CITY UNIVERSITY
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A unifying E2-quasi-exactly solvable model

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Supersymmetry in Integrable Systems - SIS'15
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S. Dey, A. Fring, T. Mathanaranjan, Ann. of Physics, 346 (2014) 28
S. Dey, A. Fring, T. Mathanaranjan, Int. J. Th. Phys. (2014) 10.1007
A. Fring, J. Phys. A: Math. Theor. 48 (2015)145301
A. Fring, Phys. Lett. A379 (2015) 873876; arXiv:1507.00611

Why study models of Euclidean Lie algebraic type?

1. Mathematical motivation:
   a) (quasi)-exactly solvable models of $sl_2(\mathbb{R})$-Lie algebraic type
      $\Rightarrow$ solutions are hypergeometric functions
   b) models of Euclidean-Lie algebraic type
      $\Rightarrow$ solutions are Mathieu functions

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      $\Rightarrow$ solutions are hypergeometric functions
   b) models of Euclidean-Lie algebraic type
      $\Rightarrow$ solutions are Mathieu functions
2. Physical motivation:
   - applications of b)-type models in optics
   - the complex Mathieu equation corresponds to the eigenvalue equation for the collision operator in a 2D Lorentz gas

Hamiltonians of $sl_2(\mathbb{R})$-Lie algebraic type

Quasi-solvable Hamiltonian of Lie algebraic type:

$$H_J = \sum_{l=0,\pm} \kappa_l J_l + \sum_{n,m=0,\pm} \kappa_{nm} : J_n J_m :, \qquad \kappa_l, \kappa_{nm} \in \mathbb{R},$$

$sl_2(\mathbb{R})$-Lie algebra

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0, \qquad J_0^\dagger, J_\pm^\dagger \notin \{J_0, J_\pm\}$$

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$\mathcal{PT}$-symmetric versions:
Rescale $J_\pm \to \tilde{J}_\pm = \pm i J_\pm$, $J_0 \to \tilde{J}_0 = J_0$
Example:
$\kappa_{00} = -4$, $\kappa_+ = -2\zeta = \kappa_-$, $\zeta \in \mathbb{R}$

$$V(x) = -\left[\zeta \sinh 2x - iM\right]^2$$

[P.E.G. Assis, A. Fring, J. Phys. A42 (2009) 015203]

Hamiltonians of Euclidean Lie algebraic type

E_2-algebra:

$$[u, J] = iv, \qquad [v, J] = -iu, \qquad [u, v] = 0$$

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Representations:

- quantizing of strings on tori

$$\Pi^{(1)}: \quad J := -i\partial_\theta, \qquad u := \sin\theta, \qquad v := \cos\theta$$

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- two dimensional representations

$$\Pi^{(2)} \quad : \qquad J := yp_x - xp_y, \quad u := x, \quad v := y,$$
$$\Pi^{(3)} \quad : \qquad J := xp_y - p_x y, \quad u := p_y, \quad v := p_x,$$

with q_j, p_j satisfying $[q_j, p_k] = i\delta_{jk}$ for $j, k = 1, 2$

Different types of "$\mathcal{PT}$-symmetries":

$$
\begin{array}{lllll}
\mathcal{PT}_1: & J \to -J, & u \to -u, & v \to -v, & i \to -i, \\
\mathcal{PT}_2: & J \to -J, & u \to u, & v \to v, & i \to -i, \\
\mathcal{PT}_3: & J \to J, & u \to v, & v \to u, & i \to -i, \\
\mathcal{PT}_4: & J \to J, & u \to -u, & v \to v, & i \to -i, \\
\mathcal{PT}_5: & J \to J, & u \to u, & v \to -v, & i \to -i.
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$$

$\mathcal{PT}_i$-invariant Hamitonians:

$$H_{\mathcal{PT}_1} = \mu_1 J^2 + i\mu_2 J + i\mu_3 u + i\mu_4 v + \mu_5 uJ + \mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + \mu_9 uv$$

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$$H_{\mathcal{PT}_2} = \mu_1 J^2 + i\mu_2 J + \mu_3 u + \mu_4 v + i\mu_5 uJ + i\mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + \mu_9 uv$$

$$H_{\mathcal{PT}_3} = \mu_1 J^2 + \mu_2 J + \mu_3 (u+v) + i\mu_4 (u-v) + \mu_5 (u+v)J + i\mu_6 (u-v)J + i\mu_7 (v^2 - u^2) + \mu_8 (v^2 + u^2) + \mu_9 uv$$

$$H_{\mathcal{PT}_4} = \mu_1 J^2 + \mu_2 J + i\mu_3 u + \mu_4 v + i\mu_5 uJ + \mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + i\mu_9 uv$$

$$H_{\mathcal{PT}5} = \mu_1 J^2 + \mu_2 J + \mu_3 u + i\mu_4 v + \mu_5 uJ + i\mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + i\mu_9 uv$$

with $\mu_i \in \mathbb{R}$ for $i = 1, \dots, 9$

Standard approach to non-Hermitian QM:

- Given $H \begin{cases} \text{either} & \text{solve } \eta H \eta^{-1} = h \quad \text{for} \quad \eta \Rightarrow \rho = \eta^\dagger \eta \\ \text{or} & \text{solve } H^\dagger = \rho H \rho^{-1} \quad \text{for} \quad \rho \Rightarrow \eta = \sqrt{\rho} \end{cases}$

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- Thus, this is not re-inventing or disputing the validity of quantum mechanics
- We only give up the restrictive requirement that Hamiltonians have to be Hermitian.

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[C. Bender, *Rep. Prog. Phys.* 70 (2007) 947]
[A. Mostafazadeh, *Int. J. Geom. Meth. Phys.* 7 (2010) 1191]
[A. Fring, *Phil. Trans. R. Soc.* A 371 (2013) 20120046]

Isospectral partner Hamitonians:

$$h_{\mathcal{PT}_5} = \mu_1 J^2 + \mu_2 J + \frac{1}{2}\left(\mu_5 - \mu_6 \tanh\frac{\lambda}{2}\right)\{u, J\}$$

$$+ \left[\frac{2\mu_5^2 \sinh^2\lambda + \mu_6^2(\operatorname{sech}^2\frac{\lambda}{2} + \cosh 2\lambda - 1) + 2(\tanh\frac{\lambda}{2} - \sinh 2\lambda)\mu_5\mu_6}{8\mu_1}\right.$$

$$\left. + \frac{\mu_8 - \mu_7}{2}\cosh(2\lambda)\right](v^2 - u^2) + \left[\operatorname{csch}\lambda\left(\mu_4 + \frac{1}{2}\mu_5\right) + \frac{\mu_2}{2\mu_1}(\mu_5\right.$$

$$\left. - \coth\lambda\mu_6)\right] u + \frac{\mu_6^2\cosh\lambda - \mu_5\mu_6\sinh\lambda}{4\mu_1(1+\cosh\lambda)} + \frac{1}{2}(\mu_7 + \mu_8)$$

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Sinusoidal optical lattices from further constraints

$$\mu_1 = 1, \;\; \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = 0, \;\; \mu_8 = -4, \;\; \mu_9 = -8V_0$$
$$V(x) = 4cos^2 x + 4iV_0 sin2x$$

[B. Midya, B. Roy, et al, Phys. Lett. A374 (2010) 2605]
[H. Jones, J. Phys. A44 (2011) 345302]

However, it is not always possible to find isospectral pairs:

For instance: $\mathcal{PT}_3$-symmetric non-Hermitian Hamiltonian

$$\mathcal{H}_{\text{Mat}} = J^2 + 2ig(u^2 - v^2) \Rightarrow \mathcal{H}_{\text{Mat}}^{\Pi^{(1)}} = -\frac{d^2}{d\theta^2} + 2ig\cos(2\theta)$$

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Consider instead

$$\mathcal{H}_N = J^2 + \zeta^2(u^2 - v^2)^2 + 2i\zeta N(u^2 - v^2),$$

and take a double scaling limit

$$\lim_{N\to\infty,\zeta\to 0} \mathcal{H}_N = \mathcal{H}_{\text{Mat}}, \qquad \text{for } g := N\zeta < \infty$$

[B. Bagchi, S. Mallik, C. Quesne, ... Phys. Lett. A289 (2001) 34]
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Relation of $\mathcal{H}_{\text{Mat}}$ to E_2:

[C. M. Bender, R. Kalveks, Int. J. Theor. Phys. 50 (2011) 955]

E_2-quasi-exact solvability

In general: $\mathcal{H} : V_n \mapsto V_n$ with $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$

For $\Pi^{(1)}$ define:

$$V_n^s = \mathrm{span}\left\{ \phi_0 \left[\sin(2\theta), \ldots, i^{n+1}\sin(2n\theta)\right] \middle| \, \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \right\}$$

$$V_n^c = \mathrm{span}\left\{ \phi_0 \left[1, i\cos(2\theta), \ldots, i^n\cos(2n\theta)\right] \middle| \, \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \right\}$$

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For $\phi_0^c = e^{i\kappa\cos 2\theta}$, $\phi_0^s = e^{\kappa\sin 2\theta}$ with $\kappa \in \mathbb{R}$ we find:

$$\begin{aligned} J &: V_n^{s,c}\left(\phi_0^c\right) \mapsto V_{n+1}^{c,s}\left(\phi_0^c\right) \\ uv &: V_n^{s,c}\left(\phi_0^c\right) \mapsto V_{n+1}^{c,s}\left(\phi_0^c\right) \\ i(u^2 - v^2) &: V_n^{s,c}\left(\phi_0^c\right) \mapsto V_{n+1}^{s,c}\left(\phi_0^c\right) \end{aligned}$$

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\end{aligned}$$

For representation $\Pi^{(2)}$ and $\Pi^{(3)}$ use polynomials in x, y.

Thus we have:

$$\mathcal{H}_N : V_n^{s,c}(\phi_0^c) \mapsto V_{n+2}^{s,c}(\phi_0^c) \oplus \zeta^2 V_{n+2}^{s,c}(\phi_0^c) \oplus V_{n+1}^{s,c}(\phi_0^c)$$

- with constraint on $V_{n+2}^{s,c}(\phi_0^c) \oplus \zeta^2 V_{n+2}^{s,c}(\phi_0^c)$
- and quantization condition on level $n+1$

$$\mathcal{H}_N : V_{(N-1)/2}^{s,c}(\phi_0^c) \mapsto V_{(N-1)/2}^{s,c}(\phi_0^c)$$

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More solutions exist:

$$\hat{\mathcal{H}}_N = J^2 + \zeta uvJ + 2i\zeta N(u^2 - v^2), \qquad \zeta, N \in \mathbb{R}$$

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More solutions exist:

$$\hat{\mathcal{H}}_N = J^2 + \zeta uvJ + 2i\zeta N(u^2 - v^2), \qquad \zeta, N \in \mathbb{R}$$

$\hat{\mathcal{H}}_N$ also reduces to $\mathcal{H}_{\text{Mat}}$ in the double scaling limit

$$\lim_{N\to\infty,\zeta\to 0} \hat{\mathcal{H}}_N = \mathcal{H}_{\text{Mat}}, \qquad \text{for } g := N\zeta < \infty$$

Can we combine the models?

Generic Ansatz:

$$\mathcal{H} = J^2 + \mu\zeta uvJ + \lambda\zeta^2(u^2 - v^2)^2 + 2i\zeta N(u^2 - v^2), \qquad \lambda, \zeta, N \in \mathbb{R},$$

leads to four-term relation.

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leads to four-term relation.

Restricting μ:

$$\mathcal{H}(N, \zeta, \lambda) = J^2 + 2(1 - \lambda)\zeta uvJ + \lambda\zeta^2(u^2 - v^2)^2 + 2i\zeta N(u^2 - v^2)$$

leads to desired three-term relation.

Can we combine the models?

Generic Ansatz:

$$\mathcal{H} = J^2 + \mu\zeta uvJ + \lambda\zeta^2(u^2 - v^2)^2 + 2i\zeta N(u^2 - v^2), \qquad \lambda, \zeta, N \in \mathbb{R},$$

leads to four-term relation.

Restricting μ:

$$\mathcal{H}(N, \zeta, \lambda) = J^2 + 2(1-\lambda)\zeta uvJ + \lambda\zeta^2(u^2 - v^2)^2 + 2i\zeta N(u^2 - v^2)$$

leads to desired three-term relation.

The limits $\lambda \to 0$, $\lambda \to 1$ yield the previous cases.

Three term recurrence relations for $\mathcal{H}(N, \zeta, \lambda)$:

Ansatz:

$$\psi_N^c(\theta) = \phi_0 \sum_{n=0}^{\infty} i^n c_n P_n(E) \cos(2n\theta)$$

$$\psi_N^s(\theta) = \phi_0 \sum_{n=0}^{\infty} i^{n+1} c_n Q_n(E) \sin(2n\theta)$$

Three term recurrence relations for $\mathcal{H}(N,\zeta,\lambda)$:

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$$\psi_N^s(\theta) = \phi_0 \sum_{n=0}^{\infty} i^{n+1} c_n Q_n(E) \sin(2n\theta)$$

$$c_n = \frac{1}{\zeta^n}(N+\lambda)(1+\lambda)^{n-1}\left[\frac{1+N+2\lambda}{1+\lambda}\right]_{n-1}, \; \phi_0 = e^{\frac{i}{2}\zeta\cos(2\theta)}$$

Three term recurrence relations for $\mathcal{H}(N, \zeta, \lambda)$:

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yields

$$P_2 = (E - \lambda\zeta^2 - 4)P_1 + \mathbf{2}\zeta^2 [N-1][N+\lambda] P_0,$$

$$P_{i+1} = (E - \lambda\zeta^2 - 4i^2)P_i + \zeta^2 [N + i\lambda + (i-1)][N - (i-1)\lambda - i] P_{i-1}$$

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$$Q_2 = (E - 4 - \lambda\zeta^2)Q_1$$

$$Q_{j+1} = (E - \lambda\zeta^2 - 4j^2)Q_j + \zeta^2 [N + j\lambda + (j-1)][N - (j-1)\lambda - j] Q_{j-1}$$

for $i = 0, 2, \ldots, j = 2, 3, 4$

Solutions:

$$
\begin{aligned}
P_0 &= 1 \\
P_1 &= E - \lambda\zeta^2 \\
P_2 &= \lambda^2\zeta^4 + 2\zeta^2\left[\lambda - \lambda E + N(\lambda + N - 1)\right] + (E-4)E \\
P_3 &= -\lambda^3\zeta^6 + \lambda\zeta^4\left(\lambda(2\lambda + 3E - 13) - 3N^2 - 3(\lambda - 1)N + 2\right) \\
&\quad +(E-16)(E-4)E + 32(\lambda + N(\lambda + N - 1)) \\
&\quad -\zeta^2\left[3\lambda E^2 + E\left(2\lambda^2 - 3N^2 - 3\lambda(N+11) + 3N + 2\right)\right] \\
Q_1 &= 1 \\
Q_2 &= E - 4 - \lambda\zeta^2, \\
Q_3 &= \lambda^2\zeta^4 + \zeta^2\left[\lambda(15 - 2\lambda - 2E) + N^2 + (\lambda - 1)N - 2\right] \\
&\quad +(E-16)(E-4)
\end{aligned}
$$

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Present case: $\hat{n} = -(1+N)/(1+\lambda)$ or $\tilde{n} = (\lambda+N)/(1+\lambda)$:

$$P_{\tilde{n}+\ell} = P_{\tilde{n}} R_\ell \qquad \text{and} \qquad Q_{\tilde{n}+\ell} = Q_{\tilde{n}} R_\ell$$

with

$$\begin{aligned} R_1 &= E - 4\tilde{n}^2 - \lambda\zeta^2, \\ R_2 &= (E - 4\tilde{n}^2 - \lambda\zeta^2)(E - 4(\tilde{n}+1)^2 - \lambda\zeta^2) - 2\tilde{n}(1+\lambda)^2\zeta^2 \end{aligned}$$

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Typical features of Bender-Dunne polynomials.
[C.M. Bender, G.V. Dunne, J. Math. Phys. 37 (1996) 6]

Energy quantization:

We find E_n from $P_{\tilde{n}}(E) = 0$ and $Q_{\tilde{n}}(E) = 0$:

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E_3^{c,\ell} &= \frac{20}{3} + \lambda\zeta^2 + \frac{4\hat{\Omega}}{3} e^{\frac{i\pi\ell}{3}} + \frac{1}{3}\left[52 - 12(1+\lambda)^2\zeta^2\right] e^{-\frac{i\pi\ell}{3}} \hat{\Omega}^{-1}
\end{aligned}
$$

with $\ell = 0, \pm 2$

$$
\hat{\Omega}^3 = \left[\left[3(\lambda+1)^2\zeta^2 - 13\right]^3 + \left[18(\lambda+1)^2\zeta^2 + 35\right]^2\right]^{\frac{1}{2}} + 35 + 18(\lambda+1)^2\zeta^2
$$

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$$

$$
\begin{aligned}
E_2^s &= 4+\lambda\zeta^2, \\
E_3^{s,\pm} &= 10+\zeta^2\lambda \pm 2\sqrt{9-(\lambda+1)^2\zeta^2}, \\
E_4^{s,\ell} &= \frac{56}{3}+\lambda\zeta^2+\frac{4\Omega}{3}e^{\frac{i\pi\ell}{3}}+\frac{1}{3}\left[196-12(1+\lambda)^2\zeta^2\right]e^{-\frac{i\pi\ell}{3}}\Omega^{-1}
\end{aligned}
$$

$$
\Omega^3 = \left[\left(3\zeta^2(\lambda+1)^2-49\right)^3+\left(18\zeta^2(\lambda+1)^2+143\right)^2\right]^{\frac{1}{2}} +143+18\zeta^2(\lambda+1)^2
$$

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Recall: discriminant $= \Delta = \prod_{1\leq i<j\leq n}(E_i - E_j)^2$

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Compute zeros of $\Delta^c_{\tilde{n}}$, $\Delta^s_{\tilde{n}}$ of $P_{\tilde{n}}(E), Q_{\tilde{n}}(E)$:

$$
\begin{aligned}
\tilde{\Delta}^c_2 &= \hat{\zeta}^2 - 1, \quad \tilde{\Delta}^s_3 = \hat{\zeta}^2 - 9, \quad \tilde{\Delta}^c_3 = \hat{\zeta}^6 - \hat{\zeta}^4 + 103\hat{\zeta}^2 - 36, \\
\tilde{\Delta}^s_4 &= \hat{\zeta}^6 - 37\hat{\zeta}^4 + 991\hat{\zeta}^2 - 3600, \\
\tilde{\Delta}^c_4 &= \hat{\zeta}^{12} + 2\hat{\zeta}^{10} + 385\hat{\zeta}^8 - 33120\hat{\zeta}^6 + 16128\hat{\zeta}^4 - 732276\hat{\zeta}^2 \\
&\quad +129600, \\
\tilde{\Delta}^s_5 &= \hat{\zeta}^{12} - 94\hat{\zeta}^{10} + 7041\hat{\zeta}^8 - 381600\hat{\zeta}^6 + 6645600\hat{\zeta}^4 \\
&\quad -78318900\hat{\zeta}^2 + 158760000,
\end{aligned}
$$

$\hat{\zeta} := \zeta(1+\lambda)$

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\end{aligned}
$$

$\hat{\zeta} := \zeta(1+\lambda)$

Computable from the determinant of the Sylvester matrix S:

$$
S_{ij} = \begin{cases} a_{n+i-j}, & \text{for } 1 \leq i \leq n-1, 1 \leq j \leq 2n-1, \\ (1+i-j)a_{1+i-j}, & \text{for } n \leq i \leq 2n-1, 1 \leq j \leq 2n-1, \end{cases}
$$

where $P(E) = \sum_{k=0}^{n} a_k E^k$

Vicinity of exceptional points:

What happens near the exceptional points?

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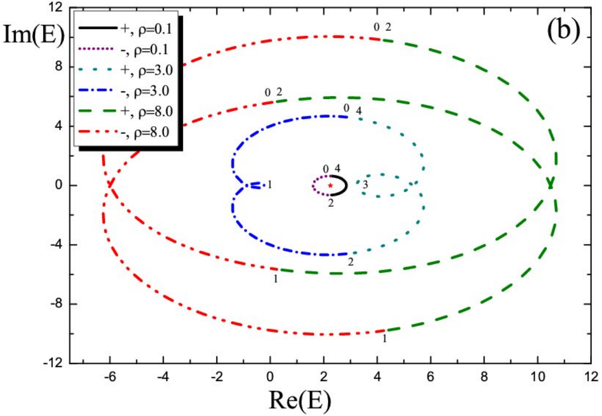
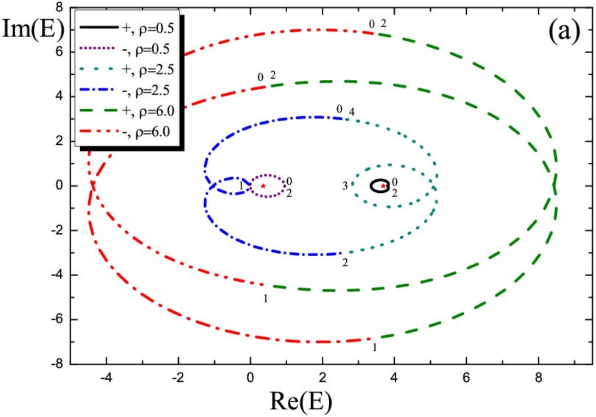
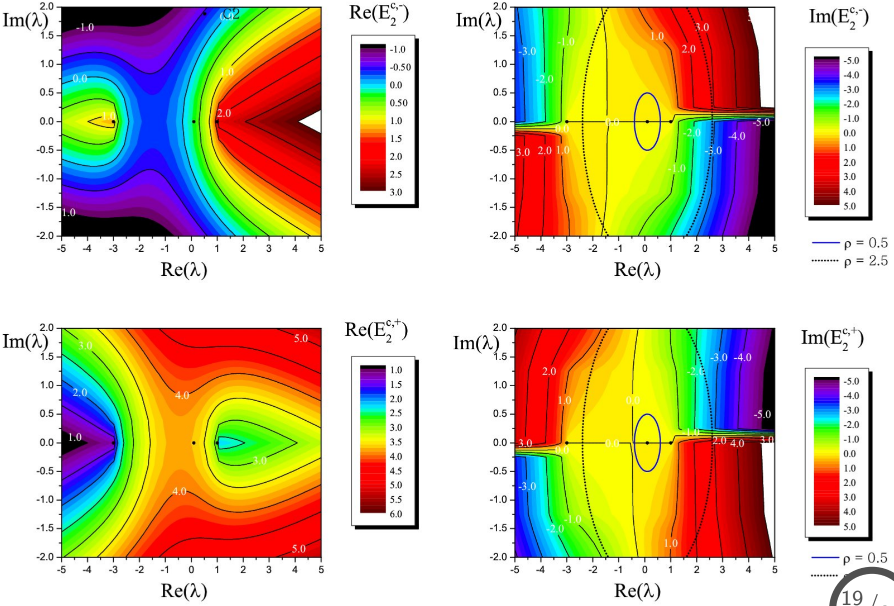
Energy loops $E(\lambda = \tilde{\lambda} + \rho e^{i\pi\phi}, \zeta)$ varying ϕ with fixed $\tilde{\lambda}$, ρ and ζ:
Around an exceptional point: $E_2^{c,\pm}$ with $E_2^{c,-} = E_2^{c,+} = 9/4$



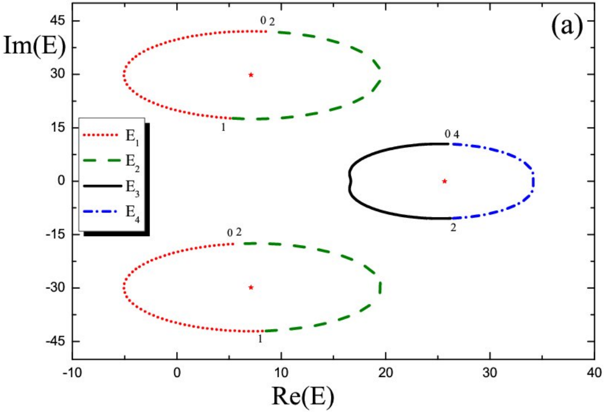
No exceptional point: $E_2^{c,\pm}$ with $E_2^{c,-} = 0.35$, $E_2^{c,+} = 3.70$



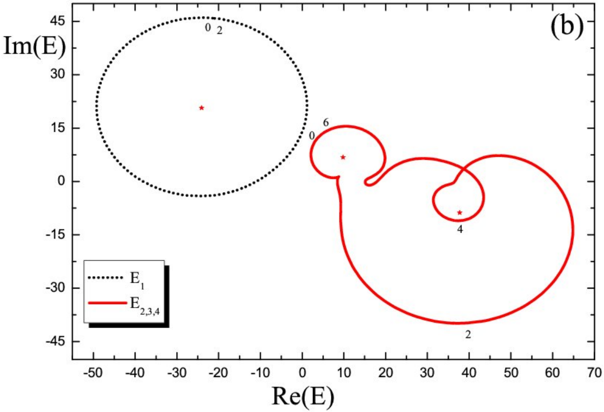
The exceptional points are branch points.

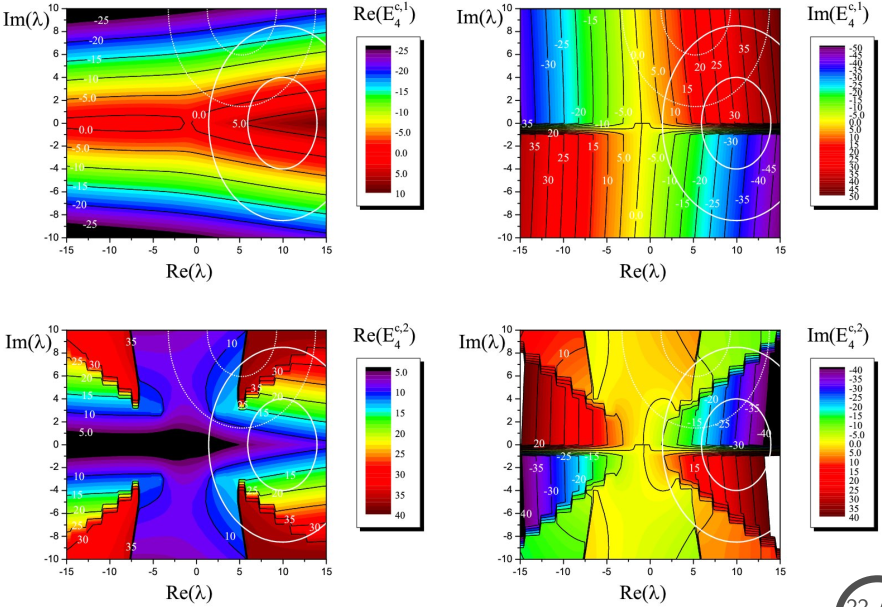


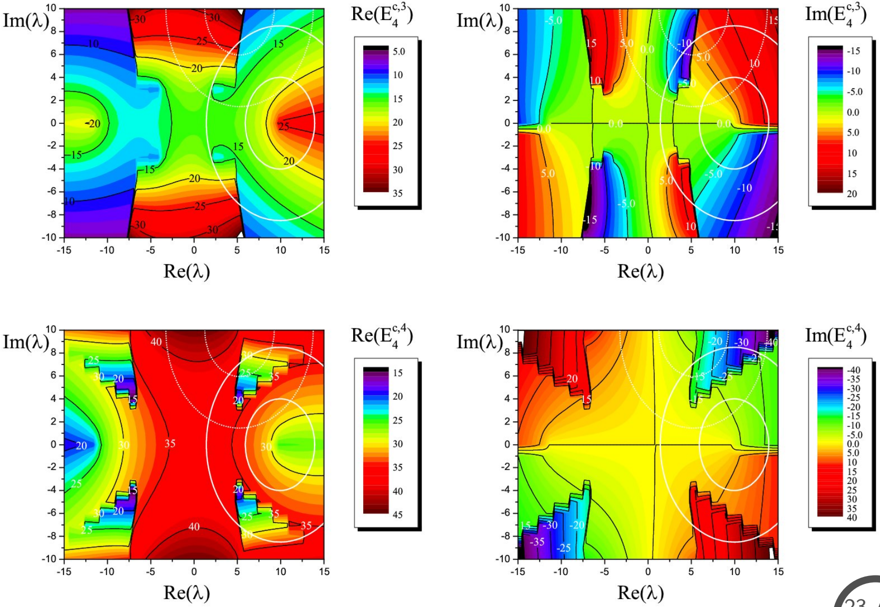
Four energies:
$E_4^{c,1} = E_4^{c,2} = 25.6613,\ E_4^{c,3} = (E_4^{c,4})^* = 7.1029 + i29.8106$



Four energies:
$E_4^{c,1} = E_4^{c,2} = 37.7449 - i8.7611$, $E_4^{c,3} = 9.8103 + i6.7668$,
$E_4^{c,4} = -24.0439 + i20.7081$




Im(λ)
Re(λ)
Re($E_4^{c,1}$)
Im($E_4^{c,1}$)
Re($E_4^{c,2}$)
Im($E_4^{c,2}$)



Re($E_4^{c,3}$)
Im($E_4^{c,3}$)
Re($E_4^{c,4}$)
Im($E_4^{c,4}$)
Im(λ)
Re(λ)

Double scaling limit to $\mathcal{H}_{\text{Mat}}$

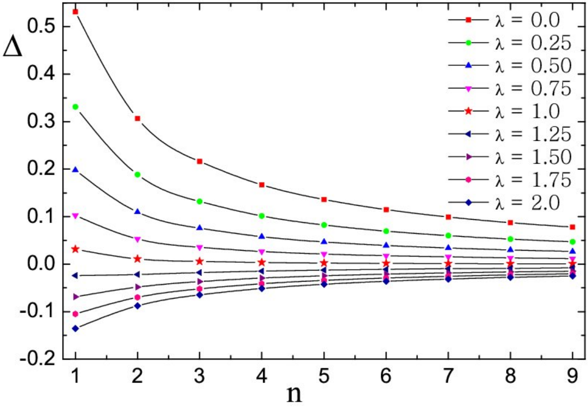
Recall:

$$\lim_{N\to\infty,\zeta\to 0} \mathcal{H}_N = \mathcal{H}_{\text{Mat}}, \qquad \text{for } g := N\zeta < \infty$$

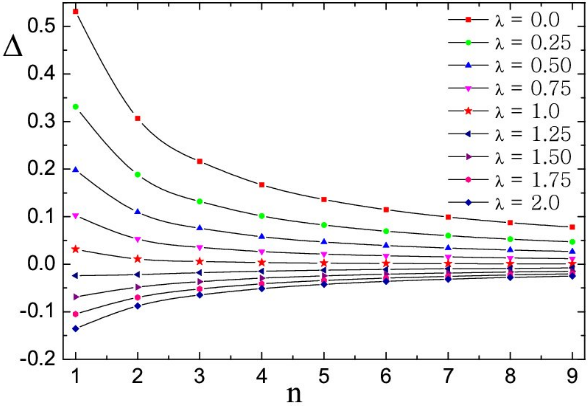
For $\lambda = 1$:

N	$\zeta_0 N$	$\zeta_0 N$	$\zeta_0 N$	$\zeta_0 N$	$\zeta_0 N$
3	1.50000				
5	1.47963	7.50000			
7	1.47426	7.19195	18.4246		
9	1.47208	7.08219	17.5098	34.4001	
11	1.47098	7.02966	17.1292	32.5974	55.4904
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	1.46877	6.92895	16.4711	30.0967	47.806

Which λ is optimal?



$$\Delta(n) = \zeta_0 N(n) - \zeta_M, \; N(n) = (n+1) + n\lambda \text{ for } n = 1, 2, 3, \ldots$$



$\Delta(n) = \zeta_0 N(n) - \zeta_M$, $N(n) = (n+1) + n\lambda$ for $n = 1, 2, 3, \ldots$
The optimal approximation for finite values of N is $\lambda = 1$.

Alternatively take the limit on the recurrence relation.
$N \to \infty$, $\zeta \to 0$, $g := N\zeta < \infty$,
$\lim_{N\to\infty,\zeta\to 0} P_n =: P_n^M$, $\lim_{N\to\infty,\zeta\to 0} Q_n =: Q_n^M$

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We obtain infinite matrices Ξ and Θ with entries

$$\begin{aligned} \Xi_{i,j} &= 4i^2\delta_{i,j} + \frac{1}{2}\delta_{j,i+1} - 2g^2\delta_{i,j+1}, & \text{for } i,j \in \mathbb{N}, \\ \Theta_{i,j} &= 4i^2\delta_{i,j} + \frac{1}{2}\delta_{j,i+1} - 2g^2\delta_{i,j+1} + \frac{1}{2}\delta_{i,0}\delta_{j,1}, & \text{for } i,j \in \mathbb{N}_0, \end{aligned}$$

acting $(Q_1^M, Q_2^M, Q_3^M, \ldots)$, $(P_0^M, P_1^M, P_1^M, \ldots)$

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\end{aligned}
$$

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Exceptional points from truncated matrices with rank ℓ:

$$
\begin{aligned}
\det(\Xi^\ell - E\mathbb{I}) &= 0 \\
\det(\Theta^\ell - E\mathbb{I}) &= 0
\end{aligned}
$$

Real zeros g_0 of the discriminant polynomials $\Delta^{\Theta}(g)$:

ℓ	g_0	g_0	g_0	g_0	g_0	g_0	g_0
2	1.41421						
3	1.46904						
4	1.46877	12.34951					
5	1.46877	17.88618					
6	1.46877	16.44658	24.21371				
7	1.46877	16.47150	29.27154				
8	1.46877	16.47116	34.30396	45.47616			
⋮	⋮	⋮	⋮	⋮			
26	1.46877	16.47117	47.80597	95.47527	125.4485	159.4792	239.8178
27	**1.46877**	**16.47117**	**47.80597**	**95.47527**	130.5181	**159.4792**	**239.8178**
26	240.9227	336.4911	341.4216	427.3330	449.3487	498.9970	
27	251.2637	**336.4911**	357.0076	448.0887	449.5057	525.2659	

Real zeros g_0 of the discriminant polynomials $\Delta^{\Xi}(g)$:

ℓ	g_0	g_0	g_0	g_0	g_0	g_0	g_0
2	6.00000						
3	6.97891						
4	6.92848	18.77091					
5	6.92896	24.29547					
6	6.92895	29.26843	29.73862				
7	6.92895	30.10798	34.30404				
8	6.92895	30.09660	39.34849	61.30789			
⋮	⋮	⋮	⋮	⋮			
26	6.928955	30.09677	69.59879	125.4354	130.5181	197.6067	251.2637
27	**6.928955**	**30.09677**	**69.59879**	**125.4354**	135.5878	**197.6067**	261.6061
26	286.1126	357.0076	390.9532	448.0887	511.0770	525.2021	
27	**286.1126**	372.5999	**390.9532**	468.8640	512.1858	551.0671	

Weakly orthogonal polynomials:

Favard's theorem [Acad. Sci. Paris 200 (1935) 2053]

For any three-term recurrence relation of the form

$$\Phi_{n+1} = (E - a_n)\,\Phi_n - b_n \Phi_{n-1},$$

with $b_n = 0$ for $n \leq 0$ and $b_K = 0$ for some K,

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with $b_n = 0$ for $n \leq 0$ and $b_K = 0$ for some K,
$\exists$ a linear functional $\mathcal{L}$ acting on polynomials p as

$$\mathcal{L}(p) = \int_{-\infty}^{\infty} p(E)\omega(E)dE,$$

such that the polynomials $\Phi_n(E)$ are orthogonal

$$\mathcal{L}(\Phi_n \Phi_m) = \mathcal{L}(E\Phi_n \Phi_{m-1}) = N_n \delta_{nm}.$$

$N_n \equiv$ squared norms of Φ_n
$\omega(E) \equiv$ measure

Norms can be computed in two alternative ways:

i) $\mathcal{L}$ (three-term relation $\times$ Φ_{n-1}):
$N_n^{\Phi} = \mathcal{L}(\Phi_n^2) = \mathcal{L}(E\Phi_{n-1}\Phi_n) = \prod_{k=1}^{n} b_k$

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E_k are the ℓ roots of the polynomial $\Phi(E)$.

Norms for the case at hand:

$$N_n^P = 2\zeta^{2n}(1+\lambda)^{2n} \left(\frac{1-N}{1+\lambda}\right)_n \left(\frac{\lambda+N}{1+\lambda}\right)_n, \qquad n = 1, 2, 3, \ldots$$

$$N_n^Q = \frac{1}{2(N+\lambda)(1-N)} N_n^P, \qquad n = 2, 3, 4, \ldots$$

with $N_0^P = N_1^Q = 1$.

Measures for the case at hand:

For $N = 3 + 2\lambda$:

$$
\begin{aligned}
\omega_1^c &= \frac{1}{3} - \frac{\left(260 - 60\hat{\zeta}^2\right)\Omega + \left(3\hat{\zeta}^2 + 4\right)\Omega^2 + 20\Omega^3}{12\left[\left(13 - 3\hat{\zeta}^2\right)^2 + \left(13 - 3\hat{\zeta}^2\right)\Omega^2 + \Omega^4\right]}, \\
\omega_2^c &= \chi_{-2}, \qquad \omega_3^c = \chi_2, \\
\chi_\ell &= \frac{1}{3} + \frac{\left(3\hat{\zeta}^2 - 20\Omega + 4\right)\left(1 + 2e^{\frac{i\pi\ell}{3}}\right)}{36(3\hat{\zeta}^2 + \Omega^2 - 13)} \\
&\quad + \frac{4 + 3\hat{\zeta}^2 - 20e^{\frac{i\pi\ell}{3}}\Omega}{12\left(1 + 2e^{\frac{i\pi\ell}{3}}\right)\left(3\hat{\zeta}^2 - 13\right) + \left(1 - e^{\frac{i\pi\ell}{3}}\right)\Omega^2}
\end{aligned}
$$

Confirm with in two alternative ways:

$$\begin{aligned} N_0^P &= \mathcal{L}(P_0^2) = \omega_1^c + \omega_2^c + \omega_3^c = 1, \\ N_1^P &= \mathcal{L}(P_1^2) = \omega_1^c P_1^2(E_3^{c,0}) + \omega_2^c P_1^2(E_3^{c,-2}) + \omega_3^c P_1^2(E_3^{c,2}) \\ &= -12\hat{\zeta}^2, \\ N_2^P &= \mathcal{L}(P_2^2) = \omega_1^c P_2^2(E_3^{c,0}) + \omega_2^c P_2^2(E_3^{c,-2}) + \omega_3^c P_2^2(E_3^{c,2}) \\ &= 48\hat{\zeta}^4 \\ \mathcal{L}(P_1 P_2) &= \omega_1^c P_1(E_3^{c,0}) P_2(E_3^{c,0}) + \omega_2^c P_1(E_3^{c,-2}) P_2(E_3^{c,-2}) \\ &\quad + \omega_3^c P_1(E_3^{c,2}) P_2(E_3^{c,2}) = 0. \end{aligned}$$

Similarly we can compute the momentum functionals in two alternative ways.

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Thank you for your attention.