. J. Geom. Meth. Phys. 7 (2010) 1191]

[PT Symmetry: In Quantum and Classical Physics, World Scientific Publishing Co., Singapore, 2018]

Theoretical framework (key equations):

Is it possible to have a consistent description of time-dependent non-Hermitian Hamiltonian systems?

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Time-dependent Schrödinger eqn for $h(t) = h^\dagger(t)$, $H(t) \neq H^\dagger(t)$

$$h(t)\phi(t) = i\hbar\partial_t\phi(t), \quad \text{and} \quad H(t)\Psi(t) = i\hbar\partial_t\Psi(t)$$

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$$h(t)\phi(t) = i\hbar\partial_t\phi(t), \quad \text{and} \quad H(t)\Psi(t) = i\hbar\partial_t\Psi(t)$$

Time-dependent Dyson operator

$$\phi(t) = \eta(t)\Psi(t)$$

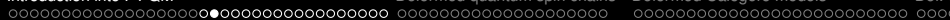
⇒ Time-dependent Dyson relation

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t)$$

⇒ Time-dependent quasi-Hermiticity relation

$$H^\dagger\rho(t) - \rho(t)H = i\hbar\partial_t\rho(t)$$

[from conjugating Dyson relation and $\rho(t) := \eta^\dagger(t)\eta(t)$]



QM framework: Non-Hermitian time-dependent Hamiltonians

$H(t)$ governs unitary time-evolution:

Hermitian:

$$\phi(t) = u(t, t')\phi(t'), \quad u(t, t') = T \exp \left[-i \int_{t'}^t ds h(s) \right]$$

with

$$h(t)u(t, t') = i\hbar\partial_t u(t, t'), \quad u(t, t')u(t', t'') = u(t, t''), \quad u(t, t) = \mathbb{I}$$

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$$\left\langle u(t, t')\phi(t') \mid u(t, t')\tilde{\phi}(t') \right\rangle = \left\langle \phi(t) \mid \tilde{\phi}(t) \right\rangle$$

Non-Hermitian:

$$\Psi(t) = U(t, t')\Psi(t'), \quad U(t, t') = T \exp \left[-i \int_{t'}^t ds H(s) \right]$$

$$H(t)U(t, t') = i\hbar\partial_t U(t, t'), \quad U(t, t')U(t', t'') = U(t, t''), \quad U(t, t) = \mathbb{I}$$

$$\left\langle U(t, t')\Psi(t') \mid U(t, t')\tilde{\Psi}(t') \right\rangle_{\rho} = \left\langle \Psi(t) \mid \tilde{\Psi}(t) \right\rangle_{\rho}$$

Relation between $u(t, t')$ and $U(t, t')$:

$$U(t, t') = \eta^{-1}(t)u(t, t')\eta(t')$$

or the generalized Duhamel's formula

$$\begin{aligned} U(t, t') &= u(t, t') - \int_{t'}^t \frac{d}{ds} [U(t, s)u(s, t')] ds \\ &= u(t, t') - i\hbar \int_{t'}^t U(t, s) [H(s) - h(s)] u(s, t') ds \end{aligned}$$



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Relation between Green's functions:

$$G_h(t, t') := -iu(t, t')\theta(t - t') \quad G_H(t, t') := -iU(t, t')\theta(t - t')$$

$$G_U(t, t') = G_u(t, t') + i \int_{-\infty}^{\infty} G_U(t, s) [H(s) - h(s)] G_u(s, t') ds$$

$H(t)$ is nonobservable and not the energy operator

Observables $o(t)$ in the Hermitian system are self-adjoint.

Observables $\mathcal{O}(t)$ in the non-Hermitian $\mathcal{O}(t)$ are quasi Hermitian

$$o(t) = \eta(t)\mathcal{O}(t)\eta^{-1}(t).$$

Then we have

$$\langle \phi(t) | o(t)\phi(t) \rangle = \langle \Psi(t) | \rho(t)\mathcal{O}(t)\Psi(t) \rangle .$$

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Since $H(t)$ is not quasi/pseudo Hermitian it is not an observable.

The observable energy operator is

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\hbar\eta^{-1}(t)\partial_t\eta(t).$$

Three scenarios:

① $\partial_t \eta = 0$, $\partial_t H \neq 0$, $\partial_t h \neq 0$

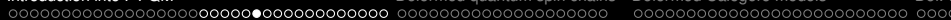
Technically reduces to time-independent case.

[C. Figueira de Morisson Faria, A. Fring; J. of Phys. A 39 (2006) 9269]

② $\partial_t \eta \neq 0$, $\partial_t H = 0$, $\partial_t h \neq 0$

Alternative representation:

- Heisenberg picture: time-dependent observables
- Schrödinger picture: time-dependent states
- Metric picture: time-dependent metric operators



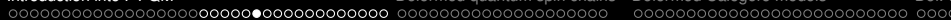
Making sense of the broken \mathcal{PT} -regime:

Two-level system

$$H = -\frac{1}{2} [\omega \mathbb{I} + \lambda \sigma_z + i\kappa \sigma_x]$$

with eigensystem

$$E_{\pm} = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}, \quad \varphi_{\pm} = \begin{pmatrix} i(-\lambda \pm \sqrt{\lambda^2 - \kappa^2}) \\ \kappa \end{pmatrix}$$



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$$[\mathcal{PT}, H] = 0, \quad \text{and} \quad \mathcal{PT}\varphi_{\pm} = e^{i\phi}\varphi_{\pm} \quad \text{for} \quad |\lambda| > |\kappa|$$



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$$[\mathcal{PT}, H] = 0, \quad \mathcal{PT}\varphi_{\pm} \neq e^{i\phi}\varphi_{\pm} \quad |\lambda| < |\kappa|$$



Claim: This system has real energies for $|\lambda(t)| < |\kappa(t)|!$



Two-dimensional system with infinite dimensional Hilbert space

$$H_K = aK_1 + bK_2 + i\lambda K_3, \quad a, b, \lambda \in \mathbb{R}$$

with Lie algebraic generators

$$K_1 = \frac{1}{2} (p_x^2 + x^2), \quad K_2 = \frac{1}{2} (p_y^2 + y^2), \quad K_3 = \frac{1}{2} (xy + p_x p_y)$$

$$K_4 = \frac{1}{2} (xp_y - yp_x)$$

$$\begin{aligned} [K_1, K_2] &= 0, & [K_1, K_3] &= iK_4, & [K_1, K_4] &= -iK_3, \\ [K_2, K_3] &= -iK_4, & [K_2, K_4] &= iK_3, & [K_3, K_4] &= i(K_1 - K_2)/2 \end{aligned}$$

• H_K is \mathcal{PT} -symmetric: $[\mathcal{PT}_\pm, H_{xy}] = 0$

$$\mathcal{PT}_\pm : x \rightarrow \pm x, \quad y \rightarrow \mp y, \quad p_x \rightarrow \mp p_x, \quad p_y \rightarrow \pm p_y, \quad i \rightarrow -i$$

Two-dimensional system with infinite dimensional Hilbert space

$$H_K = aK_1 + bK_2 + i\lambda K_3, \quad a, b, \lambda \in \mathbb{R}$$

with Lie algebraic generators

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$$h_K = \frac{1}{2}(a+b)(K_1 + K_2) + \frac{1}{2}\sqrt{(a-b)^2 - \lambda^2}(K_1 - K_2)$$

$$\text{with } \eta = e^{2\theta K_4}, \quad \theta = \text{arctanh}[\lambda/(b-a)]$$



Spontaneously broken \mathcal{PT} -symmetry for $a = b$:

Eigenenergies:

$$E_{n,m} = E_{m,n}^* = a(1 + n + m) + i\frac{\lambda}{2}(n - m)$$

Eigenfunctions:

$$\varphi_{n,m}(x, y) = \frac{e^{-\frac{x^2}{2} - \frac{y^2}{2}}}{2^{n+m} \sqrt{n! m! \pi}} \left[\sum_{k=0}^n \binom{n}{k} H_k(x) H_{n-k}(y) \right] \\ \times \left[\sum_{l=0}^m (-1)^l \binom{m}{l} H_l(y) H_{m-l}(x) \right]$$



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Claim: This system has real energies for $a(t), \lambda(t)$!



Time-dependent system:

$$H(t) = \frac{a(t)}{2} (p_x^2 + p_y^2 + x^2 + y^2) + i \frac{\lambda(t)}{2} (xy + p_x p_y), \quad a(t), \lambda(t) \in \mathbb{R}$$

Ansatz:

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Solution: $\gamma_4 = \operatorname{arcsinh}(\kappa \operatorname{sech} \gamma_3)$, $\chi(t) := \cosh \gamma_3$, $\kappa = \text{const}$
with dissipative Ermakov-Pinney equation

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda} \dot{\chi} - \lambda^2 \chi = \frac{\kappa^2 \lambda^2}{\chi^3}$$

Lewis Riesenfeld invariants:

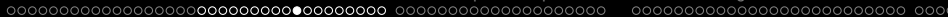
$$\frac{dI_{\mathcal{H}}(t)}{dt} = \partial_t I_{\mathcal{H}}(t) - i\hbar [I_{\mathcal{H}}(t), \mathcal{H}(t)] = 0, \quad \text{for } \mathcal{H} = h = h^\dagger, H \neq H^\dagger$$

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Solution to time-dependent Schrödinger equation:

$$\begin{aligned} I_{\mathcal{H}}(t) |\phi_{\mathcal{H}}(t)\rangle &= \Lambda |\phi_{\mathcal{H}}(t)\rangle, & |\Psi_{\mathcal{H}}(t)\rangle &= e^{i\hbar\alpha(t)} |\phi_{\mathcal{H}}(t)\rangle \\ \dot{\alpha} &= \langle \phi_{\mathcal{H}}(t) | i\hbar\partial_t - \mathcal{H}(t) | \phi_{\mathcal{H}}(t) \rangle, & \dot{\Lambda} &= 0 \end{aligned}$$

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Procedure:

- 1 Construct $I_h(t)$
- 2 Construct $I_H(t)$
- 3 Find $\eta(t)$ from similarity transformation



With Ansätze:

$$l_H(t) = \sum_{i=1}^4 \alpha_i(t) K_i, \quad l_h(t) = \sum_{i=1}^4 \beta_i(t) K_i, \quad h(t) = \sum_{i=1}^4 b_i(t) K_i,$$

where $\alpha_i = \alpha_i^r + i\alpha_i^i \in \mathbb{C}$, $b_i, \beta_i, \alpha_i^r, \alpha_i^i \in \mathbb{R}$.



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$$\gamma_3 = \arctan \left[\frac{\tanh \left[q_2 - \int_0^t \lambda(s) ds \right]}{\sqrt{1 - q_3^2 \operatorname{sech} \left[q_2 - \int_0^t \lambda(s) ds \right]^2}} \right]$$

$$\gamma_4 = -\operatorname{arccot} \left[\frac{1}{q_3} \cosh \left[q_2 - \int_0^t \lambda(s) ds \right] \right]$$

$$q_2, q_3 = \text{const}$$

With γ_3 we obtain a solution to the Ermakov-Pinney equation

$$\chi(t) = \cosh \gamma_3 = \sqrt{\frac{\cosh^2 \left[q_2 - \int_0^t \lambda(s) ds \right] - q_3^2}{1 - q_3^2}}$$

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Explicit form of the Hermitian Hamiltonian:

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Solution for time-dependent Schrödinger equation

Solution for $\tilde{h}(t) = a(t)K_1$, $a(t) \in \mathbb{R}$

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with phase

$$\alpha_n(t) = - \left(n + \frac{1}{2} \right) \int_0^t \frac{a(s)}{\varkappa^2(s)} ds$$

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We compute

$$\langle \tilde{\varphi}_n(x, t) | K_1 | \tilde{\varphi}_m(x, t) \rangle = 2^{n-2} n! (2n+1) \sqrt{\pi} \frac{a^2(1 + \varkappa^4) + \varkappa^2 \dot{\varkappa}^2}{a^2 \varkappa^2} \delta_{n,m}$$

$$\langle \tilde{\varphi}_n(x, t) | \tilde{\varphi}_n(x, t) \rangle = 2^n n! \sqrt{\pi} := N$$



right hand side does not depend on t :

$$\frac{d}{dt} \left[\frac{a^2(1 + \kappa^4) + \kappa^2 \dot{\kappa}^2}{a^2 \kappa^2} \right] = \frac{2\dot{\kappa}}{a^2} \left(\ddot{\kappa} - \frac{\dot{a}}{a} \dot{\kappa} + a^2 \kappa - \frac{a^2}{\kappa^3} \right) = 0$$



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take previous solution

$$\varkappa(t) = \sqrt{\tilde{\kappa} \cos \left[2 \int_0^t a(s) ds \right] + \sqrt{1 + \tilde{\kappa}^2}}$$

$$\frac{a^2(1 + \varkappa^4) + \varkappa^2 \dot{\varkappa}^2}{a^2 \varkappa^2} = 2\sqrt{1 + \tilde{\kappa}^2}$$

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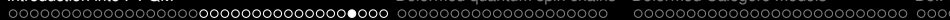
For $\hat{\varphi}_n(x, t) = \tilde{\varphi}_m(x, t)/\sqrt{N}$ we compute

$$\langle \hat{\varphi}_n(x, t) | K_1 | \hat{\varphi}_m(x, t) \rangle = \left(n + \frac{1}{2} \right) \sqrt{1 + \tilde{\kappa}^2} \delta_{n,m}$$

Solution for

$$h(t) = f_+(t)K_1 + f_-(t)K_2$$

$$f_{\pm}(t) = a(t) \pm \frac{q_3 \sqrt{1 - q_3^2} \lambda(t)}{1 + \cosh \left[2q_2 - 2 \int_0^t \lambda(s) ds \right] - 2q_3^2}$$



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$$\Psi_h^{n,m}(x, y, t) = \hat{\varphi}_n^+(x, t) \hat{\varphi}_m^-(y, t)$$

with $a \rightarrow f^{\pm}$, $\varkappa \rightarrow \varkappa_{\pm}$, $\tilde{\kappa} \rightarrow \tilde{\kappa}_{\pm}$, $\alpha_n \rightarrow \alpha_n^{\pm}$

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gives **real instantaneous energy expectation values**

$$\begin{aligned} E^{n,m}(t) &= \langle \Psi_h^{n,m}(t) | h(t) | \Psi_h^{n,m}(t) \rangle = \langle \Psi_H^{n,m}(t) | \rho(t) \tilde{H}(t) | \Psi_H^{n,m}(t) \rangle \\ &= f_+(t) \left(n + \frac{1}{2} \right) \sqrt{1 + \tilde{\kappa}_+^2} + f_-(t) \left(m + \frac{1}{2} \right) \sqrt{1 + \tilde{\kappa}_-^2} \end{aligned}$$

for any given fields $a(t)$, $\lambda(t) \in \mathbb{R}$, constants $\tilde{\kappa}_{\pm} \in \mathbb{R}$, $|q_3| < 1$



Symmetry ensuring reality of $E(t)$

Back to time-dependent two-level system

$$H(t) = -\frac{1}{2} [\omega \mathbb{I} + \alpha \kappa(t) \sigma_z + i \kappa(t) \sigma_x] \quad h(t) = -\frac{1}{2} [\omega \mathbb{I} + \chi(t) \sigma_z]$$

either known $\kappa(t)$ unknown $\chi(t)$ or unknown $\kappa(t)$ known $\chi(t)$.

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Energy operator

$$\tilde{H}(t) = -\frac{1}{2} \left\{ \omega \mathbb{I} + \frac{\chi}{\delta} \left[i(\alpha \xi - 1) \sigma_x + i \left(\hat{\xi} \sqrt{1 - \alpha^2} \right) \sigma_y + (\xi - \delta) \sigma_z \right] \right\}$$

with instantaneous expectation values

$$\tilde{E}_{\pm}(t) = \langle \psi_{\pm}(t) | \tilde{H}(t) \eta^2 \psi_{\pm}(t) \rangle = -\frac{1}{2} [\omega \pm \chi(t)]$$

Time-dependent $\widetilde{\mathcal{PT}}$ -symmetry

Solve

$$\left[\widetilde{\mathcal{PT}}, \tilde{H} \right] = 0, \quad \widetilde{\mathcal{PT}} \tilde{\varphi}_{\pm} = e^{i\tilde{\omega}_{\pm}} \tilde{\varphi}_{\pm}, \quad \widetilde{\mathcal{PT}}^2 = \mathbb{I}.$$

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We find

$$\widetilde{\mathcal{PT}} = \frac{1}{\sqrt{(\xi - \delta)^2 + (\alpha^2 - 1)\hat{\xi}^2}} \left[i \left(\sqrt{1 - \alpha^2}\hat{\xi} \right) \sigma_y + (\xi - \delta)\sigma_z \right] \tau$$

$$\tilde{\varphi}_{\pm} \sim \begin{pmatrix} (1 \mp 1)\delta - \xi \\ \sqrt{1 - \alpha^2}\hat{\xi} + i(1 - \alpha\xi) \end{pmatrix}$$

$$\tilde{\omega}_+ = \arctan \left[\frac{2\sqrt{1 - \alpha^2}(1 - \alpha\xi)\hat{\xi}}{1 + \xi(\xi - 2\alpha + \xi\alpha^2) + (\alpha^2 - 1)\hat{\xi}^2} \right],$$

$$\tilde{\omega}_- = \arctan \left[\frac{\sqrt{1 - \alpha^2}(1 - \alpha\xi)\hat{\xi}}{2\delta^2 - 3\delta\xi + \xi^2 + (\alpha^2 - 1)\hat{\xi}^2} \right] + \pi$$

Ising quantum spin chain of length N

$$\mathcal{H} = -\frac{1}{2} \sum_{i=1}^N (\quad \lambda \sigma_i^x \sigma_{i+1}^x \quad) \quad \lambda \in \mathbb{R}$$

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- $\sigma_i^{x,y,z} := \mathbb{I} \otimes \mathbb{I} \otimes \dots \otimes \sigma_i^{x,y,z} \otimes \dots \otimes \mathbb{I} \otimes \mathbb{I}$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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[G. von Gehlen, J. Phys. A24 (1991) 5371]



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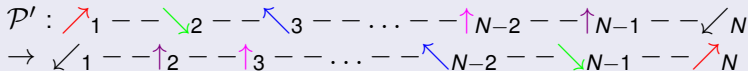
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[Castro-Alvaredo, A.F., J.Phys. A42 (2009) 465211]

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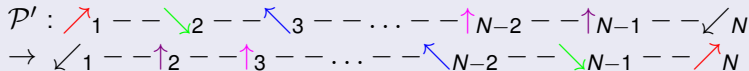
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$$\Rightarrow [\mathcal{PT}, \mathcal{H}] = 0$$

Deformed quantum spin chains (Different realizations for \mathcal{PT} -symmetry)

- Alternative definitions for parity:

$$\mathcal{P}_x := \prod_{i=1}^N \sigma_i^x \quad \mathcal{P}_y := \prod_{i=1}^N \sigma_i^y$$

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$$[\mathcal{PT}, \mathcal{H}] = 0, [\mathcal{P}_x \mathcal{T}, \mathcal{H}] \neq 0, [\mathcal{P}_y \mathcal{T}, \mathcal{H}] \neq 0, [\mathcal{P}' \mathcal{T}, \mathcal{H}] \neq 0$$

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- XXZ-spin-chain in a magnetic field

$$\mathcal{H}_{\text{XXZ}} = \frac{1}{2} \sum_{i=1}^{N-1} [(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta_+ (\sigma_i^z \sigma_{i+1}^z - 1))] + \frac{\Delta_-}{2} (\sigma_1^z - \sigma_N^z),$$

$$\Delta_{\pm} = (q \pm q^{-1})/2 \quad \Rightarrow \mathcal{H}_{\text{XXZ}}^{\dagger} \neq \mathcal{H}_{\text{XXZ}} \text{ for } q \notin \mathbb{R}$$

$$[\mathcal{PT}, \mathcal{H}_{\text{XXZ}}] \neq 0 \quad [\mathcal{P}_x \mathcal{T}, \mathcal{H}_{\text{XXZ}}] = 0 \quad [\mathcal{P}_y \mathcal{T}, \mathcal{H}_{\text{XXZ}}] = 0 \quad [\mathcal{P}' \mathcal{T}, \mathcal{H}_{\text{XXZ}}] = 0$$

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These possibilities reflect the ambiguities in the observables.

\mathcal{PT} -symmetry \Rightarrow domains in the parameter space of λ and κ

Broken and unbroken \mathcal{PT} -symmetry

$$[\mathcal{PT}, \mathcal{H}] = 0 \quad \wedge \quad \mathcal{PT}\Phi(\lambda, \kappa) \begin{cases} = \Phi(\lambda, \kappa) & \text{for } (\lambda, \kappa) \in U_{\mathcal{PT}} \\ \neq \Phi(\lambda, \kappa) & \text{for } (\lambda, \kappa) \in U_{b\mathcal{PT}} \end{cases}$$

$(\lambda, \kappa) \in U_{\mathcal{PT}} \Rightarrow$ real eigenvalues

$(\lambda, \kappa) \in U_{b\mathcal{PT}} \Rightarrow$ eigenvalues in complex conjugate pairs

- The two site Hamiltonian

$$\begin{aligned} \mathcal{H} &= -\frac{1}{2} [\sigma_1^z + \sigma_2^z + 2\lambda\sigma_1^x\sigma_2^x + i\kappa(\sigma_2^x + \sigma_1^x)] \\ &= -\frac{1}{2} [\sigma^z \otimes \mathbb{I} + \mathbb{I} \otimes \sigma^z + 2\lambda\sigma^x \otimes \sigma^x + i\kappa(\mathbb{I} \otimes \sigma^x + \sigma^x \otimes \mathbb{I})] \end{aligned}$$

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 &= - \begin{pmatrix} -1 & \frac{i\kappa}{2} & \frac{i\kappa}{2} & \lambda \\ \frac{i\kappa}{2} & 0 & \lambda & \frac{i\kappa}{2} \\ \frac{i\kappa}{2} & \lambda & 0 & \frac{i\kappa}{2} \\ \lambda & \frac{i\kappa}{2} & \frac{i\kappa}{2} & -1 \end{pmatrix}
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- domain of unbroken \mathcal{PT} -symmetry:

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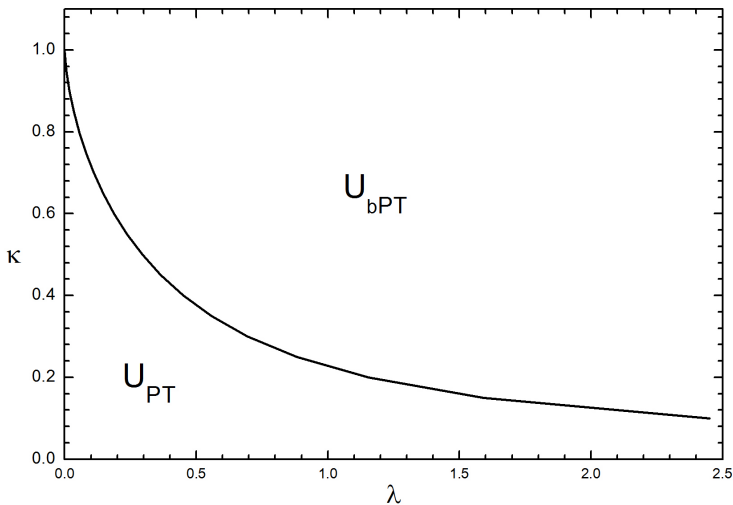
char. polynomial factorises into 1st and 3rd order

discriminant: $\Delta = r^2 - q^3$

$$q = \frac{1}{9} (-3\kappa^2 + 4\lambda^2 + 3), \quad r = \frac{\lambda}{27} (18\kappa^2 + 8\lambda^2 + 9)$$

Deformed quantum spin chains (Exact Results, $N = 2$)

$$U_{PT} = \left\{ \lambda, \kappa : \kappa^6 + 8\lambda^2\kappa^4 - 3\kappa^4 + 16\lambda^4\kappa^2 + 20\lambda^2\kappa^2 + 3\kappa^2 - \lambda^2 \leq 1 \right\}$$

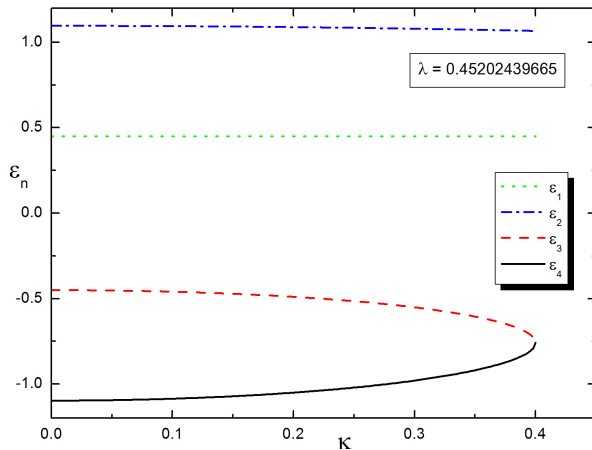


Deformed quantum spin chains (Exact Results, $N = 2$)

Real eigenvalues: $[\theta = \arccos(r/q^{3/2})]$

$$\varepsilon_1 = \lambda, \quad \varepsilon_2 = 2q^{1/2} \cos\left(\frac{\theta}{3}\right) - \frac{\lambda}{3}, \quad \varepsilon_{3/4} = 2q^{1/2} \cos\left(\frac{\theta}{3} + \pi \mp \frac{1\pi}{3}\right) - \frac{\lambda}{3}$$

Avoided level crossing:

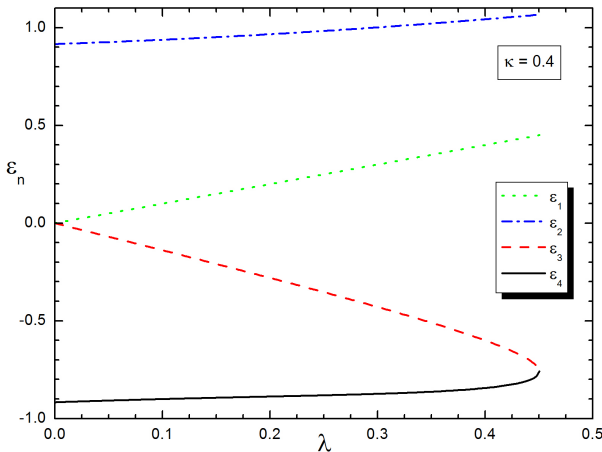


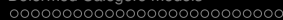
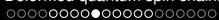
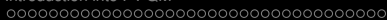
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Deformed quantum spin chains (Exact Results, $N = 2$)

- Right eigenvectors of \mathcal{H} :

$$|\Phi_1\rangle = (0, -1, -1, 0) \quad |\Phi_n\rangle = (\gamma_n, -\alpha_n, -\alpha_n, \beta_n) \quad n = 2, 3, 4$$

$$\alpha_n = i\kappa(\lambda - \varepsilon_n + 1)$$

$$\beta_n = \kappa^2 + 2\lambda^2 + 2\lambda\varepsilon_n$$

$$\gamma_n = -\kappa^2 - 2\varepsilon_n^2 + 2\lambda - 2\lambda\varepsilon_n + 2\varepsilon_n$$

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- signature: $\mathbf{s} = (+, -, +, -)$

$$\mathcal{P} |\Phi_n\rangle = \mathbf{s}_n |\Psi_n\rangle$$

from relating left and right eigenvectors

- \mathcal{C} -operator:

$$\begin{aligned} \mathcal{C} &= \sum_n s_n |\Phi_n\rangle \langle \Psi_n| \\ &= \begin{pmatrix} C_5 & -C_3 & -C_3 & C_4 \\ -C_3 & -C_1 - 1 & -C_1 & C_2 \\ -C_3 & -C_1 & -C_1 - 1 & C_2 \\ C_4 & C_2 & C_2 & 2(C_1 + 1) - C_5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} C_1 &= \frac{\alpha_4^2}{N_4^2} - \frac{\alpha_2^2}{N_2^2} - \frac{\alpha_3^2}{N_3^2} - \frac{1}{2}, & C_2 &= \frac{\alpha_4\beta_4}{N_4^2} - \frac{\alpha_2\beta_2}{N_2^2} - \frac{\alpha_3\beta_3}{N_3^2}, \\ C_3 &= \frac{\alpha_2\gamma_2}{N_2^2} + \frac{\alpha_3\gamma_3}{N_3^2} - \frac{\alpha_4\gamma_4}{N_4^2}, & C_4 &= \frac{\beta_2\gamma_2}{N_2^2} + \frac{\beta_3\gamma_3}{N_3^2} - \frac{\beta_4\gamma_4}{N_4^2}, \\ C_5 &= \frac{\gamma_2^2}{N_2^2} + \frac{\gamma_3^2}{N_3^2} - \frac{\gamma_4^2}{N_4^2} \end{aligned}$$

$$N_1 = \sqrt{2}, \quad N_n = \sqrt{2\alpha_n^2 + \beta_n^2 + \gamma_n^2} \quad \text{for } n = 2, 3, 4$$

- metric operator:

$$\rho = \mathcal{P}\mathcal{C} = \begin{pmatrix} C_5 & -C_3 & -C_3 & C_4 \\ C_3 & 1 + C_1 & C_1 & -C_2 \\ C_3 & C_1 & 1 + C_1 & -C_2 \\ C_4 & C_2 & C_2 & 2(1 + C_1) - C_5 \end{pmatrix}$$

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since $C_1 > 0$

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$$\rho = \mathcal{P}\mathcal{C} = \begin{pmatrix} C_5 & -C_3 & -C_3 & C_4 \\ C_3 & 1 + C_1 & C_1 & -C_2 \\ C_3 & C_1 & 1 + C_1 & -C_2 \\ C_4 & C_2 & C_2 & 2(1 + C_1) - C_5 \end{pmatrix}$$

- since $i\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$
 $\Rightarrow C_1, iC_2, iC_3, C_4, C_5 \in \mathbb{R}$
 $\Rightarrow \rho$ is Hermitian $\rho = \rho^\dagger$
- EV of ρ :

$$y_1 = y_2 = 1, \quad y_{3/4} = 1 + 2C_1 \pm 2\sqrt{C_1(1 + C_1)}$$

since $C_1 > 0 \Rightarrow \rho$ is positive

- square root of the metric operator:

$$\eta = \rho^{1/2} = UD^{1/2}U^{-1}$$

where $D = \text{diag}(y_1, y_2, y_3, y_4)$, $U = \{r_1, r_2, r_3, r_4\}$

$$|r_1\rangle = (0, -1, 1, 0)$$

$$|r_2\rangle = (C_4, 0, 0, 1 - C_5),$$

$$|r_{3/4}\rangle = (\tilde{\gamma}_{3/4}, \tilde{\alpha}_{3/4}, \tilde{\alpha}_{3/4}, \tilde{\beta}_{3/4})$$

$$\tilde{\alpha}_{3/4} = y_{3/4}(C_3 C_4 + C_2(-4C_1 + C_5 - 1))/2 - C_3 C_4$$

$$\tilde{\beta}_{3/4} = -C_3^2 - C_1 - C_1 C_5 + (C_3^2 + C_1(4C_1 - C_5 + 3)) y_{3/4},$$

$$\tilde{\gamma}_{3/4} = C_1 C_4 - C_2 C_3 + (C_2 C_3 + C_1 C_4) y_{3/4}$$

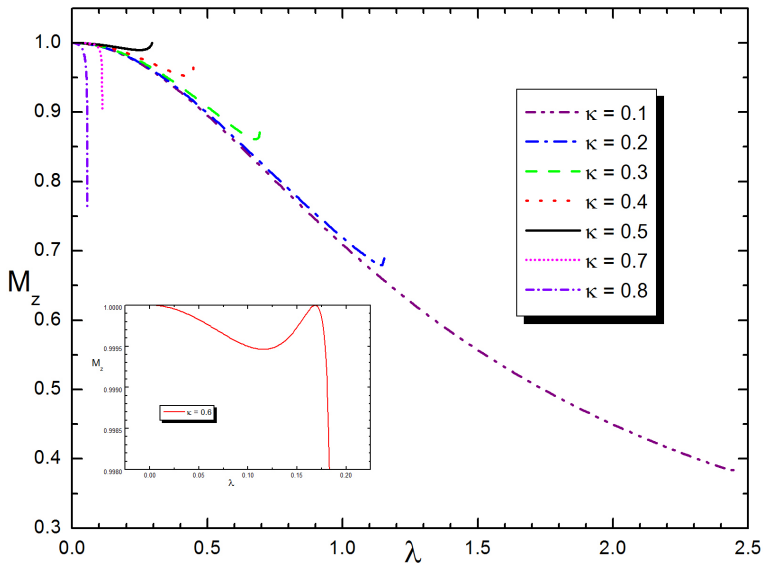
- isospectral Hermitian counterpart:

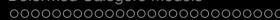
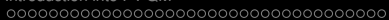
$$\begin{aligned}
 h &= \eta \mathcal{H} \eta^{-1} \\
 &= \mu_1 \sigma_x \otimes \sigma_x + \mu_2 \sigma_y \otimes \sigma_y + \mu_3 \sigma_z \otimes \sigma_z + \mu_4 (\sigma_z \otimes \mathbb{I} + \mathbb{I} \otimes \sigma_z)
 \end{aligned}$$

$$\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{R}$$

for $\lambda = 0.1, \kappa = 0.5$:

$$h = \begin{pmatrix} -0.829536 & 0 & 0 & -0.0606492 \\ 0 & -0.0341687 & -0.1341687 & 0 \\ 0 & -0.1341687 & -0.0341687 & 0 \\ -0.0606492 & 0 & 0 & 0.897873 \end{pmatrix}$$

The magnetization in the z-direction for $N = 2$:



Deformed quantum spin chains ($N \neq 2$, perturbation theory)

- Perturbation theory about the Hermitian part

$$H(\lambda, \kappa) = h_0(\lambda) + i\kappa h_1 \quad h_0 = h_0^\dagger, h_1 = h_1^\dagger \quad \kappa \in \mathbb{R}$$

assume $\eta = \eta^\dagger = e^{q/2} \Rightarrow$ solve for q

$$H^\dagger = e^q H e^{-q} = H + [q, H] + \frac{1}{2}[q, [q, H]] + \frac{1}{3!}[q, [q, [q, H]]] + \dots$$

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$$H(\lambda, \kappa) = h_0(\lambda) + i\kappa h_1 \quad h_0 = h_0^\dagger, h_1 = h_1^\dagger \quad \kappa \in \mathbb{R}$$

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for $c_q^{(\ell+1)}(h_0) = [q, \dots [q, [q, h_0]] \dots] = 0$ closed formulae:

$$h = h_0 + \sum_{n=1}^{\lfloor \frac{\ell}{2} \rfloor} \frac{(-1)^n E_n}{4^n (2n)!} c_q^{(2n)}(h_0) \quad H = h_0 - \sum_{n=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \frac{\kappa_{2n-1}}{(2n-1)!} c_q^{(2n-1)}(h_0)$$

$E_n \equiv$ Euler numbers, e.g. $E_1 = 1, E_2 = 5, E_3 = 61, \dots$

$$\kappa_n = \frac{1}{2^n} \sum_{m=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{n+m} \binom{n}{2m} E_m$$

$$\kappa_1 = 1/2, \kappa_3 = -1/4, \kappa_5 = 1/2, \kappa_7 = -17/8, \dots$$

[C. F. de Morisson Faria, A.F., J. Phys. A39 (2006) 9269]



further assumption

$$q = \sum_{k=1}^{\infty} \kappa^{2k-1} q_{2k-1}$$

solve recursively:

$$[h_0, q_1] = 2ih_1$$

$$[h_0, q_3] = \frac{i}{6}[q_1, [q_1, h_1]]$$

$$[h_0, q_5] = \frac{i}{6}[q_1, [q_3, h_1]] + \frac{i}{6}[q_3, [q_1, h_1]] - \frac{i}{360}[q_1, [q_1, [q_1, [q_1, h_1]]]]$$

Here

$$h_0(\lambda) = -\sum_{i=1}^N (\sigma_i^z + \lambda \sigma_i^x \sigma_{i+1}^x) / 2, \quad h_1 = -\sum_{i=1}^N \sigma_i^x / 2$$

further assumption

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- Perturbation theory in λ

$$H(\lambda, \kappa) = h_0(\kappa) + \lambda h_1 \quad h_0 \neq h_0^\dagger, h_1 = h_1^\dagger \quad \lambda \in \mathbb{R}$$

exact result for $N = 2$:

$\lambda = 0.1, \kappa = 0.5$:

$$h = \begin{pmatrix} -0.829536 & 0 & 0 & -0.0606492 \\ 0 & -0.0341687 & -0.1341687 & 0 \\ 0 & -0.1341687 & -0.0341687 & 0 \\ -0.0606492 & 0 & 0 & 0.897873 \end{pmatrix}$$

$\lambda = 0.9, \kappa = 0.1$:

$$h = \begin{pmatrix} -0.985439 & 0 & 0 & -0.890532 \\ 0 & -0.0094167 & -0.909417 & 0 \\ 0 & -0.909417 & -0.0094167 & 0 \\ -0.890532 & 0 & 0 & 1.00427 \end{pmatrix}$$



perturbative result 4th order for $N = 2$:

$\lambda = 0.1, \kappa = 0.5$:

$$h = \begin{pmatrix} -0.829534 & 0 & 0 & -0.0606716 \\ 0 & -0.0341688 & -0.134169 & 0 \\ 0 & -0.134169 & -0.0341688 & 0 \\ -0.0606716 & 0 & 0 & 0.897872 \end{pmatrix}$$

$\lambda = 0.9, \kappa = 0.1$:

$$h = \begin{pmatrix} -0.985439 & 0 & 0 & -0.890532 \\ 0 & -0.0094167 & -0.909417 & 0 \\ 0 & -0.909417 & -0.0094167 & 0 \\ -0.890532 & 0 & 0 & 1.00427 \end{pmatrix}$$

- new notation:

$$S_{a_1 a_2 \dots a_p}^N := \sum_{k=1}^N \sigma_k^{a_1} \sigma_{k+1}^{a_2} \dots \sigma_{k+p-1}^{a_p}, \quad a_i = x, y, z, u; i = 1, \dots, p \leq N$$

with $\sigma^u = \mathbb{I}$ to allow for non-local interactions

- for instance:

$$\begin{aligned} H(\lambda, \kappa) &= -\frac{1}{2} \sum_{j=1}^N (\sigma_j^z + \lambda \sigma_j^x \sigma_{j+1}^x + i\kappa \sigma_j^x), \quad \lambda, \kappa \in \mathbb{R} \\ &= -\frac{1}{2} (\mathbf{S}_z^N + \lambda \mathbf{S}_{xx}^N) - i\kappa \frac{1}{2} \mathbf{S}_x^N \end{aligned}$$

- perturbative result for $N = 3$:

$$\begin{aligned} h &= \mu_{xx}^3(\lambda, \kappa) \mathbf{S}_{xx}^3 + \mu_{yy}^3(\lambda, \kappa) \mathbf{S}_{yy}^3 + \mu_{zz}^3(\lambda, \kappa) \mathbf{S}_{zz}^3 + \mu_z^3(\lambda, \kappa) \mathbf{S}_z^3 \\ &\quad + \mu_{xxz}^3(\lambda, \kappa) \mathbf{S}_{xxz}^3 + \mu_{yyz}^3(\lambda, \kappa) \mathbf{S}_{yyz}^3 + \mu_{zzz}^3(\lambda, \kappa) \mathbf{S}_{zzz}^3 \end{aligned}$$

- perturbative result for $N = 4$:

$$\begin{aligned}
 h = & \mu_{xx}^4(\lambda, \kappa) \mathbf{S}_{xx}^4 + \nu_{xx}^4(\lambda, \kappa) \mathbf{S}_{xux}^4 + \mu_{yy}^4(\lambda, \kappa) \mathbf{S}_{yy}^4 + \nu_{yy}^4(\lambda, \kappa) \mathbf{S}_{yuy}^4 \\
 & + \mu_{zz}^4(\lambda, \kappa) \mathbf{S}_{zz}^4 + \nu_{zz}^4(\lambda, \kappa) \mathbf{S}_{zuz}^4 + \mu_z^4(\lambda, \kappa) \mathbf{S}_z^4 + \mu_{xzx}^4(\lambda, \kappa) \mathbf{S}_{xzx}^4 \\
 & + \mu_{xxz}^4(\lambda, \kappa) (\mathbf{S}_{xxz}^4 + \mathbf{S}_{zxx}^4) + \mu_{yyz}^4(\lambda, \kappa) (\mathbf{S}_{yyz}^4 + \mathbf{S}_{zyy}^4) \\
 & + \mu_{yzy}^4(\lambda, \kappa) \mathbf{S}_{yzy}^4 + \mu_{zzz}^4(\lambda, \kappa) \mathbf{S}_{zzz}^4 + \mu_{xxxx}^4(\lambda, \kappa) \mathbf{S}_{xxxx}^4 \\
 & + \mu_{yyyy}^4(\lambda, \kappa) \mathbf{S}_{yyyy}^4 + \mu_{zzzz}^4(\lambda, \kappa) \mathbf{S}_{zzzz}^4 + \mu_{xxyy}^4(\lambda, \kappa) \mathbf{S}_{xxyy}^4 \\
 & + \mu_{xyxy}^4(\lambda, \kappa) \mathbf{S}_{xyxy}^4 + \mu_{zzyy}^4(\lambda, \kappa) \mathbf{S}_{zzyy}^4 + \mu_{zyzy}^4(\lambda, \kappa) \mathbf{S}_{zyzy}^4 \\
 & + \mu_{xxzz}^4(\lambda, \kappa) \mathbf{S}_{xxzz}^4 + \mu_{xzxz}^4(\lambda, \kappa) \mathbf{S}_{xzxz}^4
 \end{aligned}$$

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 & + \mu_{zz}^4(\lambda, \kappa) \mathbf{S}_{zz}^4 + \nu_{zz}^4(\lambda, \kappa) \mathbf{S}_{zuz}^4 + \mu_z^4(\lambda, \kappa) \mathbf{S}_z^4 + \mu_{xzx}^4(\lambda, \kappa) \mathbf{S}_{xzx}^4 \\
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 & + \mu_{xxzz}^4(\lambda, \kappa) \mathbf{S}_{xxzz}^4 + \mu_{xzxz}^4(\lambda, \kappa) \mathbf{S}_{xzxz}^4
 \end{aligned}$$

non-local terms

Three possibilities to obtain PT-invariant Calogero models

- 1 Extended Calogero-Moser-Sutherland models
- 2 From constraint field equations
- 3 **Deformed Calogero-Moser-Sutherland models**

Calogero-Moser-Sutherland models (extended)

$$\mathcal{H}_{BK} = \frac{p^2}{2} + \frac{\omega^2}{2} \sum_i q_i^2 + \frac{g^2}{2} \sum_{i \neq k} \frac{1}{(q_i - q_k)^2} + i\tilde{g} \sum_{i \neq k} \frac{1}{(q_i - q_k)} p_i$$

with $g, \tilde{g} \in \mathbb{R}, q, p \in \mathbb{R}^{\ell+1}$

[B. Basu-Mallick, A. Kundu, Phys. Rev. B62 (2000) 9927]

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[B. Basu-Mallick, A. Kundu, Phys. Rev. B62 (2000) 9927]

- 1 Representation independent formulation?
- 2 Other potentials apart from the rational one?
- 3 Other algebras apart from A_n, B_n or Coxeter groups?
- 4 Is it possible to include more coupling constants?
- 5 Are the extensions still integrable?

- Generalize Hamiltonian to:

$$\mathcal{H}_\mu = \frac{1}{2} p^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha^2 V(\alpha \cdot q) + i \mu \cdot p$$

- Now Δ is any root system

- $\mu = 1/2 \sum_{\alpha \in \Delta} \tilde{g}_\alpha f(\alpha \cdot q) \alpha$, $f(x) = 1/x$ $V(x) = f^2(x)$

[A. F., Mod. Phys. Lett. A21 (2006) 691, Acta P. 47 (2007) 44]

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[A. F., Mod. Phys. Lett. A21 (2006) 691, Acta P. 47 (2007) 44]

- Not so obvious that one can re-write

$$\mathcal{H}_\mu = \frac{1}{2}(p+i\mu)^2 + \frac{1}{2} \sum_{\alpha \in \Delta} \hat{g}_\alpha^2 V(\alpha \cdot q), \quad \hat{g}_\alpha^2 = \begin{cases} g_s^2 + \alpha_s^2 \tilde{g}_s^2 & \alpha \in \Delta_s \\ g_l^2 + \alpha_l^2 \tilde{g}_l^2 & \alpha \in \Delta_l \end{cases}$$

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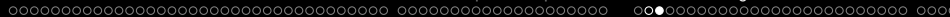
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$$\Rightarrow \mathcal{H}_\mu = \eta^{-1} h_{\text{Cal}} \eta \quad \text{with} \quad \eta = e^{-q \cdot \mu}$$



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- integrability follows trivially $\dot{L} = [L, M]: L(p) \rightarrow L(p + i\mu)$

Extended Calogero-Moser-Sutherland models

- Generalize Hamiltonian to:

$$\mathcal{H}_\mu = \frac{1}{2}p^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha^2 V(\alpha \cdot q) + i\mu \cdot p$$

· Now Δ is any root system

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- integrability follows trivially $\dot{L} = [L, M]: L(p) \rightarrow L(p + i\mu)$

- computing backwards for any CMS-potential

$$\mathcal{H}_\mu = \frac{1}{2}p^2 + \frac{1}{2} \sum_{\alpha \in \Delta} \hat{g}_\alpha^2 V(\alpha \cdot q) + i\mu \cdot p - \frac{1}{2}\mu^2$$

- $\mu^2 = \alpha_s^2 \tilde{g}_s^2 \sum_{\alpha \in \Delta_s} V(\alpha \cdot q) + \alpha_l^2 \tilde{g}_l^2 \sum_{\alpha \in \Delta_l} V(\alpha \cdot q)$ only for V rational

- From real fields to complex particle systems

- i) No restrictions

e.g. Benjamin-Ono equation

$$u_t + uu_x + \lambda Hu_{xx} = 0 \quad (*)$$

$H \equiv$ Hilbert transform, i.e. $Hu(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(z)}{z-x} dz$

Then

$$u(x, t) = \frac{\lambda}{2} \sum_{k=1}^{\ell} \left(\frac{i}{x - z_k} - \frac{i}{x - z_k^*} \right) \in \mathbb{R}$$

satisfies (*) iff z_k obeys the A_n -Calogero equ. of motion

$$\ddot{z}_k = \frac{\lambda^2}{2} \sum_{k \neq j} (z_j - z_k)^{-3}$$

[H. Chen, N. Pereira, Phys. Fluids 22 (1979) 187]

[talk by J. Feinberg, PHHQP workshop VI, 2007, London]

ii) restrict to submanifold

Theorem: [Airault, McKean, Moser, CPAM, (1977) 95]

Given a Hamiltonian $H(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n)$ with flow

$$\dot{x}_i = \partial H / \partial \dot{x}_i \quad \text{and} \quad \ddot{x}_i = -\partial H / \partial x_i \quad i = 1, \dots, n$$

and conserved charges I_j in involution with H , i.e.

$\{I_j, H\} = 0$. Then the locus of $\text{grad } I = 0$ is invariant.

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$\{I_j, H\} = 0$. Then the locus of $\text{grad } I = 0$ is invariant.

Example: Boussinesq equation

$$v_{tt} = a(v^2)_{xx} + bv_{xxxx} + v_{xx} \quad (**)$$

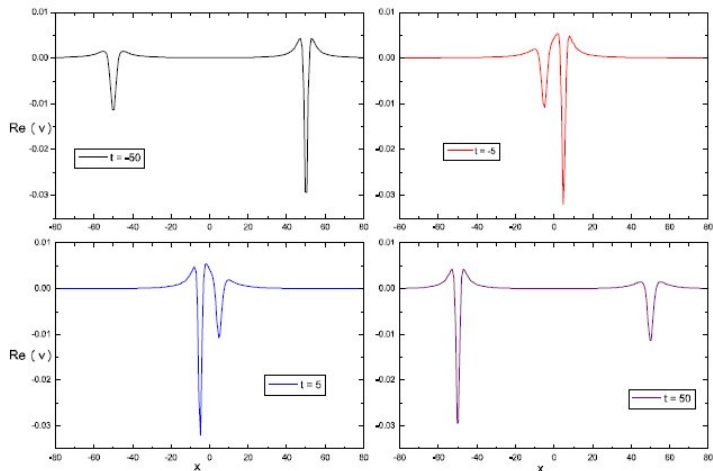
Then

$$v(x, t) = c \sum_{k=1}^{\ell} (x - z_k)^{-2}$$

satisfies (**) iff $b=1/12$, $c=-a/2$ and z_k obeys

$$\ddot{z}_k = 2 \sum_{j \neq k} (z_j - z_k)^{-3} \quad \Leftrightarrow \quad \ddot{z}_k = -\frac{\partial H}{\partial z_i}$$

$$\dot{z}_k = 1 - \sum_{j \neq k} (z_j - z_k)^{-2} \quad \Leftrightarrow \quad \text{grad}(I_3 - I_1) = 0$$

Constrained field equations \rightarrow complex Calogero models

[P. Assis and A.F., J. Phys. A42 (2009) 425206]

Consider

Antilinearly invariant deformed Calogero model

$$\mathcal{H}_{\text{PTCMS}} = \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\alpha \cdot \tilde{q})^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha V(\alpha \cdot \tilde{q}), \quad m, g_\alpha \in \mathbb{R}$$

Define deformed coordinates (A_2)

$$q_1 \rightarrow \tilde{q}_1 = q_1 \cosh \varepsilon + i\sqrt{3}(q_2 - q_3) \sinh \varepsilon$$

$$q_2 \rightarrow \tilde{q}_2 = q_2 \cosh \varepsilon + i\sqrt{3}(q_3 - q_1) \sinh \varepsilon$$

$$q_3 \rightarrow \tilde{q}_3 = q_3 \cosh \varepsilon + i\sqrt{3}(q_1 - q_2) \sinh \varepsilon$$



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$$q_3 \rightarrow \tilde{q}_3 = q_3 \cosh \varepsilon + i\sqrt{3}(q_1 - q_2) \sinh \varepsilon$$

With standard 3D representation for the simple A_2 -roots

$\alpha_1 = \{1, -1, 0\}$, $\alpha_2 = \{0, 1, -1\}$, $q_{ij} := q_i - q_j$ compute

$$\alpha_1 \cdot \tilde{q} = q_{12} \cosh \varepsilon - \frac{i}{\sqrt{3}}(q_{13} + q_{23}) \sinh \varepsilon,$$

$$\alpha_2 \cdot \tilde{q} = q_{23} \cosh \varepsilon - \frac{i}{\sqrt{3}}(q_{21} + q_{31}) \sinh \varepsilon,$$

$$(\alpha_1 + \alpha_2) \cdot \tilde{q} = q_{13} \cosh \varepsilon + \frac{i}{\sqrt{3}}(q_{12} + q_{32}) \sinh \varepsilon.$$

Define deformed coordinates (A_2)

$$q_1 \rightarrow \tilde{q}_1 = q_1 \cosh \varepsilon + i\sqrt{3}(q_2 - q_3) \sinh \varepsilon$$

$$q_2 \rightarrow \tilde{q}_2 = q_2 \cosh \varepsilon + i\sqrt{3}(q_3 - q_1) \sinh \varepsilon$$

$$q_3 \rightarrow \tilde{q}_3 = q_3 \cosh \varepsilon + i\sqrt{3}(q_1 - q_2) \sinh \varepsilon$$

With standard 3D representation for the simple A_2 -roots

$\alpha_1 = \{1, -1, 0\}$, $\alpha_2 = \{0, 1, -1\}$, $q_{ij} := q_i - q_j$ compute

$$\alpha_1 \cdot \tilde{q} = q_{12} \cosh \varepsilon - \frac{\iota}{\sqrt{3}}(q_{13} + q_{23}) \sinh \varepsilon,$$

$$\alpha_2 \cdot \tilde{q} = q_{23} \cosh \varepsilon - \frac{\iota}{\sqrt{3}}(q_{21} + q_{31}) \sinh \varepsilon,$$

$$(\alpha_1 + \alpha_2) \cdot \tilde{q} = q_{13} \cosh \varepsilon + \frac{\iota}{\sqrt{3}}(q_{12} + q_{32}) \sinh \varepsilon.$$

Symmetries:

$$\mathcal{S}_1 : \quad q_1 \leftrightarrow q_2, q_3 \leftrightarrow q_3, \iota \rightarrow -\iota,$$

$$\mathcal{S}_2 : \quad q_2 \leftrightarrow q_3, q_1 \leftrightarrow q_1, \iota \rightarrow -\iota.$$



Note, this Hamiltonian also results from deforming the roots:

$$\alpha_1 \rightarrow \tilde{\alpha}_1 = \alpha_1 \cosh \varepsilon + i\sqrt{3} \sinh \varepsilon \lambda_2$$

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Thus

$$\begin{aligned} \mathcal{H}_{PTCMS} &= \frac{p^2}{2} + \frac{m^2}{16} \sum_{\tilde{\alpha} \in \tilde{\Delta}_s} (\tilde{\alpha} \cdot q)^2 + \frac{1}{2} \sum_{\tilde{\alpha} \in \tilde{\Delta}} g_{\tilde{\alpha}} V(\tilde{\alpha} \cdot q), \quad m, g_{\tilde{\alpha}} \in \mathbb{R} \\ &= \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\alpha \cdot \tilde{q})^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_{\alpha} V(\alpha \cdot \tilde{q}), \quad m, g_{\alpha} \in \mathbb{R} \end{aligned}$$

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$$\sigma_2^{\varepsilon} : \tilde{\alpha}_2 \leftrightarrow -\tilde{\alpha}_2, \tilde{\alpha}_1 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 \quad \Leftrightarrow \quad q_2 \leftrightarrow q_3, q_1 \leftrightarrow q_1, \iota \rightarrow -\iota$$

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- Involution $\in \mathcal{W} \equiv$ Coxeter group \Rightarrow deform in antilinear way

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$$\alpha_j \in \Delta \subset \mathbb{R}^n, \quad \tilde{\alpha}_j(\varepsilon) \in \tilde{\Delta}(\varepsilon) \subset \mathbb{R}^n \oplus i\mathbb{R}^n, \quad \varepsilon \in \mathbb{R}$$

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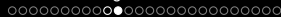
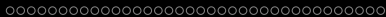
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- Find a second map that leaves $\tilde{\Delta}(\varepsilon)$ invariant

$$\varpi : \tilde{\Delta}(\varepsilon) \rightarrow \tilde{\Delta}(\varepsilon), \quad \tilde{\alpha} \mapsto \omega \tilde{\alpha}$$

- (i) $\varpi : \tilde{\alpha} = \mu_1 \alpha_1 + \mu_2 \alpha_2 \mapsto \mu_1^* \omega \alpha_1 + \mu_2^* \omega \alpha_2$ for $\mu_1, \mu_2 \in \mathbb{C}$
- (ii) $\varpi \circ \varpi = \mathbb{I}$



Make the following assumptions

(i) ω decomposes as

$$\omega = \tau\hat{\omega} = \hat{\omega}\tau$$

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Many solutions were constructed

$\tilde{\Delta}(\varepsilon)$ for A_3

$$\theta_\varepsilon = r_0 \mathbb{I} + r_2 \sigma^2 + \imath r_1 (\sigma - \sigma^3)$$

with explicit representation

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \sigma = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix},$$

$$\sigma_- = \sigma_1 \sigma_3, \sigma_+ = \sigma_2, \sigma = \sigma_- \sigma_+$$

$$\theta_\varepsilon = \begin{pmatrix} r_0 - \imath r_1 & -2\imath r_1 & -\imath r_1 - r_2 \\ 2\imath r_1 & r_0 - r_2 + 2\imath r_1 & 2\imath r_1 \\ -\imath r_1 - r_2 & -2\imath r_1 & r_0 - \imath r_1 \end{pmatrix}$$



all constraints require

$$\begin{aligned} (r_0 + r_2) \left[(r_0 + r_2)^2 - 4r_1^2 \right] &= 1 \\ r_0 - r_2 + 2r_1 &= (r_0 - r_2 + 2r_1)(r_0 + r_2) \\ (r_0 + r_2) &= (r_0 - r_2)^2 - 4r_1^2 \end{aligned}$$

these are solved by

$$r_0(\varepsilon) = \cosh \varepsilon, \quad r_1(\varepsilon) = \pm \sqrt{\cosh^2 \varepsilon - \cosh \varepsilon}, \quad r_2(\varepsilon) = 1 - \cosh \varepsilon$$

\Rightarrow simple deformed roots

$$\tilde{\alpha}_1 = \cosh \varepsilon \alpha_1 + (\cosh \varepsilon - 1) \alpha_3 - i\sqrt{2}\sqrt{\cosh \varepsilon} \sinh\left(\frac{\varepsilon}{2}\right) (\alpha_1 + 2\alpha_2 + \alpha_3),$$

$$\tilde{\alpha}_2 = (2 \cosh \varepsilon - 1) \alpha_2 + 2i\sqrt{2}\sqrt{\cosh \varepsilon} \sinh\left(\frac{\varepsilon}{2}\right) (\alpha_1 + \alpha_2 + \alpha_3),$$

$$\tilde{\alpha}_3 = \cosh \varepsilon \alpha_3 + (\cosh \varepsilon - 1) \alpha_1 - i\sqrt{2}\sqrt{\cosh \varepsilon} \sinh\left(\frac{\varepsilon}{2}\right) (\alpha_1 + 2\alpha_2 + \alpha_3).$$

remaining positive roots

$$\tilde{\alpha}_4 := \tilde{\alpha}_1 + \tilde{\alpha}_2, \quad \tilde{\alpha}_5 := \tilde{\alpha}_2 + \tilde{\alpha}_3, \quad \tilde{\alpha}_6 := \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3.$$

$\tilde{\Delta}(\varepsilon)$ for A_{4n-1} -subseries

closed solution

$$\theta_\varepsilon = r_0 \mathbb{I} + r_{2n} \sigma^{2n} + i r_n (\sigma^n - \sigma^{-n}),$$

- with $r_{2n} = 1 - r_0$, $r_n = \pm \sqrt{r_0^2 - r_0}$
- useful choice $r_0 = \cosh \varepsilon$

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 $\tilde{\Delta}(\varepsilon)$ for E_6

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$$r_2 = \pm 1/\sqrt{3} \sqrt{r_0^2 - 1}, \quad r_0 = \cosh \varepsilon$$



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$\tilde{\Delta}(\varepsilon)$ for B_{2n+1} -subseries

no solution based on factorisation of the Coxeter element



with different ω_j we find for instance for B_{2n+1}

$$\tilde{\alpha}_{2j-1} = \cosh \varepsilon \alpha_{2j-1} + i \sinh \varepsilon \left(\alpha_{2j-1} + 2 \sum_{k=2j}^{\ell} \alpha_k \right) \quad \text{for } j = 1, \dots,$$

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Solutions for complex root systems

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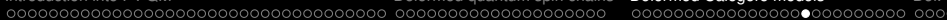
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in dual space

$$\theta_{\varepsilon}^* = \begin{pmatrix} R & & & & \\ & R & & 0 & \\ & & R & & \\ & & & \ddots & \\ & 0 & & & 1 \end{pmatrix} \quad \text{with } R = \begin{pmatrix} \cosh \varepsilon & i \sinh \varepsilon \\ -i \sinh \varepsilon & \cosh \varepsilon \end{pmatrix}$$



- Physical properties (A_2 , G_2)
 - The deformed model can be solved by separation of variables as the undeformed case.
 - Some restrictions cease to exist, as the wavefunctions are now regularized.
 - \Rightarrow modified energy spectrum:

$$E = 2|\omega|(2n + \lambda + 1)$$

becomes

$$E_{n\ell}^{\pm} = 2|\omega| [2n + 6(\kappa_s^{\pm} + \kappa_l^{\pm} + \ell) + 1] \quad \text{for } n, \ell \in \mathbb{N}_0,$$

$$\text{with } \kappa_{s/l}^{\pm} = (1 \pm \sqrt{1 + 4g_{s/l}})/4$$

[A. Fring and M. Znojil, J. Phys. A41 (2008) 194010]



The generic case

- generalized Calogero Hamiltonian (undeformed)

$$\mathcal{H}_C(p, q) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\alpha \in \Delta^+} (\alpha \cdot q)^2 + \sum_{\alpha \in \Delta^+} \frac{g_\alpha}{(\alpha \cdot q)^2},$$

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- define the variables

$$z := \prod_{\alpha \in \Delta^+} (\alpha \cdot q) \quad \text{and} \quad r^2 := \frac{1}{\hat{h} t_\ell} \sum_{\alpha \in \Delta^+} (\alpha \cdot q)^2,$$

$\hat{h} \equiv$ dual Coxeter number, $t_\ell \equiv \ell$ -th symmetrizer of l

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- Ansatz:

$$\psi(q) \rightarrow \psi(z, r) = z^{\kappa+1/2} \varphi(r)$$

\Rightarrow solution for $\kappa = 1/2\sqrt{1+4g}$.

$$\varphi_n(r) = c_n \exp\left(-\sqrt{\frac{\hat{h}t_\ell}{2}} \frac{\omega}{2} r^2\right) L_n^a\left(\sqrt{\frac{\hat{h}t_\ell}{2}} \omega r^2\right).$$

$L_n^a(x) \equiv$ Laguerre polynomial, $a = (2 + h + h\sqrt{1+4g})l/4 - 1$

The generic case

- eigenenergies

$$E_n = \frac{1}{4} \left[\left(2 + h + h\sqrt{1 + 4g} \right) l + 8n \right] \sqrt{\frac{\hat{h}t_\ell}{2}} \omega$$

- anyonic exchange factors

$$\psi(q_1, \dots, q_i, q_j, \dots, q_n) = e^{i\pi s} \psi(q_1, \dots, q_j, q_i, \dots, q_n), \quad \text{for } 1 \leq i, j \leq n,$$

with

$$s = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4g}$$

$\therefore r$ is symmetric and z antisymmetric



The generic case

The construction is based on the identities:

$$\sum_{\alpha, \beta \in \Delta^+} \frac{\alpha \cdot \beta}{(\alpha \cdot \mathbf{q})(\beta \cdot \mathbf{q})} = \sum_{\alpha \in \Delta^+} \frac{\alpha^2}{(\alpha \cdot \mathbf{q})^2},$$

$$\sum_{\alpha, \beta \in \Delta^+} (\alpha \cdot \beta) \frac{(\alpha \cdot \mathbf{q})}{(\beta \cdot \mathbf{q})} = \frac{\hat{h} h \ell}{2} t_\ell,$$

$$\sum_{\alpha, \beta \in \Delta^+} (\alpha \cdot \beta) (\alpha \cdot \mathbf{q})(\beta \cdot \mathbf{q}) = \hat{h} t_\ell \sum_{\alpha \in \Delta^+} (\alpha \cdot \mathbf{q})^2,$$

$$\sum_{\alpha \in \Delta^+} \alpha^2 = \ell \hat{h} t_\ell.$$

Strong evidence on a case-by-case level, but no rigorous proof.



The generic case

- antilinearly deformed Calogero Hamiltonian

$$\mathcal{H}_{adC}(p, q) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\tilde{\alpha} \in \tilde{\Delta}^+} (\tilde{\alpha} \cdot q)^2 + \sum_{\tilde{\alpha} \in \Delta^+} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha} \cdot q)^2}$$

- define the variables

$$\tilde{z} := \prod_{\tilde{\alpha} \in \tilde{\Delta}^+} (\tilde{\alpha} \cdot q) \quad \text{and} \quad \tilde{r}^2 := \frac{1}{\hat{h}t_\ell} \sum_{\tilde{\alpha} \in \tilde{\Delta}^+} (\tilde{\alpha} \cdot q)^2$$



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when identities still hold \Rightarrow

$$\psi(q) = \psi(\tilde{z}, r) = \tilde{z}^s \varphi_n(r)$$

eigenenergies with different constraints (only performed for ground state)

Deformed A_3 -models

- potential from deformed Coxeter group factors

$$\alpha_1 = \{1, -1, 0, 0\}, \alpha_2 = \{0, 1, -1, 0\}, \alpha_3 = \{0, 0, 1, -1\}$$

$$\tilde{\alpha}_1 \cdot q = q_{43} + \cosh \varepsilon (q_{12} + q_{34}) - i\sqrt{2} \cosh \varepsilon \sinh \frac{\varepsilon}{2} (q_{13} + q_{24})$$

$$\tilde{\alpha}_2 \cdot q = q_{23}(2 \cosh \varepsilon - 1) + i2\sqrt{2} \cosh \varepsilon \sinh \frac{\varepsilon}{2} q_{14}$$

$$\tilde{\alpha}_3 \cdot q = q_{21} + \cosh \varepsilon (q_{12} + q_{34}) - i\sqrt{2} \cosh \varepsilon \sinh \frac{\varepsilon}{2} (q_{13} + q_{24})$$

$$\tilde{\alpha}_4 \cdot q = q_{42} + \cosh \varepsilon (q_{13} + q_{24}) + i\sqrt{2} \cosh \varepsilon \sinh \frac{\varepsilon}{2} (q_{12} + q_{34})$$

$$\tilde{\alpha}_5 \cdot q = q_{31} + \cosh \varepsilon (q_{13} + q_{24}) + i\sqrt{2} \cosh \varepsilon \sinh \frac{\varepsilon}{2} (q_{12} + q_{34})$$

$$\tilde{\alpha}_6 \cdot q = q_{14}(2 \cosh \varepsilon - 1) - i\sqrt{2} \cosh \varepsilon \sinh \frac{\varepsilon}{2} q_{23}$$

notation $q_{ij} = q_i - q_j$,

Deformed A_3 -models

- potential from deformed Coxeter group factors

$$\alpha_1 = \{1, -1, 0, 0\}, \alpha_2 = \{0, 1, -1, 0\}, \alpha_3 = \{0, 0, 1, -1\}$$

$$\tilde{\alpha}_1 \cdot q = q_{43} + \cosh \varepsilon (q_{12} + q_{34}) - i\sqrt{2} \cosh \varepsilon \sinh \frac{\varepsilon}{2} (q_{13} + q_{24})$$

$$\tilde{\alpha}_2 \cdot q = q_{23}(2 \cosh \varepsilon - 1) + i2\sqrt{2} \cosh \varepsilon \sinh \frac{\varepsilon}{2} q_{14}$$

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notation $q_{ij} = q_i - q_j$, **No longer singular for $q_{ij} = 0$**

- \mathcal{PT} -symmetry for $\tilde{\alpha}$

$$\sigma_-^\varepsilon : \tilde{\alpha}_1 \rightarrow -\tilde{\alpha}_1, \tilde{\alpha}_2 \rightarrow \tilde{\alpha}_6, \tilde{\alpha}_3 \rightarrow -\tilde{\alpha}_3, \tilde{\alpha}_4 \rightarrow \tilde{\alpha}_5, \tilde{\alpha}_5 \rightarrow \tilde{\alpha}_4, \tilde{\alpha}_6 \rightarrow \tilde{\alpha}_2$$

$$\sigma_+^\varepsilon : \tilde{\alpha}_1 \rightarrow \tilde{\alpha}_4, \tilde{\alpha}_2 \rightarrow -\tilde{\alpha}_2, \tilde{\alpha}_3 \rightarrow \tilde{\alpha}_5, \tilde{\alpha}_4 \rightarrow \tilde{\alpha}_1, \tilde{\alpha}_5 \rightarrow \tilde{\alpha}_3, \tilde{\alpha}_6 \rightarrow \tilde{\alpha}_6$$

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- \mathcal{PT} -symmetry in dual space

$$\sigma_-^{\mathcal{E}} : q_1 \rightarrow q_2, q_2 \rightarrow q_1, q_3 \rightarrow q_4, q_4 \rightarrow q_3, \iota \rightarrow -\iota$$

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\Rightarrow

$$\sigma_-^\varepsilon \tilde{z}(q_1, q_2, q_3, q_4) = \tilde{z}^*(q_2, q_1, q_4, q_3) = \tilde{z}(q_1, q_2, q_3, q_4)$$

$$\sigma_+^\varepsilon \tilde{z}(q_1, q_2, q_3, q_4) = \tilde{z}^*(q_1, q_3, q_2, q_4) = -\tilde{z}(q_1, q_2, q_3, q_4)$$

$$\psi(q_1, q_2, q_3, q_4) = e^{i\pi S} \psi(q_2, q_4, q_1, q_3).$$

Anyonic exchange factors in the 4-particle scattering process

$$\begin{array}{cccc}
 w & x & y & z \\
 \bullet & \bullet & \bullet & \bullet \\
 q_1 & q_2 & q_3 & q_4
 \end{array}
 = e^{2\pi S}
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 \end{array}$$

$$\begin{array}{cc}
 x & y \\
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 \begin{array}{cc}
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 \bullet & \bullet \\
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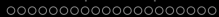
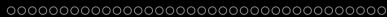
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Find Hermitian counterpart h , Dyson map η and metric ρ :

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \Leftrightarrow H^\dagger \rho = \rho H \quad \text{with} \quad \rho = \eta^\dagger \eta$$

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Some B_ℓ -models correspond to complex rotations

$$\begin{pmatrix} \tilde{z}_i \\ \tilde{z}_j \end{pmatrix} = R_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} = \eta_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} \eta_{ij}^{-1}, \quad \text{for } z \in \{x, p\}, \eta_{ij} = e^{\varepsilon(x_i p_j - x_j p_i)}$$

Hermitian isospectral counterparts and metric

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For instance for:

$$\theta_\varepsilon^* = \begin{pmatrix} R & & & \\ & R & & 0 \\ & & R & \\ & 0 & & \ddots \\ & & & & 1 \end{pmatrix} \quad \text{with } R = \begin{pmatrix} \cosh \varepsilon & i \sinh \varepsilon \\ -i \sinh \varepsilon & \cosh \varepsilon \end{pmatrix}$$

we have

$$\mathcal{H}_0(p, x) = \eta \mathcal{H}_\varepsilon(p, x) \eta^{-1}$$

with

$$\eta = \eta_{12}^{-1} \eta_{34}^{-1} \eta_{56}^{-1} \cdots \eta_{(\ell-2)(\ell-1)}^{-1}$$

For B_5

$$\theta_\varepsilon^* = \begin{pmatrix} r_0 & -i\vartheta & i\vartheta & 1 - r_0 & 0 \\ i\vartheta & r_0 & 1 - r_0 & -i\vartheta & 0 \\ -i\vartheta & 1 - r_0 & r_0 & i\vartheta & 0 \\ 1 - r_0 & i\vartheta & -i\vartheta & r_0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

we find

$$\tilde{X} = \theta_\varepsilon^* X = R_{24}^{-1} R_{13} R_{34} R_{12}^{-1} X = \eta X \eta^{-1}, \quad \text{with } \eta = \eta_{24}^{-1} \eta_{13} \eta_{34} \eta_{12}^{-1}.$$

In general this is an open problem.

General deformation prescription:

\mathcal{PT} -anti-symmetric quantities:

$$\mathcal{PT} : \phi(x, t) \mapsto -\phi(x, t) \quad \Rightarrow \quad \delta_\varepsilon : \phi(x, t) \mapsto -i[i\phi(x, t)]^\varepsilon$$

Two possibilities for the KdV Hamiltonian

$$\delta_\varepsilon^+ : u_x \mapsto u_{x,\varepsilon} := -i(iu_x)^\varepsilon \quad \text{or} \quad \delta_\varepsilon^- : u \mapsto u_\varepsilon := -i(iu)^\varepsilon,$$

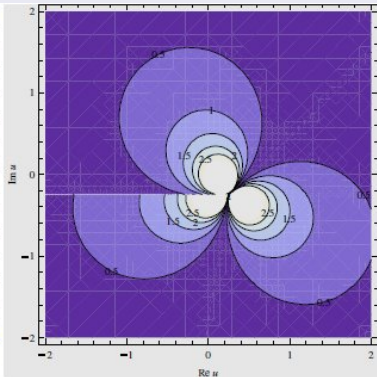
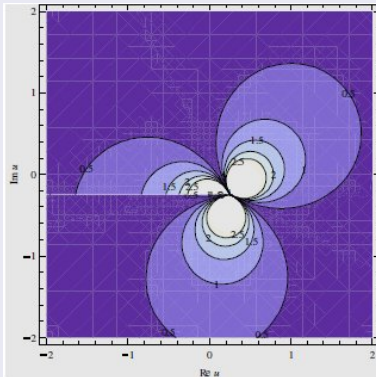
such that

$$\mathcal{H}_\varepsilon^+ = -\frac{\beta}{6}u^3 - \frac{\gamma}{1+\varepsilon}(iu_x)^{\varepsilon+1} \quad \mathcal{H}_\varepsilon^- = \frac{\beta}{(1+\varepsilon)(2+\varepsilon)}(iu)^{\varepsilon+2} + \frac{\gamma}{2}u_x^2$$

with equations of motion

$$u_t + \beta uu_x + \gamma u_{xxx,\varepsilon} = 0 \quad u_t + i\beta u_\varepsilon u_x + \gamma u_{xxx} = 0$$

Broken \mathcal{PT} -symmetric rational solutions for $\mathcal{H}_{1/3}^+$

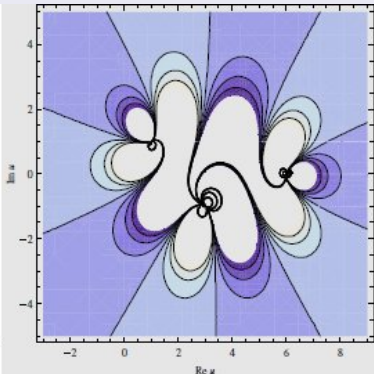
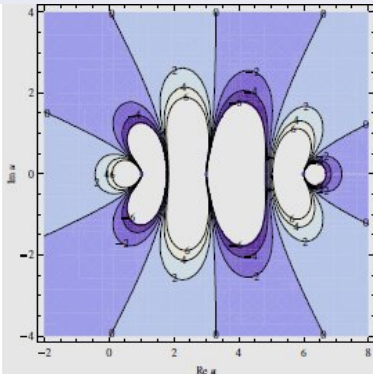


Different Riemann sheets for $A = (1 - i)/4$, $c = 1$, $\beta = 2 + 2i$
and $\gamma = 3$

(a) $u^{(1)}$

(b) $u^{(2)}$

Elliptic solutions for $\mathcal{H}_{-1/2}^+$:



(a) \mathcal{PT} -symmetric with $A = 1$, $B = 3$, $C = 6$, $\beta = 3/10$, $\gamma = -3$ and $c = 1$

(b) spontaneously broken \mathcal{PT} -symmetry with $A = 1 + i$, $B = 3 - i$, $C = 6$, $\beta = 3/10$, $\gamma = -3$ and $c = 1$

The $\mathcal{H}_\varepsilon^-$ -models

Integrating twice gives now:

$$u_\zeta^2 = \frac{2}{\gamma} \left(\kappa_2 + \kappa_1 u + \frac{c}{2} u^2 - \beta \frac{i^\varepsilon}{(1+\varepsilon)(2+\varepsilon)} u^{2+\varepsilon} \right) =: \lambda Q(u)$$

where

$$\lambda = -\frac{2\beta i^\varepsilon}{\gamma(1+\varepsilon)(2+\varepsilon)}$$

For $\kappa_1 = \kappa_2 = 0$

$$u(\zeta) = \left(\frac{c(\varepsilon+1)(\varepsilon+2)}{i^\varepsilon \beta \left[\cosh \left(\frac{\sqrt{c\varepsilon}(\zeta-\zeta_0)}{\sqrt{\gamma}} \right) + 1 \right]} \right)^{1/\varepsilon}$$

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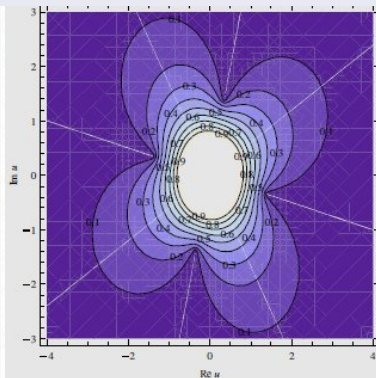
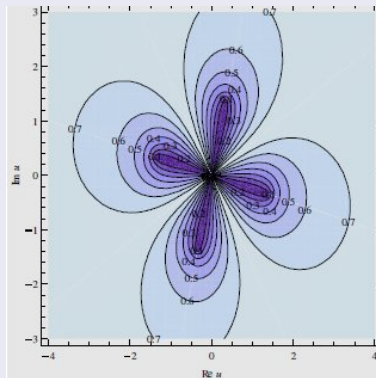
$$\kappa_1 = \kappa_2 = 0, \quad B = iC \quad \text{and} \quad C^4 = \frac{15c}{\beta}$$

eigenvalues of Jacobian:

$$j_1 = \pm i\sqrt{r_\lambda} r_B^2 \exp\left[\frac{i}{2}(4\theta_B + \theta_\lambda)\right]$$

$$j_2 = \mp i\sqrt{r_\lambda} r_B^2 \exp\left[-\frac{i}{2}(4\theta_B + \theta_\lambda)\right]$$

Broken \mathcal{PT} -symmetric solution for \mathcal{H}_4^- :



(a) star node at the origin for $c = 1$, $\beta = 2 + i3$, $\gamma = 1$ and $B = (15/2 + i3)^{1/4}$

(b) centre at the origin for $c = 1$, $\beta = 2 + i3$, $\gamma = -1$ and $B = (30/13 - i45/13)^{1/4}$

Reduction to quantum mechanical Hamiltonians:

Again we can relate to simple quantum mechanical models:

The identification

$$u \rightarrow x, \quad \zeta \rightarrow t, \quad \kappa_1 = 0, \quad \kappa_2 = \gamma E, \quad \text{and} \quad \beta = \gamma g(1+\varepsilon)(2+\varepsilon)$$

relates $\mathcal{H}_\varepsilon^-$ to

$$H = E = \frac{1}{2}p^2 - \frac{c}{2\gamma}x^2 + gx^2(ix)^\varepsilon$$

For $c = 0$ these are the "classical models" studied in

[C. Bender, S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243]

Reduction of the \mathcal{H}_2^- -model

$$\mathcal{H}_2^-[u] = \frac{\beta}{12}u^4 + \frac{\gamma}{2}u_x^2$$

Twice integrated equation of motion:

$$u_\zeta^2 = \frac{2}{\gamma} \left(\kappa_2 + \kappa_1 u + \frac{c}{2}u^2 + \beta \frac{1}{12}u^4 \right) =: \lambda Q(u)$$

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Reduction $u \rightarrow x, \zeta \rightarrow t$

$$\kappa_1 = -\gamma\tau, \quad \kappa_2 = \gamma E_x, \quad \beta = -3\gamma g \quad \text{and} \quad c = -\gamma\omega^2$$

Quartic harmonic oscillator of the form

$$H = E_x = \frac{1}{2}p^2 + \tau x + \frac{\omega^2}{2}x^2 + \frac{g}{4}x^4$$

Boundary cond.: $\kappa_1 = \tau = 0, \lim_{\zeta \rightarrow \infty} u(\zeta) = 0, \lim_{\zeta \rightarrow \infty} u_x(\zeta) = \sqrt{2E_x}$

[A.G. Anderson, C. Bender, U. Morone, arXiv:1102.4822]

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Note: $E_x \neq E_u(a)$

Ito type systems and its deformations

Coupled nonlinear system

$$\begin{aligned}u_t + \alpha v v_x + \beta u u_x + \gamma u_{xxx} &= 0, & \alpha, \beta, \gamma \in \mathbb{C}, \\v_t + \delta(uv)_x + \phi v_{xxx} &= 0, & \delta, \phi \in \mathbb{C}\end{aligned}$$

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Hamiltonian for $\delta = \alpha$

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\mathcal{PT} -symmetries:

$$\begin{aligned} \mathcal{PT}_{++} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto u, v \mapsto v & \quad \text{for } \alpha, \beta, \gamma, \phi \in \mathbb{R} \\ \mathcal{PT}_{+-} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto u, v \mapsto -v & \quad \text{for } \alpha, \beta, \gamma, \phi \in \mathbb{R} \\ \mathcal{PT}_{-+} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto -u, v \mapsto v & \quad \text{for } i\alpha, i\beta, \gamma, \phi \in \mathbb{R} \\ \mathcal{PT}_{--} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto -u, v \mapsto -v & \quad \text{for } i\alpha, i\beta, \gamma, \phi \in \mathbb{R} \end{aligned}$$

Deformed models

$$\mathcal{H}_{\varepsilon,\mu}^{++} = -\frac{\alpha}{2}uv^2 - \frac{\beta}{6}u^3 - \frac{\gamma}{1+\varepsilon}(iu_x)^{\varepsilon+1} - \frac{\phi}{1+\mu}(iv_x)^{\mu+1}$$

$$\mathcal{H}_{\varepsilon,\mu}^{+-} = \frac{\alpha}{1+\mu}u(iv)^{\mu+1} - \frac{\beta}{6}u^3 - \frac{\gamma}{1+\varepsilon}(iu_x)^{\varepsilon+1} + \frac{\phi}{2}v_x^2$$

$$\mathcal{H}_{\varepsilon,\mu}^{-+} = -\frac{\alpha}{2}uv^2 - \frac{i\beta}{(1+\varepsilon)(2+\varepsilon)}(iu)^{2+\varepsilon} + \frac{\gamma}{2}u_x^2 - \frac{\phi}{1+\mu}(iv_x)^{\mu+1}$$

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- Non-Hermitian Hamiltonians describe physical systems within a self-consistent quantum mechanical framework.
- One can use this possibility to explore deformations of well studied models, e.g. integrable systems.
- There exist now experiments, especially in optics, for the broken PT-regime.

Thank you for your attention
आपका बहुत बहुत धन्यवाद