Goldstone's theorem and the Higgs mechanism in non-Abelian non-Hermitian quantum field theories

Andreas Fring

Virtually at University of York, 30^{th} of April 2020

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A. Fring and T. Taira, Nucl. Phys. B 950 (2020) 114834;
A. Fring and T. Taira, Phys. Rev. D 101 (2020) 045014;
A. Fring and T. Taira, arXiv:2004.00723

Outline

- Brief introduction to $\mathcal{PT}$-quantum mechanics

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- Non-Abelian gauge theories and Higgs mechanism
- Conclusions and Outlook

$\mathcal{PT}$-quantum mechanics (real eigenvalues)

- $\mathcal{PT}$-symmetry: $\mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$
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$\mathcal{PT}$-symmetry is only an example of an antilinear involution
[E. Wigner, *J. Math. Phys.* 1 (1960) 409]
[C. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243]

$\mathcal{H}$ is Hermitian with respect to a new metric

- Assume pseudo-Hermiticity:

$$h = \eta \mathcal{H} \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger \mathcal{H}^\dagger \eta^\dagger \quad \Leftrightarrow \quad \mathcal{H}^\dagger \eta^\dagger \eta = \eta^\dagger \eta \mathcal{H}$$

$$\Phi = \eta^{-1}\phi \qquad \eta^\dagger = \eta$$

$\Rightarrow \mathcal{H}$ is Hermitian with respect to the new metric

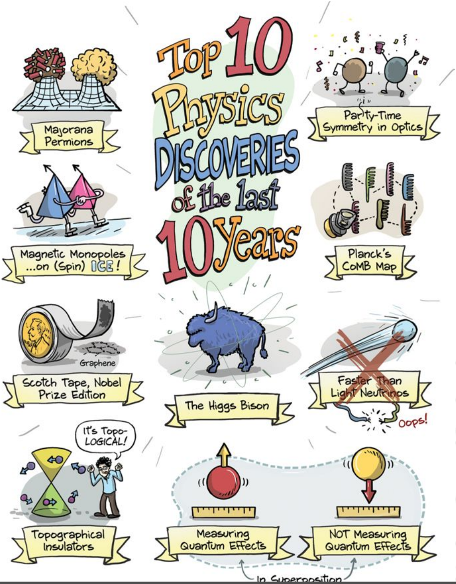
Proof:

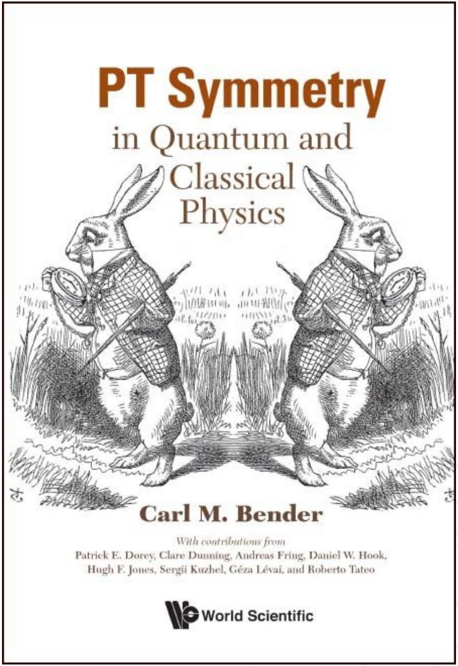
$$\langle \Psi | \mathcal{H}\Phi \rangle_\eta = \langle \Psi | \eta^2 \mathcal{H}\Phi \rangle = \langle \eta^{-1}\psi | \eta^2 \mathcal{H} \eta^{-1} \phi \rangle = \langle \psi | \eta \mathcal{H} \eta^{-1} \phi \rangle =$$
$$\langle \psi | h\phi \rangle = \langle h\psi | \phi \rangle = \langle \eta \mathcal{H} \eta^{-1} \psi | \phi \rangle = \langle \mathcal{H}\Psi | \eta\phi \rangle = \langle \mathcal{H}\Psi | \eta^2 \Phi \rangle$$
$$= \langle \mathcal{H}\Psi | \Phi \rangle_\eta$$

$\Rightarrow$ Eigenvalues of $\mathcal{H}$ are real, eigenstates are orthogonal

Many applications in optics

Nature Physics volume 11, page 799 (2015)




PT Symmetry
in Quantum and
Classical
Physics
Carl M. Bender
With contributions from
Patrick E. Dorey, Clare Dunning, Andreas Fring, Daniel W. Hook,
Hugh F. Jones, Sergii Kuzhel, Géza Lévai, and Roberto Tateo
World Scientific

Problem with non-Hermitain field theory

Consider action of the general form

$$\mathcal{I} = \int d^4x \left[\partial_\mu \phi \partial^\mu \phi^* - V(\phi)\right],$$

complex scalar fields $\phi = (\phi_1, \ldots, \phi_n)$, potential $V(\phi) \neq V^\dagger(\phi)$

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$\mathcal{PT}$-papers in quantum field theory ≈ 60 versus the rest ≈ 4400

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  [J. Alexandre, J. Ellis, P. Millington, D. Seynaeve]
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- Pseudo-Hermitian approach: [A. Fring, T. Taira]

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Key findings:

Goldstone theorem in non-Hermitian field theories

- The GT holds in the $\mathcal{PT}$-symmetric regime
- The GT breaks down in the broken $\mathcal{PT}$ regime
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- The G boson is not identifiable at the zero exceptional point

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Non-Hermitian systems posses intricate physical parameter spaces

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Each generator of a global continuous symmetry group that is broken by the vacuum gives rise to a massless particle.

$$\mathcal{I} = \int d^4x \left[\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi^* - V(\Phi)\right]$$

Vacua Φ_0:

$$\left.\frac{\partial V(\Phi)}{\partial \Phi}\right|_{\Phi=\Phi_0} = 0$$

Symmetry $\Phi \to \Phi + \delta\Phi$: $V(\Phi) = V(\Phi) + \nabla V(\Phi)^T \delta\Phi$,

$$\frac{\partial V(\Phi)}{\partial \Phi_i}\delta\Phi_i(\Phi) = 0$$

Differentiating with respect to Φ_j at a vacuum Φ_0

$$\left.\frac{\partial^2 V(\Phi)}{\partial \Phi_j \partial \Phi_i}\right|_{\Phi=\Phi_0} \delta\Phi_i(\Phi_0) + \left.\frac{\partial V(\Phi)}{\partial \Phi_i}\right|_{\Phi=\Phi_0} \left.\frac{\partial \delta\Phi_i(\Phi)}{\partial \Phi_j}\right|_{\Phi=\Phi_0} = 0$$

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invariant vacuum: $\delta\Phi_i(\Phi_0) = 0 \quad \Rightarrow \quad$ no restriction on M^2

broken vacuum: $\delta\Phi_i(\Phi_0) \neq 0 \quad \Rightarrow \quad M^2$ has zero eigenvalue

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Non-Hermitian version:

$$\hat{\mathcal{I}} = \int d^4x \left[\frac{1}{2}\partial_\mu\Phi\hat{I}\partial^\mu\Phi^* - \hat{V}(\Phi)\right]$$

$$\hat{I}\hat{H}(\Phi_0)\delta\Phi_i(\Phi_0) = \hat{M}^2\delta\Phi_i(\Phi_0) = 0$$

M^2 is no longer Hermitian

A simple model with three complex scalar field:

$$\mathcal{I}_3 = \int d^4x \sum_{i=1}^{3} \partial_\mu \phi_i \partial^\mu \phi_i^* - V_3$$

$$V_3 = -\sum_{i=1}^{3} c_i m_i^2 \phi_i \phi_i^* + c_\mu \mu^2 (\phi_1^* \phi_2 - \phi_2^* \phi_1) + c_\nu \nu^2 (\phi_2 \phi_3^* - \phi_3 \phi_2^*) + \frac{g}{4} (\phi_1 \phi_1^*)^2$$

with $m_i, \mu, \nu, g \in \mathbb{R}$ and $c_i, c_\mu, c_\nu = \pm 1$

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with $m_i, \mu, \nu, g \in \mathbb{R}$ and $c_i, c_\mu, c_\nu = \pm 1$

Three key properties:

- discrete modified $\mathcal{CPT}$-transformations

$$\mathcal{CPT}_1 : \phi_i(x_\mu) \to (-1)^{i+1} \phi_i^*(-x_\mu)$$
$$\mathcal{CPT}_2 : \phi_i(x_\mu) \to (-1)^{i} \phi_i^*(-x_\mu), \qquad i = 1, 2, 3$$

- continuous global $U(1)$-symmetry

$$\phi_i \to e^{i\alpha} \phi_i, \qquad \phi_i^* \to e^{-i\alpha} \phi_i^*, \qquad i = 1, 2, 3, \ \alpha \in \mathbb{R}$$

- non-Hermitian potential $V_3 \neq V_3^\dagger$

(incompatible) equations of motion:

$$\begin{aligned}
\Box\phi_1 - c_1 m_1^2\phi_1 - c_\mu\mu^2\phi_2 + \frac{g}{2}\phi_1^2\phi_1^* &= 0 \\
\Box\phi_2 - c_2 m_2^2\phi_2 + c_\mu\mu^2\phi_1 + c_\nu\nu^2\phi_3 &= 0 \\
\Box\phi_3 - c_3 m_3^2\phi_3 - c_\nu\nu^2\phi_2 &= 0 \\
\Box\phi_1^* - c_1 m_1^2\phi_1^* + c_\mu\mu^2\phi_2^* + \frac{g}{2}\phi_1(\phi_1^*)^2 &= 0 \\
\Box\phi_2^* - c_2 m_2^2\phi_2^* - c_\mu\mu^2\phi_1^* - c_\nu\nu^2\phi_3^* &= 0 \\
\Box\phi_3^* - c_3 m_3^2\phi_3^* + c_\nu\nu^2\phi_2^* &= 0
\end{aligned}$$

This can be fixed with an equal-time similarity transformation:

$$\eta = \exp\left[\frac{\pi}{2}\int d^3x \Pi_2^{\varphi}(\mathbf{x},t)\varphi_2(\mathbf{x},t)\right]\exp\left[\frac{\pi}{2}\int d^3x \Pi_2^{\chi}(\mathbf{x},t)\chi_2(\mathbf{x},t)\right]$$

$$\eta\phi_i\eta^{-1} = (-i)^{\delta_{2i}}\phi_i, \qquad \eta\phi_i^*\eta^{-1} = (-i)^{\delta_{2i}}\phi_i^*$$

Equivalent version ($\hat{\mathcal{I}}_3 = \eta \mathcal{I}_3 \eta^{-1}$) $\phi_i = 1/\sqrt{2}(\varphi_i + i\chi_i)$

$$\hat{\mathcal{I}}_3 = \int d^4x \sum_{i=1}^{3} \frac{1}{2}(-1)^{\delta_{2i}} \left[\partial_\mu \varphi_i \partial^\mu \varphi_i + \partial_\mu \chi_i \partial^\mu \chi_i + c_i m_i^2 \left(\varphi_i^2 + \chi_i^2\right)\right]$$
$$+ c_\mu \mu^2 \left(\varphi_1 \chi_2 - \varphi_2 \chi_1\right) + c_\nu \nu^2 \left(\varphi_3 \chi_2 - \varphi_2 \chi_3\right) - \frac{g}{16}(\varphi_1^2 + \chi_1^2)^2$$

(compatible) equations of motion:

$$\begin{aligned}
-\Box\varphi_1 &= -c_1 m_1^2 \varphi_1 - c_\mu \mu^2 \chi_2 + \frac{g}{4}\varphi_1(\varphi_1^2 + \chi_1^2) \\
-\Box\chi_2 &= -c_2 m_2^2 \chi_2 + c_\mu \mu^2 \varphi_1 + c_\nu \nu^2 \varphi_3 \\
-\Box\varphi_3 &= -c_3 m_3^2 \varphi_3 - c_\nu \nu^2 \chi_2 \\
-\Box\chi_1 &= -c_1 m_1^2 \chi_1 + c_\mu \mu^2 \varphi_2 + \frac{g}{4}\chi_1(\varphi_1^2 + \chi_1^2) \\
-\Box\varphi_2 &= -c_2 m_2^2 \varphi_2 - c_\mu \mu^2 \chi_1 - c_\nu \nu^2 \chi_3 \\
-\Box\chi_3 &= -c_3 m_3^2 \chi_3 + c_\nu \nu^2 \varphi_2
\end{aligned}$$

Hessian matrix $H\left(\Phi = (\varphi_1, \chi_2, \varphi_3, \chi_1, \varphi_2, \chi_3)^T\right)$:

$$\begin{pmatrix} \frac{g(3\varphi_1^2+\chi_1^2)}{4} - c_1 m_1^2 & -c_\mu \mu^2 & 0 & \frac{g}{2}\varphi_1\chi_1 & 0 & 0 \\ -c_\mu\mu^2 & c_2 m_2^2 & -c_\nu \nu^2 & 0 & 0 & 0 \\ 0 & -c_\nu\nu^2 & -c_3 m_3^2 & 0 & 0 & 0 \\ \frac{g}{2}\varphi_1\chi_1 & 0 & 0 & \frac{g(\varphi_1^2+3\chi_1^2)}{4} - c_1 m_1^2 & c_\mu\mu^2 & 0 \\ 0 & 0 & 0 & c_\mu\mu^2 & c_2 m_2^2 & c_\nu\nu^2 \\ 0 & 0 & 0 & 0 & c_\nu\nu^2 & -c_3 m_3^2 \end{pmatrix}$$

Hessian matrix $H\left(\Phi=(\varphi_1,\chi_2,\varphi_3,\chi_1,\varphi_2,\chi_3)^T\right)$:

$$\begin{pmatrix} \frac{g(3\varphi_1^2+\chi_1^2)}{4}-c_1m_1^2 & -c_\mu\mu^2 & 0 & \frac{g}{2}\varphi_1\chi_1 & 0 & 0 \\ -c_\mu\mu^2 & c_2m_2^2 & -c_\nu\nu^2 & 0 & 0 & 0 \\ 0 & -c_\nu\nu^2 & -c_3m_3^2 & 0 & 0 & 0 \\ \frac{g}{2}\varphi_1\chi_1 & 0 & 0 & \frac{g(\varphi_1^2+3\chi_1^2)}{4}-c_1m_1^2 & c_\mu\mu^2 & 0 \\ 0 & 0 & 0 & c_\mu\mu^2 & c_2m_2^2 & c_\nu\nu^2 \\ 0 & 0 & 0 & 0 & c_\nu\nu^2 & -c_3m_3^2 \end{pmatrix}$$

No Goldstone bosons for $U(1)$-invariant vacuum (no zero EV of M^2)

$$\Phi_s^0=(0,0,0,0,0,0)$$

Hessian matrix $H\left(\Phi = (\varphi_1, \chi_2, \varphi_3, \chi_1, \varphi_2, \chi_3)^T\right)$:

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No Goldstone bosons for $U(1)$-invariant vacuum (no zero EV of M^2)

$$\Phi_s^0 = (0,0,0,0,0,0)$$

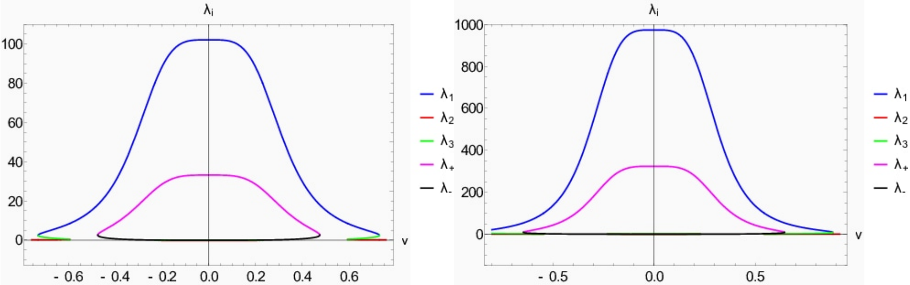
One Goldstone bosons for $U(1)$-broken vacuum (one zero EV of M^2)

$$\Phi_b^0 = \left(\varphi_1^0, \frac{c_3 c_\mu m_3^2 \mu^2 \varphi_1^0}{\kappa}, -\frac{c_\nu c_\mu \nu^2 \mu^2 \varphi_1^0}{\kappa}, -K(\varphi_1^0), \frac{c_3 c_\mu m_3^2 \mu^2 K(\varphi_1^0)}{\kappa}, \frac{c_\nu c_\mu \nu^2 \mu^2 K(\varphi_1^0)}{\kappa}\right)$$

with $K(x) := \pm\sqrt{\frac{4c_3 m_3^2 \mu^4}{g\kappa} + \frac{4c_1 m_1^2}{g} - x^2}$, $\quad \kappa := c_2 c_3 m_2^2 m_3^2 + \nu^4$

Physical parameter space (Eigenvalue spectra of M^2)

The physical parameter space is bounded by exceptional points, zero exceptional points and singularities



$c_1 = c_2 = c_3 = 1$, $m_1 = 1$, $m_2 = 1/2$ and $m_3 = 1/5$

left panel: $\mu = 1.7$ no physical region
right panel: $\mu = 3$ physical regions $\nu \in (\pm 0.64468, \pm 0.54490)$

Identification of the Goldstone boson field

Diagonalisation of M^2:

$$\hat{\Phi}_r^T (M_2^2)_r \hat{\Phi}_r = \sum_{k=0,\pm} m_k^2 \psi_k^2 = \sum_{k=0,\pm} m_k^2 (\hat{\Phi}_r^T I U)_k (U^{-1}\Phi_r)_k$$

- $\mathcal{PT}$- symmetric regime ($U = (v_0, v_+, v_-)$)

$$\psi_{\text{Gb}}^{\mathcal{PT}} = \frac{1}{\sqrt{N}} \left(-\kappa \hat{\chi}_1 - c_3 c_\mu m_3^2 \mu^2 \hat{\varphi}_2 + c_\mu c_\nu \mu^2 \nu^2 \hat{\chi}_3 \right)$$

- standard exceptional point (bring into Jordan form)

$$\psi_{\text{Gb}}^{\text{e}} = \frac{1}{\kappa c_3 m_3^2 \lambda_{\text{e}}^2} \left(-\kappa \hat{\chi}_1 - m_3 \mu_{\text{e}}^2 \hat{\varphi}_2 + \nu^2 \mu_{\text{e}}^2 \hat{\chi}_3 \right)$$

- zero exceptional point
  The identification is not possible $\rightarrow$ restrict parameter space?

Non-Abelian, non-Hermitian version of the Goldstone Theorem

A simple model with two complex scalar fields

$$\mathcal{L}_{su2} = \sum_{i=1}^{2} \left(|\partial_\mu \phi_i|^2 + m_i^2 |\phi_i|^2 \right) - \mu^2 \left(\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1 \right) - \frac{g}{4} |\phi_1|^4$$

with fields ϕ_i in the fundamental representation of $SU(2)$

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Similarity transformed version:

$$\hat{\mathcal{L}}_{su2} = \partial_\mu F \hat{I} \partial^\mu F + \frac{1}{2} F^T \hat{H} F - \frac{g}{16} \left(F^T \hat{E} F \right)^2$$

where

$$H_\pm = \begin{pmatrix} m_1^2 & \pm\mu^2 \\ \pm\mu^2 & m_2^2 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Phi_j = \begin{pmatrix} \varphi_1^j \\ \chi_2^j \end{pmatrix}, \quad \Psi_j = \begin{pmatrix} \chi_1^j \\ \varphi_2^j \end{pmatrix}, \quad \phi_j^k = \frac{1}{\sqrt{2}} (\varphi_j^k + i \chi_j^k)$$

$F = (\Phi, \Psi) = (\varphi_1^1, \chi_2^1, \varphi_1^2, \chi_2^2, \chi_1^1, \varphi_2^1, \chi_1^2, \varphi_2^2)$, $\Phi = (\Phi_1, \Phi_2)$, $\Psi = (\Psi_1, \Psi_2)$, $\operatorname{diag} \hat{I} = \{I, I, I, I\}$, $\operatorname{diag} \hat{H} = \{H_+, H_+, H_-, H_-\}$, $\operatorname{diag} \hat{E} = \{E, E, E, E\}$.

Continuous global and discrete antilinar symmetries

$SU(2)$-symmetry: $\delta\phi_j^k = i\alpha_a T_a^{kl}\phi_j^l$, with $T_a = \sigma_a$

$$\delta\Phi = -\alpha_1(\sigma_1\otimes\sigma_3)\Psi + i\alpha_2(\sigma_2\otimes\mathbb{I})\Phi - \alpha_3(\sigma_3\otimes\sigma_3)\Psi$$
$$\delta\Psi = \alpha_1(\sigma_1\otimes\sigma_3)\Phi + i\alpha_2(\sigma_2\otimes\mathbb{I})\Psi + \alpha_3(\sigma_3\otimes\sigma_3)\Phi$$
$$\delta F = i\left[-\alpha_1(\sigma_2\otimes\sigma_1\otimes\sigma_3) + \alpha_2(\mathbb{I}\otimes\sigma_2\otimes\mathbb{I}) - \alpha_3(\sigma_2\otimes\sigma_3\otimes\sigma_3)\right]F$$

$\mathcal{CPT}_{\pm}$-symmetry:

Continuous global and discrete antilinar symmetries

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\delta\Phi &= -\alpha_1 (\sigma_1 \otimes \sigma_3)\Psi + i\alpha_2 (\sigma_2 \otimes \mathbb{I})\Phi - \alpha_3 (\sigma_3 \otimes \sigma_3)\Psi \\
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\delta F &= i\left[-\alpha_1 (\sigma_2 \otimes \sigma_1 \otimes \sigma_3) + \alpha_2 (\mathbb{I} \otimes \sigma_2 \otimes \mathbb{I}) - \alpha_3 (\sigma_2 \otimes \sigma_3 \otimes \sigma_3)\right] F
\end{aligned}
$$

$\mathcal{CPT}_{\pm}$-symmetry:

$$
\begin{aligned}
&\Phi(x_\mu) \to \pm\Phi(-x_\mu), \quad \Psi(x_\mu) \to \mp\Psi(-x_\mu), \\
&F(x_\mu) \to \pm(\sigma_3 \otimes \mathbb{I} \otimes \mathbb{I})\, F(-x_\mu),
\end{aligned}
$$

No Goldstone bosons for $SU(2)$-symmetry invariant vacuum

$$F_0^s = (0,0,0,0,0,0,0,0)$$

$$M_s^2 = \begin{pmatrix} -m_1^2 & \mu^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mu^2 & -m_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -m_1^2 & \mu^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu^2 & -m_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_1^2 & -\mu^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^2 & -m_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -m_1^2 & -\mu^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu^2 & -m_2^2 \end{pmatrix},$$

two fourfold degenerate eigenvalues

$$\lambda_\pm^s = -\frac{1}{2}\left(m_1^2 + m_2^2 \pm \sqrt{(m_1^2 - m_2^2)^2 - 4\mu^4}\right)$$

Goldstone bosons for the $SU(2)$-symmetry breaking vacuum

$$F_0^b = (x, -ax, y, -ay, z, az, \pm R, \pm aR)$$

$x := \varphi_1^{0,1}$, $y := \varphi_1^{0,2}$, $z := \chi_1^{0,1}$, $r := 4\left(\mu^2 + m_1^2 m_2^2\right)/gm_2^2$, $a := \mu^2/m_2^2$, $R := \sqrt{r^2 - (x^2 + y^2 + z^2)}$,

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$a := \mu^2/m_2^2$, $R := \sqrt{r^2 - (x^2 + y^2 + z^2)}$, $X := \frac{gx^2}{2} + \frac{\mu^4}{m_2^2}$

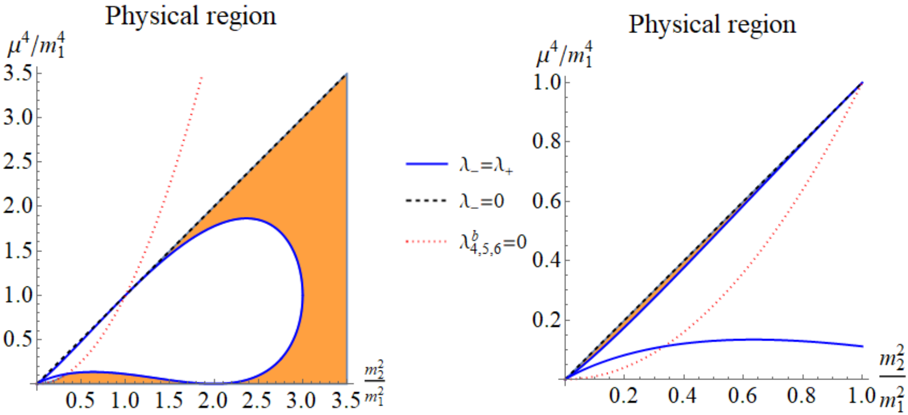
$$M_b^2 = \begin{pmatrix} X & \mu^2 & \frac{gx\varphi_1^2}{2} & 0 & \frac{gxz}{2} & 0 & -\frac{xgR}{2} & 0 \\ -\mu^2 & -m_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{gxy}{2} & 0 & X & \mu^2 & \frac{gyz}{2} & 0 & -\frac{ygR}{2} & 0 \\ 0 & 0 & -\mu^2 & -m_2^2 & 0 & 0 & 0 & 0 \\ \frac{gxz}{2} & 0 & \frac{gyz}{2} & 0 & X & -\mu^2 & -\frac{zgR}{2} & 0 \\ 0 & 0 & 0 & 0 & \mu^2 & -m_2^2 & 0 & 0 \\ -\frac{xgR}{2} & 0 & -\frac{xgR}{2} & 0 & -\frac{zgR}{2} & 0 & X & -\mu^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu^2 & -m_2^2 \end{pmatrix}$$

eigenvalues: $(K := 3\mu^4/2m_2^2 + m_1^2 - m_2^2/2,\ L := \mu^4 + m_1^2 m_2^2)$

$$\lambda_{1,2,3}^b = 0, \quad \lambda_{4,5,6}^b = \frac{\mu^4}{m_2^2} - m_2^2, \quad \lambda_\pm^b = K \pm \sqrt{K^2 + 2L}$$

Physical Regions

Now take $m_i^2 \to c_i m_i^2$ with $c_i = \pm 1$



left panel: $c_1 = -c_2 = 1$, right panel $c_1 = -c_2 = -1$
no physical regime for $c_1 = c_2 = \pm 1$

Goldstone bosons

$\mathcal{PT}$-symmetric regime:

mass squared term:

$$F^T M_b^2 F = \sum_{k=1}^{8} m_k^2 \psi_k^2 = \sum_{k=1}^{8} m_k^2 (F^T IU)_k (U^{-1}F)_k.$$

Hence

$$\psi_\ell^{\text{Gb}} := \sqrt{(F^T IU)_\ell (U^{-1}F)_\ell}, \quad \ell = 1, 3, 5$$

$$\psi_1^{\text{Gb}} = \frac{\mu^2 \varphi_2^1 - m_2^2 \chi_1^1}{\sqrt{m_2^4 - \mu^4}}, \;\; \psi_3^{\text{Gb}} = \frac{m_2^2 \varphi_1^2 + \mu^2 \chi_2^2}{\sqrt{m_2^4 - \mu^4}}, \;\; \psi_5^{\text{Gb}} = \frac{m_2^2 \varphi_1^1 + \mu^2 \chi_2^1}{\sqrt{m_2^4 - \mu^4}}$$

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standard exceptional points:

same form, but identified using Jordan normal form

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standard exceptional points:

same form, but identified using Jordan normal form

zero exceptional points:

identification is not possible

Gauged model in $SU(2)$-fundamental representation

$$\mathcal{L}_2 = \sum_{i=1}^{2} |D_\mu \phi_i|^2 + m_i^2 |\phi_i|^2 - \mu^2(\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1) - \frac{g}{4}(|\phi_1|^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$D_\mu := \partial_\mu - ieA_\mu,\ A_\mu := \tau^a A_\mu^a,\ F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]$$

The gauge vector boson acquires a mass: $m_{gb} := \frac{eR}{m_2^2}\sqrt{m_2^4 - \mu^4}$

$$e^2 (A_\mu \Psi_0)^* \mathcal{I} (A^\mu \Psi_0) = m_{gb}^2 A_\mu^a A^{a\mu},$$

combined with the "would be Goldstone bosons":

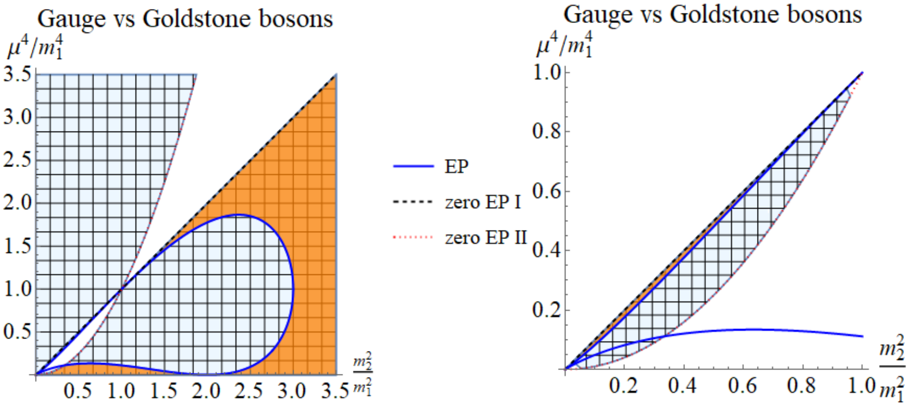
$$G^1 = \frac{e}{m_{gb}} \left(\Psi_0^2\right)^T \Phi^1,\ G^3 = \frac{e}{m_{gb}} \left(\Psi_0^2\right)^T \Phi^2,\ G^2 = -\frac{e}{m_{gb}} \left(\Psi_0^2\right)^T \mathcal{I}\Psi^1$$

$$\sum_{a=1}^{3} \frac{1}{2}\partial_\mu G^a \partial^\mu G^a - m_g A_\mu^1 \partial^\mu G^1 + m_g A_\mu^2 \partial^\mu G^2 - m_g A_\mu^3 \partial^\mu G^3 + \frac{1}{2} m_g^2 A_\mu^a A^{a\mu}$$

$$= \frac{1}{2} \sum_{a=1}^{3} m_g^2 B_\mu^a B^{a\mu} \qquad B_\mu^a := A_\mu^a - \frac{1}{m_g}\partial_\mu G^a$$

$\Rightarrow$ massive gauge vector bosons $\Longleftrightarrow$ vanishing Goldstone bosons

Massive gauge vector bosons versus massless Goldstone bosons



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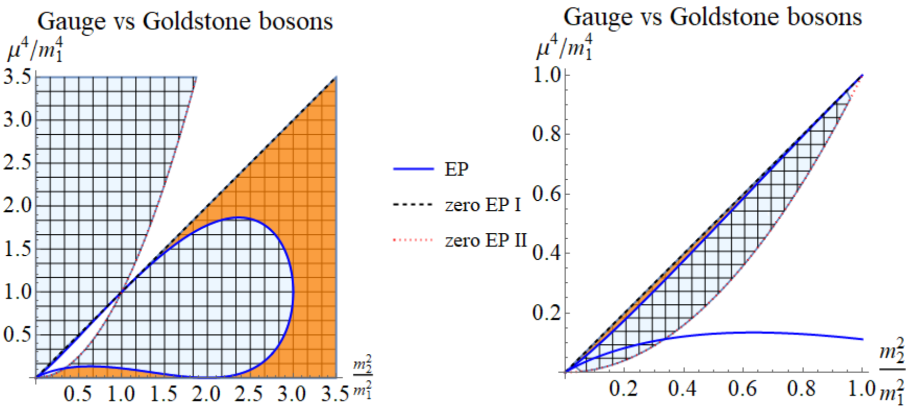

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	$\mathcal{CPT}$	broken $\mathcal{CPT}$	EP	zero EP I	zero EP II
gauge b.	massive	massive	massive	massless	massless
Gold. b.	$\exists$	$\exists$	$\exists$	$\nexists$	$\nexists$

Conclusions, work in progress, future work

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