CHAPTER 4

Categories, functors, and equivalences

In this chapter we will introduce the basic language of categories. This will allow us to make more precise the notion of two algebras having the same representation theory, which we have already appealed to on several occasions.

4.1. Categories

We have already considered classes of objects and the morphisms between them on a number of occasions. The notion of a category abstracts this idea.

DEFINITION 4.1.1. *A* category $\mathscr{C}$ *is made up a pair:* $\mathrm{Ob}(\mathscr{C})$, *a class of* objects *and* $\mathrm{Hom}_{\mathscr{C}}$, *a class of* morphisms. *Each morphism f is associated to a unique pair of objects (a,b) where a is the* source *of f and b is the* target *of f. Usually we write this as $f : a \to b$ or*

$$a \xrightarrow{f} b\ .$$

The class of all morphisms from a to b is denote $\mathrm{Hom}_{\mathscr{C}}(a,b)$, *or just* $\mathrm{Hom}(a,b)$ *if the category is clear. For each triple of objects a,b,c there exists a binary operation*

$$\mathrm{Hom}_{\mathscr{C}}(a,b)\times\mathrm{Hom}_{\mathscr{C}}(b,c)\to\mathrm{Hom}_{\mathscr{C}}(a,c)$$

which we call composition *which takes the pair of morphisms (f,g) to the morphism denoted $g\circ f$. To be a category, the following pair of conditions must hold:*

(1) *(Associativity): If $f : a \to b$, $g : b \to c$ and $h : c \to d$ are three morphisms then*

$$h\circ(g\circ f) = (h\circ g)\circ f.$$

(2) *(Identity): For each $a \in \mathrm{Ob}(\mathscr{C})$ there exists a morphism $\mathrm{id}_a \in \mathrm{Hom}_{\mathscr{C}}(a,a)$ such that for every pair of objects a and b and morphism $f :\in \mathrm{Hom}_{\mathscr{C}}(a,b)$ we have*

$$\mathrm{id}_b\circ f = f = f\circ\mathrm{id}_a\,.$$

Many authors will refer to morphisms as *arrows* or *maps*; we will avoid the former as we have already used this terminology for quivers.

We have used the word “class” rather than “set” to avoid worrying about whether we are working in a setting where our functions form a set. Abstract category theory is closely bound up with set theory, and can involve the same subtle problems about sets versus classes. We wish to avoid worrying about such things.

DEFINITION 4.1.2. *A* small category *is a category where* $\mathrm{Ob}(\mathscr{C})$ *and* $\mathrm{Hom}_{\mathscr{C}}$ *are sets. A category which is not small is called a* large *category.*

To avoid having large categories we often assume that we have some universe U, and describe an object as small if it is a member of this universe. Then the class of objects is actually a set. If we talk about "the (small) category of X" we mean that all objects in X belong to some fixed universe U. Sometimes the word "small" is omitted, and it is assumed that everything happens inside some universe U. If we want to refer to only those objects which lie inside U we call these the small objects of that type.

EXAMPLE 4.1.3. *Let **SET** denote the large category of Sets. Here* $\mathrm{Ob}(\boldsymbol{SET})$ *is the class of all sets, and* $\mathrm{Hom}_{\boldsymbol{SET}}$ *is the class of all functions between sets. This is the standard example which illustrates why classes are necessary, as it is well-known that the set of all sets cannot exist (by Russell's paradox). If we fix some universe U then **Set** denotes the category whose objects are all sets contained in U, and whose morphisms are all functions between such sets.*

Many categories can be obtained by taking a subset of the objects in **Set** which have some extra properties, and then restricting to those morphisms which preserve such properties. Here are some examples.

EXAMPLE 4.1.4. *(a) The category of all small groups **Grp** with morphisms given by group homomorphisms.*
*(b) The category of all small vector spaces **Vect** with morphisms given by linear maps.*
*(c) The category **Top** of small topological spaces with morphisms given by continuous maps.*

One can also construct purely abstract examples of categories by picking a set to call the objects, and another to call morphisms, and then defining composition of the morphisms in such a way that it satisfies the axioms. This is closer to what we did when we considered quivers in Chapter 1.

EXAMPLE 4.1.5. *A* group *can be considered as a category with one object* $*$. *Here the elements of the group are precisely the set of morphism from* $*$ *to itself, and composition of morphisms is given by multiplication in the group. The group axioms now imply that this is a category. Note that in a group every morphism has an inverse. If we consider a category with one object with identity morphism but without requiring that maps are invertible we get a* monoid*. On the other hand, if we keep invertibility but allow several objects instead then we get a* groupoid*.*

Notice that when considering a category both the objects *and* the morphisms are crucial. The same set of objects can lie in two quite different categories; for example we can consider the small set of groups but with morphisms all functions between the underlying sets, or two different groups both coming from categories with only one object. Thus when we work with categories we are interested in both objects and morphisms equally, and so a function from one category to another should respect both levels of structure. These will be the functors which we introduce next.

However, before we do this, we note that algebras and quivers give rise to categories.

EXAMPLE 4.1.6. *Given an algebra A we denote by A-mod the category of all (small) left A-modules, with morphisms the set of module homomorphisms. There is a similar version for right A-modules which is denoted mod-A. Similarly, for a quiver Q we denote by* $kQ-\mod$ *the category of all (small) representations of Q.*

Some people distinguish between A-mod and A-Mod, where the former denotes only the category of *finite dimensional* modules. However this is not particularly standard, and there are other notations (such as A-fdmod) that are also used.

4.2. Functors

As has already been indicated in the previous section, a functor is just a morphism of categories. However, we will need to be quite careful to unpack exactly what that means. We have two kinds of structure in a category, the class of objects, and the class of morphisms together with a composition rule. A functor should preserve any relationships in such a structure.

DEFINITION 4.2.1. *A* functor $F : \mathscr{C} \to \mathscr{D}$ *between two categories* $\mathscr{C}$ *and* $\mathscr{D}$ *is a pair of functions (both denoted* F*) called the* object *and* morphism *functions, where for each object* $a \in \mathscr{C}$ *the object function associates to it an object* $F(a)$ *in* $\mathscr{D}$*, and for each morphism* $f : a \to b$ *in* $\mathscr{C}$*, the morphism functor associates a corresponding morphism*

$$F(f) : F(a) \to F(b) \tag{2}$$

in $\mathscr{D}$*, such that*

$$F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$$

for all $a \in \mathrm{Ob}(\mathscr{C})$ *and*

$$F(g \circ f) = F(g) \circ F(f) \tag{3}$$

for all $f, g \in \mathrm{Hom}_{\mathscr{C}}$ *for which the composite* gf *is defined.*

It is quite common to want to modify the definition of a category, to allow for "functors" which reverse the direction of a morphism. For this reason the above is sometimes called a *covariant* functor, and a *contravariant* functor is then defined by replacing (2) by

$$F(f) : F(b) \to F(a)$$

and (3) by the condition that

$$F(g \circ f) = F(f) \circ F(g).$$

EXAMPLE 4.2.2. *Given any of the small categories in Example 4.1.4, which we will denote by* $\mathscr{C}$*, there is a functor from* $\mathscr{C}$ *to* **Set** *called the* forgetful functor*. This is the functor that takes an object in* $\mathscr{C}$ *(be it a group, a vector space, or a topological space) to the set of elements of that object. Any morphism in* $\mathscr{C}$ *is mapped to the same morphism (which is still a map between sets). Any functor which 'forgets structure' is called a forgetful functor; for example the functor from A-mod to* ***Vect*** *which forgets the module structure on a vector space.*

EXAMPLE 4.2.3. *The* power set *functor from* ***Set*** *to* ***Set*** *takes each set to its power set, and a function* $f : X \to Y$ *between sets is taken to the function which takes* $U \subseteq X$ *to* $f(U) \subseteq Y$*.*

Contravariant functors often occur when there is some kind of duality in the picture.

EXAMPLE 4.2.4. *Let* $\mathbf{Vect}_k$ *denote the category of small vector spaces over some fixed field* k*. The map which takes each vector space* V *to its dual* $V^* = \mathrm{Hom}(V, k)$ *and each linear map to its dual is a contravariant functor from* $\mathbf{Vect}_k$ *to* $\mathbf{Vect}_k$*.*

We want to treat functors as the morphisms that hold between categories. Just as for other kinds of morphisms we can compose functors.

DEFINITION 4.2.5. *Given* $F : \mathscr{C} \to \mathscr{D}$ *and* $G : \mathscr{D} \to \mathscr{E}$ *there is a composite functor* $GF : \mathscr{C} \to \mathscr{E}$ *which is given on objects by* $GF(a) = G(F(a))$ *and on morphisms by* $GF(f) = G(F(f))$.

There is an identity functor $\mathrm{Id}_{\mathscr{C}}$ from each category $\mathscr{C}$ to itself, which acts as the identity on objects and on morphisms. Thus we can consider the category **Cat** of all small categories, whose morphisms are the functors between categories.

DEFINITION 4.2.6. *A functor* $F : \mathscr{C} \to \mathscr{D}$ *is an* isomorphism *if is bijective on both objects and morphisms. Equivalently,* F *is an isomorphism if there is a functor* $G : \mathscr{D} \to \mathscr{C}$ *for which* $GF = \mathrm{Id}_{\mathscr{C}}$ *and* $FG = \mathrm{Id}_{\mathscr{D}}$. *In this case we say that the two categories are* isomorphic.

You might expect that the notion of isomorphic categories would be the natural way in which to consider two categories as being the same. However, we will see that this is too restrictive a notion, and instead introduce a weaker notion of an equivalence of categories.

For functions we have notions of injectivity and surjectivity. For functors things are a little more complicated.

DEFINITION 4.2.7. *(a) A functor* $F : \mathscr{C} \to \mathscr{D}$ *is* full *if for every pair* $a, b \in \mathrm{Ob}(\mathscr{C})$ *and every function* $f : F(a) \to F(b)$ *in* $\mathscr{D}$ *there exists a function* $g : a \to b$ *in* $\mathscr{C}$ *with* $F(g) = f$. *In other words, when a full functor maps a pair of objects into a new category, they do not gain any more functions between them than they had in the old category.*
(b) A functor $F : \mathscr{C} \to \mathscr{D}$ *is* faithful *(or an* embedding*) if for every pair* $a, b \in \mathrm{Ob}(\mathscr{C})$ *and every pair of morphisms* $f, g : a \to b$ *in* $\mathscr{C}$, *the images of the morphisms* f *and* g *under* F *are distinct.*
(c) A functor is fully faithful *if it is full and faithful.*

It is easy to see that the composite of two full functors is full, and of two faithful functors is faithful.

EXAMPLE 4.2.8. *Consider the forgetful functor from* ***Grp*** *to* ***Set***. *This is faithful, as the equality of two morphisms is determined by their action as maps of sets, but is not full, as there are fewer group morphisms than set morphisms in general.*

If a functor $F : \mathscr{C} \to \mathscr{D}$ is is fully faithful then it induces bijections between $\mathrm{Hom}_{\mathscr{C}}(a, b)$ and $\mathrm{Hom}_{\mathscr{D}}(F(a), F(b))$. However, the categories are not necessarily isomorphic as F may not be surjective on $\mathrm{Ob}(\mathscr{D})$.

DEFINITION 4.2.9. *A* subcategory $\mathscr{D}$ *of a category* $\mathscr{C}$ *is a pair:* $\mathrm{Ob}(\mathscr{D}) \subseteq \mathrm{Ob}(\mathscr{C})$ *a class of objects and* $\mathrm{Hom}_{\mathscr{D}} \subseteq \mathrm{Hom}_{\mathscr{C}}$ *a class of morphisms, such that*

(1) *If* $f \in \mathrm{Hom}_{\mathscr{D}}$ *and* $f : a \to b$ *in* $\mathscr{C}$ *then* $a, b \in \mathrm{Ob}(\mathscr{D})$.
(2) *For all* $a \in \mathrm{Ob}(\mathscr{D})$, $\mathrm{id}_a \in \mathrm{Hom}_{\mathscr{C}}$ *is an element of* $\mathrm{Hom}_{\mathscr{D}}$.
(3) *If* f *and* g *in* $\mathrm{Hom}_{\mathscr{D}}$ *are composable in* $\mathscr{C}$ *then their composite is an element of* $\mathrm{Hom}_{\mathscr{D}}$.

The obvious injection from $\mathscr{D}$ *to* $\mathscr{C}$ *is a faithful functor, called the* inclusion functor. *If this functor is full then we say that* $\mathscr{D}$ *is a* full *subcategory of* $\mathscr{C}$.

Notice that a full subcategory is determined by the set of objects it contains, as then the morphisms are precisely the morphisms between such objects in the original category.

EXAMPLE 4.2.10. *(a) The categories **Grp** and **Vect** are subcategories of the category **Set**. In neither case are these full subcategories, as there are set morphisms that are not group homomorphisms/linear maps.*
*(b) The category **Abn** of abelian groups is a full subcategory of the category **Grp**.*

4.3. Equivalences of categories

Consider the (small) category **FinSet** of finite sets. We can define a second category **FinOrd** of all finite ordinals, which is the full subcategory of **FinSet** with objects those sets of the form $n = \{0,1,2,n-1\}$ with $n \in \mathbb{N}$. Clearly there is a functor

$$F : \textbf{FinOrd} \to \textbf{FinSet}$$

given by inclusion. Given a finite set X, there is some n with $|X| = n$, and so we can chose a bijection $i_X : X \to n$. Let us define a functor

$$G : \textbf{FinSet} \to \textbf{FinOrd}$$

by setting $G(X) = |X|$ for $X \in \mathrm{Ob}(\textbf{FinSet})$, and for $f : X \to Y$ setting $G(f) = i_Y \circ f \circ (i_X)^{-1}$. (It is easy to check that this is indeed a functor.)

These two categories are not the same, but one might regard them as sharing the same fundamental properties, with the extra complexity in **FinSet** being caused by having extra but isomorphic objects. We have

$$GF = \mathrm{Id}_{\textbf{FinOrd}}$$

but FG is not the identity functor, as it maps each set of size n in **FinSet** to a single set of that cardinality. We do however have that for every $f : X \to Y$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{i_X} & FG(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle FG(f)} \\ Y & \xrightarrow{i_Y} & FG(Y) \end{array} \tag{4}$$

commutes. This diagram can be regarded as illustrating a nice relation between the action of the identity functor $\mathrm{Id}_{\textbf{FinSet}}$ (the first column) and that of the functor FG (the second column). This example should be regarded as a motivating example for the notion of *equivalence* of categories which we are about to introduce.

The definition of equivalence is quite complicated, and so we will proceed in stages. First, we want to formalise the notion illustrated by the diagram in (4).

DEFINITION 4.3.1. *Given (covariant) functors F and G from $\mathscr{C}$ to $\mathscr{D}$, a* natural transformation

$$\theta : F \to G$$

is a function θ which assigns to each object $a \in \mathrm{Ob}(\mathscr{C})$ a morphism $\theta_a : F(a) \to G(a)$ in such a way that each morphism $f : a \to b$ in $\mathscr{C}$ gives rise to a commutative diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{\theta_a} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow{\theta_b} & G(b). \end{array} \tag{5}$$

If for each $a \in \mathrm{Ob}(\mathscr{C})$ the morphism θ_a is an isomorphism in $\mathscr{D}$ then we call θ a natural equivalence *or* natural isomorphism *and we may write $\theta : F \cong G$.*

For contravariant functors we reverse the direction of the arrows labelled $F(f)$ and $G(f)$ in the diagram.

One way to think of a natural transformation is as a *morphism of functors*. It maps one functor to another while preserving the composition of morphisms in the underlying category.

EXAMPLE 4.3.2. *Consider the category of commutative rings* ***CommRng****. We can define two different functors from this category to* ***Grp****. First, let GL_n be the functor which assigns to each commutative ring R the group of $n \times n$ invertible matrices over R. A ring homomorphism f becomes a function $GL_n(f)$ on the matrix entries.*

Alternatively, consider the functor $-^\times$ which assigns to each commutative ring R the group of invertible elements $R^\times$. This is a functor as ring homomorphisms take invertible elements to invertible elements.

When a matrix M over R is invertible the determinant $\det M$ is a unit in R. As the formula for the determinant of a matrix does not depend on the underlying ring R, the function $\det_R$ from $GL_n(R)$ to $R^\times$ leads to a commutative diagram

$$\begin{array}{ccc} GL_n(R) & \xrightarrow{\det_R} & R^\times \\ {\scriptstyle GL_n(f)}\downarrow & & \downarrow{\scriptstyle f} \\ GL_n(S) & \xrightarrow{\det_S} & S^\times \end{array}$$

This shows that det *defines a natural transformation between GL_n and $-^\times$.*

EXAMPLE 4.3.3. *Given a finite dimensional vector space V, we can consider the dual vector space V^* and the double dual V^{**}. It turns out that there is a natural isomorphism from V to V^{**} but no such natural isomorphism from V to V^*. This is because any choice of a map from V to V^* depends on a choice of basis, while a map F from V to V^{**} can be defined by setting $F(v)$ to be the linear function $F(v) = v^{**}$ on V^* where $v^{**}(f) = f(v)$ for all $f \in V^*$.*

Natural isomorphisms are very important, but we still have not quite reached the stage where we have formalised our example of the relation between **FinOrd** and **FinSet**. For this we need

DEFINITION 4.3.4. *An* equivalence *between two categories $\mathscr{C}$ and $\mathscr{D}$ is a pair of functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ together with natural isomorphisms*

$$\mathrm{Id}_{\mathscr{C}} \cong GF \quad \text{and} \quad \mathrm{Id}_{\mathscr{D}} \cong FG.$$

If you return to the example of **FinOrd** and **FinSet** you can check that what we found was an equivalence between these two categories. The correspondence that we gave between bound representations of a quiver Q and kQ/I-modules in Section 1.4 can now be better expressed as an equivalence of categories. We have also now explained the meaning of Theorem 3.2.6.

In representation theory we want to apply this idea to say that two algebras have the 'same' representation theory.

DEFINITION 4.3.5. *Two algebras A and B are* Morita equivalent *if there is an equivalence of their module categories.*

4.4. Abelian categories and the Freyd-Mitchell embedding theorem

In this section we will briefly sketch one of the fundamental results in category theory, which says that any category which has certain basic properties similar to those found in a module category is in fact a subcategory of some module category. Unfortunately the technical definitions of the basic properties which are needed become rather lengthy, so we will just sketch the main ideas involved.

As an example of how the definitions which we need can be a little technical, despite corresponding to intuitively simple notions, let us consider the definition of the kernel of a morphism.

DEFINITION 4.4.1. *Given a morphism* $f : a \to b$ *in a category* $\mathscr{C}$*, the* kernel *of* f *is defined to be a morphism* $g : c \to a$ *for some object* c *such that all morphisms of the form* $h : x \to a$ *such that* $f \circ h = 0$ *factorise through* c*. Similarly we can define the* cokernel *of a morphism.*

Note that the definition does not guarantee that kernels or cokernels exist. Products and coproducts can also be defined in such a manner, by appealing to a *universal property* which characterises them.

DEFINITION 4.4.2. *An object* a *in a category* $\mathscr{C}$ *is* initial *if for each* $b \in \mathrm{Ob}(\mathscr{C})$ *there exists precisely one morphism from* a *to* b*. An object* b *is called* terminal *if for each* $a \in \mathrm{Ob}(\mathscr{C})$ *there exists precisely one morphism from* a *to* b*.*

We can now define our class of 'nice' categories that we wish to compare to module categories.

DEFINITION 4.4.3. *An* additive category *is a category* $\mathscr{C}$ *in which*

(1) *There exists an object (which we denote by* 0*) which is both initial and terminal.*
(2) *Products and coproducts of finite collections of objects always exist.*
(3) *For each pair* $a, b \in \mathrm{Ob}(\mathscr{C})$ *the set* $\mathrm{Hom}_{\mathscr{C}}(a,b)$ *has the structure of an abelian group. Further, these structures are compatible in the sense that for all objects* a, b, c*, composition of functions induces a bilinear map*

$$\mathrm{Hom}_{\mathscr{C}}(b,c) \times \mathrm{Hom}_{\mathscr{C}}(a,b) \to \mathrm{Hom}_{\mathscr{C}}(a,c).$$

An abelian category *is an additive category in which kernels and cokernels always exist, and for which if* f *is a morphism whose kernel is* 0 *then* f *is the kernel of its cokernel, and if* f *is a morphism whose cokernel is* 0*, then* f *is the cokernel of its kernel.*

This is quite a tricky definition, and we will not be able to check any examples in the time available. However, here are a few to illustrate that the concept is useful.

EXAMPLE 4.4.4. *(a) Given a ring R, the categories of R-modules is an abelian category. (We defined modules for algebras, but it is easy to see that the definition extends to rings.)*
(b) The category of vector bundles (or of sheaves) over a topological space is abelian.

We can now state the fundamental result about abelian categories.

THEOREM 4.4.5 (Freyd-Mitchell embedding theorem). *Every small abelian category is a full subcategory of some category of modules over a ring.*

Freyd proved this statement without the word full, and so the result is referred to variously as due to Freyd or to Mitchell. The importance of this result (apart from highlighting the central nature of module categories when considering abelian categories) is that allows one to argue in abstract abelian categories with the language and techniques from the more familiar setting of module categories.

4.5. Exercises

(1) Prove that for each object in a category $\mathscr{C}$, the identity morphism is unique.

(2) Given a group G, let $[G,G]$ denote the commutator subgroup of G — the set of all products of commutators of the form $ghg^{-1}h^{-1}$ with $g,h \in G$. This is in fact a normal subgroup of G. Show that the assignment $G \to [G,G]$ defines a functor from **Grp** to **Grp**, and that the assignment $G \to G/[G,G]$ defines a functor from **Grp** to **Abn**.

(3) Show that there is no functor from **Grp** to **Abn** sending each group G to its centre $Z(G)$. (Hint: Consider maps between the symmetric groups on two and three elements of the form $S_2 \to S_3 \to S_2$.)

(4) Given a group G with multiplication $*$, we define the opposite group G^{op} to be the group with the same set of elements, but with multiplication $*^{op}$ given by $g *^{op} h = h * g$. This define a covariant functor $-^{op}$ from **Grp** to **Grp** if we set $f^{op} = f$. Show that f^{op} is indeed a group homomorphism and prove that the identity functor $\mathrm{Id}_{\mathbf{Grp}}$ is naturally isomorphic to the opposite functor $-^{op}$.

(5) Given two groups A and B, regarded as categories with a single object, suppose that F and G are two functors from A to B. First note that this means that F and G are group homomorphisms. Show that there is a natural transformation from F to G if and only if there is an element $g \in B$ such that $F(h) = gG(h)g^{-1}$ for all $h \in A$.

(6) Given a field k, let $\mathbf{Mat}_k$ denote the category of all rectangular matrices with entries in k, where the objects are all natural numbers and each $m \times n$ matrix is regarded as a morphism from n to m with the usual matrix product as composition. Prove that $\mathbf{Mat}_k$ is equivalent to the category $\mathbf{FinVect}_k$ of finite dimensional vector spaces over k.