On some applications of infinitesimal methods to quantum groups and related algebras

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Abstract

The quantum general linear group, as defined by Dipper and Donkin, is a certain non-commutative deformation of the coordinate algebra of the corresponding classical group. In this thesis we study representations of this quantum group, and in particular develop an infinitesimal theory mimicking that of the classical case.

Our first main result generalises a theorem of Erdmann. This determines, using infinitesimal methods, precisely when a non-split extension exists between two Weyl modules for $\mathrm{SL}(2, k)$, in prime characteristic. We extend this result to quantum $\mathrm{GL}(2, k)$. As a corollary of this result we see that, when such extensions exist, they are unique up to isomorphism. We determine the structure of these in certain small cases, and consider a conjecture as to the structure of a much larger class of such.

Our particular quantum group was designed to provide a means of studying the q-Schur algebra of Dipper and James, and the rest of the thesis is concerned with this. In the classical case, the blocks of the Schur algebra have been determined by Donkin, and we verify that the appropriate modification of this result holds in the quantum case (with the same proof). This requires us to prove a quantum version of the strong linkage principle.

Doty, Nakano and Peters have defined an infinitesimal version of the Schur algebra, and we next consider a quantum analogue of this. After developing some of the basic representation theory of this algebra, we prove an infinitesimal version of Kostka duality. Finally, we determine the blocks of the infinitesimal Schur algebras corresponding to $\mathrm{GL}(2, k)$, and conclude by verifying that a similar result also holds in the quantum case.

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Contents

Introduction

In this thesis, we are concerned with the representation theory of quantum groups. More precisely, we will study representations of the quantum general linear group, as defined by Dipper and Donkin [5]. However, in order to motivate this study, we shall begin with a brief review of the context in which they arose.

It has long been known that representations of the symmetric and general linear groups are intimately related. This relationship was first explored by Schur in his doctoral thesis of 1901 [42], and he later refined and simplified this approach in [41]. In 1980, Green [29] further clarified the relationship between these two theories. He introduced certain finite dimensional algebras, the Schur algebras, which play an intermediate role between the general linear and symmetric groups. All of this development took place in an essentially combinatorial framework, ignoring the more geometric aspects of the algebraic group.

Independently of this, a much more geometric approach to the representations of algebraic groups has been developed. This is a rather more sophisticated theory, requiring considerably more technical machinery. From this point of view, we regard the general linear group as a group scheme, and relate its module theory to its geometry — for example, by identifying certain modules with the cohomology groups of corresponding bundles over a quotient scheme. Using these methods, and others, much information can be deduced about the structure of these representations.

From this standpoint, there are again certain algebras that are naturally related to the group. The most familiar of these are the associated Lie algebra and universal enveloping algebra, but for our purposes the coordinate algebra of the group will be of more importance. This is the Hopf algebra of regular functions on the group, and it can be shown to have an equivalent representation theory to that of the group itself.

There is also another important class of finite dimensional algebras related to our group scheme, namely those arising from the finite group schemes associated to the kernel of some power of the Frobenius morphism. These infinitesimal subgroups, along with their infinitesimal thickenings (the Jantzen subgroups), have played an important role in the development of the representation theory of the group in prime characteristic.

These two approaches to the general linear group can of course be combined. Results obtained using the more geometric approach can then be translated, via the Schur algebra, to obtain results concerning the symmetric group. The quantum groups studied in this thesis were introduced precisely so as to be able to generalise this strategy to a broader class of algebras.

The Hecke algebra (of type A) can be regarded as a deformation of the group algebra for the symmetric group. This deformation depends on one parameter, q say, and reduces to the symmetric group case when $q = 1$. While studying the representations of these, Dipper and James [7] introduced a corresponding deformation of the Schur algebra, which they called the q-Schur algebra. This was designed to parallel the relation between Schur algebras and symmetric groups in the context of Hecke algebras, and further provided a link to the representations of the finite general linear group in non-defining characteristic.

It is natural to ask if there is a corresponding deformation of the general linear group, but unfortunately such an object does not exist. However, as we have already remarked, the representation theory of the general linear group is equivalent to that of the corresponding coordinate algebra, and this can be appropriately deformed. Unfortunately there exist many such deformations, and so we shall restrict our attention to that introduced by Dipper and Donkin [5]. As we shall see later, it does not matter greatly which deformation we choose to study, as in a certain sense they are all equivalent (as they all give rise to the same q-Schur algebra).

Although the geometric aspects of the group do not survive under this deformation, it transpires that analogues of most of the classical results can be proved using purely algebraic means — as we need only verify them for the case corresponding to the general linear group. Thus a theory similar to that for the ordinary algebraic group can be developed. In this thesis we shall continue this development, with an emphasis on the infinitesimal theory.

We end this introduction with a brief survey of the rest of this thesis. The first chapter is devoted to a review of some of the basic properties of our quantum group, and is largely based on the results in [20] and [10]. It concludes with a comparison of the various different quantisations, and a proposition allowing us to translate results between them.

In the second chapter, we turn our attention to a paper of Erdmann [26]. This determines precisely when there exists a non-trivial extension between two Weyl modules for $\mathrm{SL}(2,k)$, and we generalise this to solve the corresponding problem for q-$\mathrm{GL}(2,k)$. To accomplish this, it is first necessary to calculate the blocks of our quantum group, a special case of the main theorem in chapter four. With this, the proof essentially reduces, via a standard spectral sequence argument, to a lengthy infinitesimal calculation.

By this last result, we have that when such a non-trivial extension exists, it is unique up to isomorphism. Chapter three is devoted to determining the structure of these extensions (in the classical case). Unfortunately, we can only describe this structure in certain special cases, but the chapter concludes with a conjecture as to the structure of a large class of such extensions, and a sufficient condition is given for an extension to satisfy this.

The fourth chapter is devoted to a generalisation of a result of Donkin [18], in which he determines the blocks of the Schur algebra. To prove this, we have to verify that a number of other standard results also hold, most notably the strong linkage principle. Once these are established, the rest of the chapter merely checks that the argument of the proof in the classical case now carries over essentially unchanged. From this result, it is then straightforward to determine the blocks of our quantum group.

Doty, Nakano and Peters [22] have defined an infinitesimal version of the Schur algebra. This allows infinitesimal methods to be carried over directly to the Schur algebra setting. In chapter five we generalise this construction to produce the infinitesimal q-Schur algebras, and develop some of their basic representation theory. Most of this chapter is based heavily on [22].

In the last chapter, we turn our attention to the blocks of the infinitesimal Schur algebras. These are completely determined for the cases corresponding to $\mathrm{GL}(2,k)$, from the corresponding blocks of both the Schur algebra and the Jantzen subgroups. The chapter ends by verifying that, using the main theorem of chapter four, this result also holds for the quantum case.

Chapter 1

Preliminaries

In this chapter we review some of the background theory that will be needed in this thesis. We begin in the first section by defining the quantum general linear group that will be our basic object of study. Just as in the classical case, we can reduce our study to that of certain finite dimensional algebras, the q-Schur algebras, and these are defined in the second section. An alternative approach is to consider certain infinitesimal submonoids of the quantum group, and we next develop this point of view. We leave until a later chapter the task of combining these two approaches (which will lead us to define the infinitesimal q-Schur algebras).

After a quick review of induction and restriction for quantum groups, and a short section reviewing weights and root systems, we briefly summarise some elementary results on representations of our quantum group. In the following section we outline how a similar theory can be developed in the infinitesimal case. After a brief survey of block theory for our various groups and algebras, we conclude with a discussion of some other quantisations of the general linear group, and the relationship between them.

1.1 The quantum general linear group

In this section we begin with a brief discussion of the general notion of a quantum group. This is motivated by the example of the classical algebraic groups discussed in the introduction. The section ends with the definition of the particular quantum group, q-$\mathrm{GL}(n,k)$, that will be our main object of study.

Henceforth we fix an algebraically closed field k. Unless stated otherwise, this will have characteristic $p > 0$. We define the *category of quantum groups* to be the dual of the category of k-Hopf algebras. (If we restrict to the category of finitely generated, commutative, reduced k-Hopf algebras, the corresponding category is that of the linear algebraic groups over k — see [10, Section 0.7] for details.) The *category of modules for a quantum group* will be identified with the category of comodules for the corresponding Hopf algebra.

Thus to say that H is a *quantum group* means that we have in mind some Hopf algebra over k, called the *coordinate algebra* of H, and denoted $k[H]$. If instead of a Hopf algebra we merely have a bialgebra structure, we call H a *quantum monoid*. We shall usually denote the comultipication and counit maps of a bialgebra by δ and ϵ respectively. If further it is a Hopf algebra then we denote the antipode by σ.

If H and K are quantum groups (respectively quantum monoids), then to say that $\phi : H \longrightarrow K$ is a *morphism of quantum groups* (respectively *of quantum monoids*) means that we have in mind a Hopf algebra (respectively bialgebra) map $\hat{\phi} : k[K] \longrightarrow k[H]$, called the *comorphism of* ϕ. We call K a *(quantum) subgroup* (respectively *submonoid*) of H if there exists a Hopf ideal (respectively biideal) I_K of $k[H]$ such that $k[K] = k[H]/I_K$. In this case we have a morphism $\phi : K \longrightarrow H$ whose associated comorphism is the natural map. We call ϕ the *inclusion* map and $\hat{\phi}$ the *restriction* map.

Given a k-coalgebra C, we will denote by $\mathrm{Comod}(C)$ the category of right C-comodules. There is a full subcategory of $\mathrm{Comod}(C)$ consisting of the finite dimensional right C-comodules, which we denote by $\mathrm{comod}(C)$. For a comodule $V \in \mathrm{Comod}(C)$, we will usually denote the *structure map* $V \longrightarrow V \otimes C$ by τ. Now let H be a quantum monoid over k. Then by a *left* (respectively *right*) H-module, we mean a right (respectively left) $k[H]$-comodule. We set $\mathrm{Mod}(H) = \mathrm{Comod}\ (k[H])$, and $\mathrm{mod}(H) = \mathrm{comod}\ (k[H])$. If C is a subcoalgebra of $k[H]$, we denote by $\mathrm{Mod}_C(H)$ the subcategory of objects V in $\mathrm{Mod}(H)$ such that the image of the comodule structure map $V \longrightarrow V \otimes k[H]$ lies in $V \otimes C$.

If H is a quantum group, then for all $V \in \mathrm{mod}(H)$, we can define the *dual module* V^*. Given a basis $v_1, \ldots, v_n$ of V, consider the dual basis $\alpha_1, \ldots, \alpha_n$ of $V^* = \mathrm{Hom}(V, k)$. If the structure map of V is given by $\tau_V(v_i) = \sum_{j=1}^n v_j \otimes f_{ji}$ then we give V^* a comodule structure via $\tau_{V^*}(\alpha_i) = \sum_{j=1}^n \alpha_j \otimes \sigma(f_{ij})$. If W is another H-

module, with basis $w_1, \ldots, w_m$ and structure map $\tau_W(w_i) = \sum_{j=1}^{m} w_j \otimes g_{ji}$, then we give the *tensor product* $V \otimes W$ the structure of a H-module via the structure map $\tau_{V\otimes W}(v_i \otimes w_r) = \sum_{j,s} v_j \otimes w_s \otimes f_{ji} g_{sr}$. As multiplication in $k[H]$ is not in general commutative, it is not necessarily the case that $V \otimes W \cong W \otimes V$.

The above constructions are quite general. However, for the rest of this thesis we will mainly restrict our attention to certain very particular quantum groups and monoids, which can be regarded as 'deformations' of the general linear group and various related subgroups and monoids. We thus conclude this section by defining a quantum general linear group, and some important subgroups.

We fix $q \in k\backslash\{0\}$, and define $A_q(n)$ to be the k-algebra generated by the n^2 indeterminates c_{ij}, with $1 \leq i, j \leq n$, subject to the relations

$$\begin{array}{lll} c_{ij}c_{rs} & = q c_{rs} c_{ij} & \text{for } i > r \text{ and } j \leq s, \\ c_{ij}c_{rs} & = c_{rs}c_{ij} + (q-1) c_{rj} c_{is} & \text{for } i > r \text{ and } j > s, \\ c_{ij}c_{il} & = c_{il}c_{ij} & \text{for all } i, j, l. \end{array}$$

We note that when $q = 1$ these relations just say that the c_{ij} commute; in this case we will usually denote the c_{ij} by x_{ij}. Now consider the algebra maps δ and ϵ given on generators by

$$\delta : c_{ij} \longmapsto \sum_{t=1}^{n} c_{it} \otimes c_{tj},$$

$$\epsilon : c_{ij} \longmapsto \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Then it is routine to check that

Theorem 1.1.1 *$A_q(n)$ is a bialgebra with comultiplication δ and counit ϵ.*

Proof: See [5, 1.4.2 Theorem].

We will denote by q-M(n,k) (or just M) the quantum monoid associated to this bialgebra.

We now define a *q-determinant* in $A_q(n)$. This is the element

$$d_q = \sum_{\pi \in \Sigma_n} \mathrm{sgn}(\pi) c_{1,1\pi} c_{2,2\pi} \ldots c_{n,n\pi},$$

where Σ_n denotes the symmetric group on n symbols, and sgn denotes the sign of a permutation. As remarked in [5, Section 4.2], it is straightforward to verify that the set $\{d^i \mid i \geq 0\}$ satisfies the left and right Ore conditions. Hence we can localise at this set, and consider the k-algebra $A_q(n)(d_q^{-1})$.

Lemma 1.1.2 *The bialgebra structure given on $A_q(n)$ above can be canonically extended to a bialgebra structure on $A_q(n)(d_q^{-1})$, by defining*

$$\delta(d_q^{-1}) = d_q^{-1} \otimes d_q^{-1}$$

and $\epsilon(d_q^{-1}) = 1$.

Proof: See [5, 4.2.1 Lemma].

With this lemma, a rather technical calculation now gives

Theorem 1.1.3 *There exists a k-algebra anti-endomorphism σ which, along with δ and ϵ, endows $A_q(n)(d_q^{-1})$ with a Hopf algebra structure.*

Proof: See [5, 4.2.21 Theorem].

We will denote by q-GL(n,k) (or just G where this is unambiguous) the quantum group associated to this Hopf algebra. Note that in the case $q=1$ we recover the classical group GL(n,k).

The following subgroups will also play an important role in the theory we shall develop later. We define the *torus*, q-T(n,k), to be the subgroup of G with defining ideal the Hopf ideal of $k[G]$ generated by all c_{ij} with $i \neq j$. Similarly, we define the *(negative) Borel subgroup* corresponding to the lower triangular matrices, q-B(n,k), to be the subgroup of G with defining ideal the Hopf ideal generated by all c_{ij} with $i < j$. These will often be denoted just by T and B respectively. We also define the *positive Borel subgroup* corresponding to the upper triangular matrices similarly, and denote it by q-B$^+(n,k)$, or just B^+.

More details on the construction of these subgroups can be found in [20, Section 2]. However, the results there only show that the defining ideal is a biideal; to see that it is in fact a Hopf ideal one can use the corresponding arguments in [39, (6.1.1) Lemma], replacing references to [39, (5.3.2) Theorem] and [39, (4.1.1) Lemma] by [5, 4.2.20 Lemma] and [5, Proof of 4.2.12 Lemma] respectively. We will also need to consider the corresponding submonoids of M. The submonoid corresponding to T and B will be denoted q-$\mathrm{D}(n,k)$ and q-$\mathrm{L}(n,k)$ (or just D and L) respectively.

Finally, we note that in what follows we shall often consider sub- and quotient algebras of $k[G]$ and $k[M]$. We shall consistently abuse notation and write just c_{ij} for the standard generators, or their images, in these various objects.

1.2 The q-Schur algebra

In this section we will construct the q-Schur algebras from the coordinate algebra of the quantum monoid q-$\mathrm{M}(n,k)$. This procedure is an exact analogue of the construction of the ordinary Schur algebra in [29]. However, the original definition of the q-Schur algebras due to Dipper and James (see [7]) is quite different, as they are there defined as certain centralising algebras of an action of the Hecke algebras. We conclude this section by noting that these two approaches can be reconciled.

We begin by introducing a grading on $A_q(n)$. If we define the degree of each of the generators c_{ij} to be one (for $1 \leq i,j \leq n$), then it is clear that the defining relations are homogeneous, and hence that $A_q(n)$ becomes a graded algebra. The subspace of elements of degree d will be denoted $A_q(n,d)$. It is also straightforward to verify that the $A_q(n,d)$ are in fact subcoalgebras of $A_q(n)$ for all d.

Given a finite dimensional coalgebra C, there is a natural way to give the dual k-module $\mathrm{Hom}_k(C,k)$ the structure of an algebra, with multiplication given by the dual of comultiplication in C (see for example [9] or [43]). Applying this to the case above, $A_q(n,d)^*$ becomes an algebra, which we denote by $S_q(n,d)$, and call the *q-Schur algebra.*

Alternatively, we can consider the Hecke algebra $\mathcal{H}_q = \mathcal{H}_q(\Sigma_d)$ associated with Σ_d. This is the free k-module with basis $\{T_w \mid w \in \Sigma_d\}$ and relations

$$T_sT_w = \begin{cases} T_{sw} & l(sw) = l(w)+1, \\ qT_{sw} + (q-1)T_w & \text{otherwise,} \end{cases}$$

for all $w, s \in \Sigma_d$ with $l(s) = 1$, where l is the usual length function. Now if we consider the free n-dimensional k-module E, we can define an action of $\mathcal{H}_q$ on $E^{\otimes d}$ analogous to the action by place permutation of Σ_d on $E^{\otimes d}$. With this one can then show

Theorem 1.2.1 $S_q(n,d) \cong \mathrm{End}_{\mathcal{H}_q}(E^{\otimes d})$.

Proof: See [5, 3.2.6 Corollary].

This allows us to identify the definition of the q-Schur algebra above with the original definition in [7].

In the above we have remained deliberately vague as to the precise action of the Hecke algebra on $E^{\otimes d}$, and we refer the reader to [5, Section 3] for a fuller discussion of this. The quantum general linear group of the previous section was constructed precisely so as to satisfy the above theorem — however, other non-isomorphic quantisations exist with the same property. These will be considered in more detail in Section 1.9. For now we merely note that an excellent discussion of the relationship between the two constructions of the q-Schur algebra (for the Manin quantisation) can be found in [2], starting from the point of view of the Hecke algebra.

1.3 Infinitesimal subgroups

In this section we will consider generalisations of the infinitesimal subgroups to the quantum setup. Classically, these are usually defined as the kernel of the Frobenius map to some power. A quantum analogue of G_1 is defined in [10, Section 3.1], and also of the corresponding Jantzen subgroup. These definitions can be easily generalised to provide analogues to the higher kernels, which will be needed in later sections.

Henceforth we shall restrict our attention to the case when q is a primitive lth root of unity. Note that for such a root to exist we must have $(l,p)=1$; else $l=ap$ for some a, and $q^{ap}-1=0$ implies that $(q^a-1)^p=1$, contradicting primitivity. By [25, (3.1) Theorem, taking $\alpha=1$ and $\beta=q$], we have

Theorem 1.3.1 *There exists a Hopf algebra homomorphism*

$$\hat{F} : k[\mathrm{GL}(n,k)] \longrightarrow k[q\text{-}\mathrm{GL}(n,k)]$$

taking x_{ij} to c_{ij}^l, where these are the usual generators from Section 1.1.

Thus we define the *Frobenius morphism* $F : q\text{-}\mathrm{GL}(n,k) \longrightarrow \mathrm{GL}(n,k)$ to be the morphism of quantum groups with associated comorphism $\hat{F}$. We also have the usual Frobenius map F for $\mathrm{GL}(n,k)$ taking x_{ij} to x_{ij}^p, and so we may consider the composition of these maps. Henceforth we will abuse notation and write F^r for $\mathrm{F}^{r-1}F$, which is a morphism of quantum groups.

Following [20], we say that a quantum group $\bar{H}$ is a *factor group* of a quantum group H if $k[\bar{H}]$ is a subHopf algebra of $k[H]$. Given a factor group $\bar{H}$ of H whose coordinate algebra is central in $k[H]$, we obtain a subgroup H_1 of H with defining ideal $I_{H_1} = k[H].(\ker(\epsilon_H) \cap k[\bar{H}])$.

We now return to the case $G = q\text{-}\mathrm{GL}(n,k)$. Consider the subHopf algebra of $k[G]$ generated by the elements $c_{ij}^{lp^{r-1}}$ for $1 \leq i,j \leq n$, and $d_q^{-lp^{r-1}}$. This is isomorphic to $k[\mathrm{GL}(n,k)]$ via F^r. The corresponding factor group will be denoted $\bar{G}^r$, or just $\bar{G}$ in the case $r = 1$. Then by the previous paragraph, there is a subgroup of G with defining ideal generated by the elements $c_{ij}^{lp^{r-1}} - \delta_{ij}$ for $1 \leq i,j \leq n$, and $d_q^{-lp^{r-1}} - 1$. This subgroup will be denoted G_r, and called the *rth Frobenius kernel*. We also need to consider $B_r = B \bigcap G_r$, and $T_r = T \bigcap G_r$.

Finally, we introduce quantum analogues of the *Jantzen subgroups*, which can be regarded as infinitesimal thickenings of the Frobenius kernels by the torus. Consider the ideal of $k[G]$ generated by the elements $c_{ij}^{lp^{r-1}}$ for $1 \leq i \neq j \leq n$. This is clearly a biideal, and by the isomorphism of $\bar{G}^r$ with $\mathrm{GL}(n,k)$ above, along with the description of the antipode in [5, Lemmas 4.2.20 and 4.2.12], it is easy to verify that it is in fact a Hopf ideal. We denote the subgroup of G with this as defining ideal by G_rT, and the corresponding intersection with B by B_rT. Similarly one can show that the ideal generated by the elements $c_{ij}^{lp^{r-1}}$ for $1 \leq i < j \leq n$ is a Hopf ideal of $k[G]$; we denote the subgroup corresponding to this by G_rB.

1.4 Induction for quantum groups

In this section we review the theory of induction and restriction functors for coalgebras, as originally developed in [11, Section 3]. This is then applied to the category of quantum groups, and we conclude with a brief summary of some basic properties of these functors.

Let C be a k-coalgebra with comultiplication δ and counit ϵ, C' be another such, and $\hat{\phi} : C \longrightarrow C'$ be a morphism of coalgebras. We first define the *restriction* functor

$\hat{\phi}_0 : \mathrm{Comod}(C) \longrightarrow \mathrm{Comod}(C')$. For $V \in \mathrm{Comod}(C)$ with structure map τ, set $\hat{\phi}_0(V)$ to be the k-space V regarded as a C'-comodule via the structure map $(\mathrm{id} \otimes \hat{\phi})(\tau)$. For a morphism α of C-comodules, we set $\hat{\phi}_0(\alpha)$ to be the k-map α regarded as a C'-comodule morphism.

The definition of the *induction* functor $\hat{\phi}^0 : \mathrm{Comod}(C') \longrightarrow \mathrm{Comod}(C)$ is a little more complicated. For $M \in \mathrm{Comod}(C)$ with structure map τ, and a vector space V, we denote by $|V| \otimes M$ the vector space $V \otimes M$ regarded as a C-comodule via the structure map $(\mathrm{id} \otimes \tau)$. Then for $W \in \mathrm{Comod}(C')$ with structure map τ, we define $\hat{\phi}^0(V)$ to be the C-subcomodule of $|W| \otimes C$ consisting of the elements f such that

$$(\tau \otimes \mathrm{id})(f) = (\mathrm{id} \otimes (\hat{\phi} \otimes \mathrm{id})\delta)(f).$$

For a morphism α of C'-comodules, we set $\hat{\phi}^0(\alpha)$ to be the restriction of $(\alpha \otimes \mathrm{id})$. It is easily verified that the map $(\mathrm{id} \otimes \epsilon)$ restricts to a C-comodule morphism $\nu : \hat{\phi}_0(\hat{\phi}^0(W)) \longrightarrow W$, and that for any $V \in \mathrm{Comod}(C)$,and any C'-comodule homomorphism $\alpha : \hat{\phi}_0(V) \longrightarrow W$ there exists a unique C-comodule map $\tilde{\alpha} : V \longrightarrow \hat{\phi}^0(W)$ such that $\nu\tilde{\alpha} = \alpha$. Thus we have *Frobenius reciprocity*, that is a k-space isomorphism

$$\mathrm{Hom}_{C'}(\hat{\phi}_0(V), W) \cong \mathrm{Hom}_C(V, \hat{\phi}^0(W)).$$

Now $\hat{\phi}_0$ is exact, while $\hat{\phi}^0$ is left exact and takes injectives to injectives.

Now suppose that $\phi : K \longrightarrow H$ is a morphism of quantum monoids. Set $\phi^* = \hat{\phi}_0 : \mathrm{Mod}(H) \longrightarrow \mathrm{Mod}(K)$, and $\phi_* = \hat{\phi}^0 : \mathrm{Mod}(K) \longrightarrow \mathrm{Mod}(H)$. In the case where ϕ is inclusion, we denote ϕ^* by Res_K^H and ϕ_* by Ind_K^H, and call them *restriction* and *induction* respectively. If K is a subgroup of H we may also denote $\mathrm{Res}_K^H(V)$ by $V|_K$ or just V. If $\hat{\phi} : k[H] \longrightarrow k[K]$ is injective, we write $\phi^*(V)$ as $\mathrm{Inf}_K^H(V)$, and call this *inflation*. By analogy with the classical case, when $H = q\text{-}\mathrm{GL}(n,k)$ and $K = q\text{-}\mathrm{B}(n,k)$ we will often denote $R^i\mathrm{Ind}_K^H(V)$ by $H^i(V)$.

The H-module $k[H]$ is injective, and every H-module embeds into a direct sum of copies of this (see [39, (2.4.4)]). Also, by [39, (2.8.1)], the category $\mathrm{Mod}(H)$ has enough injectives. Hence for V, $W \in \mathrm{Mod}(H)$ and $i \geq 0$, we may consider the ith derived functor of $\mathrm{Hom}_H(V, -)$ evaluated at W, which we denote by $\mathrm{Ext}_H^i(V, W)$. We also write $H^i(H, V)$ for $\mathrm{Ext}_H^i(k, V)$, where k denotes the trivial module.

We conclude with some standard results on induction, that will be used in later sections.

Theorem 1.4.1 (The generalised tensor identities) *Let K be a subgroup of a quantum group H, and let $V \in \mathrm{Mod}(K)$ and $W \in \mathrm{Mod}(H)$. Then for all $i \geq 0$,*

$$R^i\mathrm{Ind}_K^H(V|_K \otimes W) \cong V \otimes R^i\mathrm{Ind}_K^H(W).$$

If further the antipode of H is bijective then for all $i \geq 0$,

$$R^i\mathrm{Ind}_K^H(W \otimes V|_K) \cong R^i\mathrm{Ind}_K^H(W) \otimes V.$$

Proof: See [20, Proposition 1.3(ii)].

Theorem 1.4.2 *Let $\bar{H}$ be a factor group of a quantum group H such that $k[\bar{H}]$ is central in $k[H]$. Let $\pi : H \longrightarrow \bar{H}$ be the quotient map and let H_1 be the corresponding subgroup of H. Suppose further that $k[H]$ is a faithfully flat $k[\bar{H}]$-comodule.*

i) If $V \in Mod(H)$ then V^{H_1} is an H-submodule of V and is isomorphic to π^*W for some $W \in Mod(\bar{H})$.

ii) $k[H]^{H_1} \cong k[\bar{H}]$.

iii) $I|_{H_1}$ is injective for any injective H-module I.

iv) We have a Grothendieck spectral sequence with E_2 term $H^i(\bar{H}, H^j(H_1, V))$ converging to $H^*(H, V)$.

v) If W is an H-module that is trivial as an H_1-module, then for all $i \geq 0$ we have $H^i(H_1, V \otimes W) \cong H^i(H_1, V) \otimes W$ as $\bar{H}$-modules.

Proof: See [20, Propositions 1.5 and 1.6].

Part (iv) of the above is known as the *Lyndon–Hochschild–Serre spectral sequence*, and will play a vital role in the following chapter. We conclude by noting that the factor groups corresponding to the Frobenius kernels of the last section satisfy the hypotheses of the last theorem (see [5, Theorem 1.3.3]).

1.5 Weights and root systems

In this section we review the theory of weights and roots, as it applies to this setup. This will all follow just as in the classical case, and we shall make the usual choices for root systems etc. (compare with [34, II 1.21]).

First consider $k[T]$. Clearly the defining relations for $k[G]$ imply that the generators c_{ii} of $k[T]$ all commute. Thus T is just the ordinary (classical) n-dimensional torus. We denote the *character group* of T by $X(T)$, which is isomorphic to $\mathbb{Z}^n$. To define a root system in $X(T)$, we set $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with the 1 in the ith position) for $1 \leq i \leq n$. Then a *root* is an element of $X(T)$ of the form $\varepsilon_i - \varepsilon_j$ with $1 \leq i \neq j \leq n$. We denote the set of roots by Φ. A root $\varepsilon_i - \varepsilon_j$ is called *positive* (respectively *negative*) if $i < j$ (respectively $i > j$). The set of positive roots will be denoted by Φ^+, and the negatives by Φ^-. The elements $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i < n$ will be called the *simple roots*, and the set of them will be denoted by Π. We also define the *fundamental dominant weights* $\varpi_i = \varepsilon_1 + \ldots + \varepsilon_i$. Sometimes ϖ_n will simply be denoted ϖ. There is a $\mathbb{Z}$-bilinear form $\langle -, - \rangle$ on $X(T)$ satisfying $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$. We will also use the usual *dominance (partial) order* on $X(T)$, given by $\lambda \leq \mu$ if $\mu - \lambda \in \mathbb{N}\Phi^+$.

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in X(T)$ the one-dimensional T-module with structure map taking a basis element a to $a \otimes c_{11}^{\lambda_1} \ldots c_{nn}^{\lambda_n}$ will be denoted by k_λ. Given $\lambda \in X(T)$ and $V \in \mathrm{Mod}(T)$, we set V^λ to be the sum of all submodules of V isomorphic to k_λ, and call this the λ *weight space* of V. As every T-module is completely reducible, $V = \oplus_{\lambda \in X(T)} V^\lambda$ for any $V \in \mathrm{Mod}(T)$.

Suppose that H is some quantum group containing T as a subgroup. Denote by $\mathbb{Z}X(T)$ the ring with $\mathbb{Z}$-basis consisting of the set of formal elements $\{e^\lambda \mid \lambda \in X(T)\}$ and multiplication satisfying $e^\lambda e^\mu = e^{\lambda+\mu}$. Then we define the *(formal) character* of $V \in \mathrm{Mod}(H)$ by

$$\mathrm{ch}V = \sum_{\lambda \in X(T)} \left(\dim V^\lambda\right) e^\lambda,$$

where for this we regard V as a T-module via restriction.

We next wish to define an analogue of the *Weyl group* for G. Identifying Σ_n with the group of permutations of $\{1, \ldots, n\}$, we define an action of Σ_n on $X(T)$ by

$w\lambda = (\lambda_{w(1)}, \ldots, \lambda_{w(n)})$. In these circumstances we will denote Σ_n by W, and the element of Σ_n corresponding to the transposition $(i, i+1)$ by s_{α_i}. This natural action of W on $X(T)$ induces an action on $\mathbb{Z}X(T)$ given by

$$w\left(\sum_{\lambda \in X(T)} a_\lambda e^\lambda\right) = \sum_{\lambda \in X(T)} a_\lambda e^{w\lambda}.$$

By [20, Lemma 3.1(iv)], the character of any finite dimensional G-module is W-invariant.

Finally, we conclude this section by recording another action of W on $X(T)$ that will prove important. This is the 'dot' action $w.\lambda = w(\lambda + \rho) - \rho$, where $\rho = (n-1, n-2, \ldots, 0)$. With this we can define the *affine Weyl group*, W_l associated to G. This is the transformation group on $X(T)$ generated by W with the dot action, and the translations $\lambda \longmapsto \lambda + l\alpha$ for all $\alpha \in \Phi$. Occasionally we will also need the element $\bar{\rho}$, which will equal half the sum of the positive roots. The main reason for considering ρ instead of $\bar{\rho}$ is that the latter does not always lie in $X(T)$.

1.6 The simple G-modules

Much as in the classical case, one can classify the simple G-modules by considering those modules induced from the one-dimensional B-modules. We give a brief outline of this, and consider in detail the $n = 2$ case that will be needed later. The section concludes with a standard result relating this theory to that of the q-Schur algebra.

We begin by considering the modules k_λ (for $\lambda \in X(T)$) of the last section. These can be regarded as B-modules by inflation, and one can prove (see [20, Lemma 2.6(i)]) that they form a complete set of inequivalent irreducible B-modules. The modules $H^i(k_\lambda)$ with $i \geq 0$ play a crucial role in what follows; we will usually denote them just by $H^i(\lambda)$, or in the case $i = 0$ by $\nabla(\lambda)$. The ∇'s have the usual universal property (by arguments as in [39, (8.3.1) Theorem]). We also set $\Delta(\lambda) = \nabla(\lambda^*)^*$, where $\lambda^* = -w_0\lambda$ and w_0 is the longest element of W.

Theorem 1.6.1 *Let $X(T)^+$ denote the set of $\lambda \in X(T)$ such that $\nabla(\lambda) \neq 0$. Then*

$$X(T)^+ = \{\lambda \in X(T) \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}.$$

For each $\lambda \in X(T)^+$ the induced module $\nabla(\lambda)$ has simple socle which we denote by $L(\lambda)$. These satisfy dim $L(\lambda)^\lambda = 1$, and $\mu < \lambda$ for all other weights μ of $L(\lambda)$. Furthermore, the collection $\{L(\lambda) \mid \lambda \in X(T)^+\}$ is a complete set of inequivalent, irreducible G-modules.

Proof: See [20, Theorem 2.10 and Lemma 3.2].

The set $X(T)^+$ above will be called the set of *dominant weights*. It is now clear that the characters of the $L(\lambda)$'s are linearly independent, so for any finite dimensional G-module V we can define the *composition multiplicity* $[V : L(\lambda)]$ to be the number of times that $L(\lambda)$ occurs in a composition series for V. On occasion it will be necessary to consider both classical and quantum modules simultaneously; to avoid confusion in such cases we will denote the induced and simple modules for the classical group by $\bar{\nabla}(\lambda)$ and $\bar{L}(\lambda)$ respectively.

A number of vanishing theorems also hold in this context (see [20, Section 3]). The most important of these for our purposes is

Theorem 1.6.2 (Kempf's vanishing theorem) *If $\lambda+\rho \in X(T)^+$ then $H^i(\lambda) = 0$ for all $i > 0$.*

Proof: See [20, Theorem 3.4].

For $\lambda \in X(T)^+$, the character of $\nabla(\lambda)$ will be denoted by $\chi(\lambda)$. Then we have

Theorem 1.6.3 (Weyl's character formula) *For $\lambda \in X(T)^+$ we have*

$$\chi(\lambda) = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \operatorname{sgn}(w) e^{w\rho}}.$$

Proof: See [20, Theorem 3.6].

In the next chapter, we will consider extensions of Weyl modules. In this context the following result will prove useful.

Proposition 1.6.4 *For all $\tau, \tau' \in X(T)^+$, if $\operatorname{Ext}^1_G(\Delta(\tau), \Delta(\tau')) \neq 0$ then we must have $\tau < \tau'$.*

Proof: This follows just as in the classical case (see [4, 3.2 Corollary]).

In the case $n = 2$ we can give a complete classification of the induced modules above. Quantum analogues, $S_q^r(E)$, of the symmetric powers of the natural n-dimensional module are constructed in [5, Theorem 2.1.9]. As remarked in [20, Remark 3.7], there is for all $r \geq 0$ an obvious isomorphism $\nabla(r,0) \cong S_q^r(E)$. We shall denote the analogue of the one-dimensional determinant module by q-det; then we have for $\lambda = (\lambda_1, \lambda_2) \in X(T)^+$ that

$$\nabla(\lambda) \cong S_q^{\lambda_1 - \lambda_2}(E) \otimes q\text{-det}^{\lambda_2}.$$

Now in general $U \otimes V$ is not isomorphic to $V \otimes U$, but by the generalised tensor identity (see (1.4.1)) we have that $\nabla(r,0) \otimes q\text{-det}^a \cong q\text{-det}^a \otimes \nabla(r,0)$.

We conclude this section with some remarks on the polynomial representation theory of G. Following [29], we say that $V \in \mathrm{Mod}(G)$ is a *polynomial module* if it is an M-module with the same structure map. If further it is an $A_q(n,d)$-comodule we say that it is a *homogeneous polynomial module of degree d*. As in the classical case the arguments in [28, Section 1.6] give

Theorem 1.6.5 *Any polynomial G-module is a direct sum of homogeneous polynomial submodules.*

Thus any indecomposable polynomial module is homogeneous. Clearly, submodules and quotients of homogeneous modules are homogeneous of the same degree.

Now the category of left $S_q(n,d)$ modules is naturally isomorphic to the full subcategory of $\mathrm{Mod}(G)$ consisting of the homogeneous polynomial modules of degree d, and we will identify these categories. We conclude with a classification of the irreducible $S_q(n,d)$ modules. A weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(T)$ will be called *polynomial* if $\lambda_i \geq 0$ for all $1 \leq i \leq n$. The *degree* of such a weight is $\sum_{i=1}^n \lambda_i$. We set $\Lambda(n,d)$ to be the set of polynomial weights of degree d. The intersection $\Lambda(n,d) \cap X(T)^+$ will be denoted by $\Lambda^+(n,d)$. Then we have

Theorem 1.6.6 *If $\lambda \in \Lambda^+(n,d)$ then the G-modules $\nabla(\lambda)$ and $L(\lambda)$ are homogeneous polynomial modules of degree d. Thus*

$$\{L(\lambda) \mid \lambda \in \Lambda^+(n,d)\}$$

is a complete set of inequivalent, irreducible q-$S(n,d)$ modules.

Proof: This follows as in [39, (11.1.2)], replacing reference to [39, (10.4.1)(2)] by [20, Section 4 (3)(ii)].

1.7 The simple infinitesimal modules

This section will review some of the basic infinitesimal theory developed in [10, Sections 3.1 and 3.2]. Unfortunately, [10] only considers the case $r = 1$; however the arguments given there all still hold, *mutatis mutandis*, in the general case. Indeed, when results are cited from [10, Sections 3.1 and 3.2], they will be (tacitly) assumed to be stated there in the required generality.

We define the set of lp^{r-1}*-restricted weights*

$$X_r(T) = \{\lambda \in X(T) \mid 0 \leq \lambda_i - \lambda_{i+1} \leq lp^{r-1} - 1 \text{ for } 1 \leq i \leq n\},$$

where we set $\lambda_{n+1} = 0$. Then by [10, 3.1(1) and (2)], we have $\{k_\lambda|_{B_rT} : \lambda \in X(T)\}$ is a full set of irreducible B_rT-modules, and $\{k_\lambda|_{B_r} : \lambda \in X_r(T)\}$ is a full set of irreducible B_r-modules. Setting $Z_r(\lambda) = \mathrm{ind}_{B_r}^{G_r} k_\lambda$ and $\hat{Z}_r(\lambda) = \mathrm{ind}_{B_rT}^{G_rT} k_\lambda$, we have, by [10, 3.1(13)(i)], that for each $\lambda \in X(T)$ the socles of $Z_r(\lambda)$ and $\hat{Z}_r(\lambda)$ are both simple. These will be denoted by $L_r(\lambda)$ and $\hat{L}_r(\lambda)$ respectively. Then we obtain

Proposition 1.7.1 *The set $\{L_r(\lambda) : \lambda \in X_r(T)\}$ is a full set of inequivalent, irreducible G_r-modules, and $\{\hat{L}_r(\lambda) : \lambda \in X(T)\}$ is a full set of inequivalent, irreducible G_rT-modules.*

Proof: See [10, 3.1(13)(iii)].

It is straightforward to check that for all $\lambda \in X(T)$, the module $\hat{L}_r(lp^{r-1}\lambda)$ is one-dimensional, and trivial as a G_r-module. Thus we obtain as in [10, 3.1(17)] that for all λ and $\mu \in X(T)$, we have $\hat{L}_r(\lambda + lp^{r-1}\mu) \cong \hat{L}_r(\lambda) \otimes k_{lp^{r-1}\mu}$. We shall usually just write $lp^{r-1}\mu$ for the module $k_{lp^{r-1}\mu}$. The various simple modules are related by the following result.

Proposition 1.7.2 *For all $\lambda \in X(T)$ we have $\hat{L}_r(\lambda)|_{G_r} \cong L_r(\lambda)$. If $\lambda \in X_r(T)$ then $L(\lambda)|_{G_rT} \cong \hat{L}_r(\lambda)$.*

Proof: See [10, 3.1(18)(ii) and 3.2(3)].

We conclude this brief review of infinitesimal theory with two 'global' results which are proved by infinitesimal methods. Let St_r denote the *Steinberg module* $L((lp^{r-1}-1)\rho)$. Then

Proposition 1.7.3 *We have* $St_r \cong \nabla((lp^{r-1}-1)\rho)$*, and* $St_r|_{G_rT} \cong \hat{Z}_r((lp^{r-1}-1)\rho)$*.*

Proof: By [10, 3.2(8)] we have that $\hat{Z}_r((lp^{r-1}-1)\rho) \cong \hat{L}_r((lp^{r-1}-1)\rho)$. (In fact [10, 3.2(8)] defines ρ to be our $\rho + \varpi$, but this does not matter as both sides of our isomorphism are obtained from there by tensoring with an appropriate power of the one-dimensional module q-det. Similar remarks hold whenever we refer to [10] for a result involving ρ.) By the last proposition this is isomorphic to St_r. Now from [10, 3.1(20)(ii)] we have that $\mathrm{ch}\hat{Z}_r((lp^{r-1}-1)\rho) = \chi((lp^{r-1}-1)\rho)$. Hence as $St_r \leq \nabla((lp^{r-1}-1)\rho)$, and they have the same character, we must have equality as desired.

Theorem 1.7.4 (Steinberg's tensor product theorem) *For all* $\lambda \in X_r(T)$ *and* $\mu \in X(T)^+$ *we have*

$$L(\lambda + lp^{r-1}\mu) \cong L(\lambda) \otimes \bar{L}(\mu)^{F^r}.$$

Proof: See [10, 3.2(5)].

1.8 Tilting modules and blocks

In this section we give a brief review of the theory of tilting modules and of blocks. In particular, we note several equivalent criteria for two modules to lie in the same block that will be used throughout what follows.

Let $V \in \mathrm{Mod}(G)$. We say that G has a *good filtration* if there is a G-module filtration $0 = V_0 \leq V_1 \leq \ldots$ with $V = \cup_{i\geq 0} V_i$ such that each quotient V_i/V_{i-1} is either 0 or isomorphic to $\nabla(\lambda_i)$ for some $\lambda_i \in X(T)^+$. The multiplicity of $\nabla(\lambda)$ in such a filtration is independent of the choice of filtration, and we denote it by $(V : \nabla(\lambda))$. If a module has both a good filtration and a filtration by $\Delta(\lambda)$'s then we say that it is a *tilting module*.

As noted in [20, Section 4], we have by a result of Ringel (see [40, Section 5 Proposition 2] and remarks in [17, Section 1] for the classical case), that there exists for each $\lambda \in X(T)^+$ an indecomposable tilting module $T(\lambda)$ such that $\dim T(\lambda)^\lambda = 1$ and all other weights μ of $T(\lambda)$ satisfy $\mu < \lambda$. Furthermore, these form a complete set of indecomposable tilting modules.

For a quantum group H, we define a relation $\sim$ on the (set of isomorphism classes of) irreducible H-modules to be the smallest equivalence relation such that for all irreducible H-modules L and L', we have $L \sim L'$ whenever $\mathrm{Ext}^1_H(L, L') \neq 0$. The equivalence classes under this relation are called the *blocks* of H. We may equivalently say that two simple modules $L(\lambda)$ and $L(\mu)$ lie in the same block if there exists a chain $\lambda = {}_0\lambda, \ldots, {}_t\lambda = \mu$ such that for all $1 \leq i < t$ either $[I({}_i\lambda) : L({}_{i+1}\lambda)] \neq 0$ or $[I({}_{i+1}\lambda) : L({}_i\lambda)] \neq 0$.

Throughout this thesis we will tacitly make use of the following results, which give alternative criteria for block membership. We here denote by $I(\lambda)$ the injective hull of $L(\lambda)$ as a G-module, and by $I_S(\lambda)$ the injective hull of $L(\lambda)$ as an $S_q(n,d)$-module. The proof of [13, Theorem 2.6] also holds in the quantum case, and so we obtain

Proposition 1.8.1 *For $\lambda \in X(T)^+$, the module $I(\lambda)$ has a good filtration with multiplicities given by*

$$(I(\lambda) : \nabla(\mu)) = [\nabla(\mu) : L(\lambda)]$$

for $\mu \in X(T)^+$. Thus two elements λ, $\mu \in X(T)^+$ belong to the same block of G if, and only if, there exists a chain $\lambda = {}_1\lambda, \ldots, {}_t\lambda = \mu$ of elements of $X(T)^+$ such that for each $1 \leq i < t$ we have either $[\nabla({}_i\lambda) : L({}_{i+1}\lambda)] \neq 0$ or $[\nabla({}_{i+1}\lambda) : L({}_i\lambda)] \neq 0$.

Then, as noted in [20, Section 4(6)], we obtain

Corollary 1.8.2 *For $\lambda \in \Lambda^+(n,d)$, the module $I_S(\lambda)$ has a good filtration with multiplicities given by*

$$(I_S(\lambda) : \nabla(\mu)) = [\nabla(\mu) : L(\lambda)]$$

for $\mu \in \Lambda^+(n,d)$. Thus two elements λ, $\mu \in \Lambda^+(n,d)$ belong to the same block of $S_q(n,d)$ if, and only if, there exists a chain $\lambda = {}_1\lambda, \ldots, {}_t\lambda = \mu$ of elements of $\Lambda^+(n,d)$ such that for each $1 \leq i < t$ we have either $[\nabla({}_i\lambda) : L({}_{i+1}\lambda)] \neq 0$ or $[\nabla({}_{i+1}\lambda) : L({}_i\lambda)] \neq 0$.

Finally we note that there is an infinitesimal version of this result. Writing $\hat{Q}_r(\lambda)$ for the injective hull of $\hat{L}_r(\lambda)$ as a G_rT-module we have, by [10, 3.2(11)(v)],

Theorem 1.8.3 *For $\lambda \in X(T)$, the module $\hat{Q}_r(\lambda)$ has a filtration by $\hat{Z}_r(\mu)$'s with multiplicities given by*

$$(\hat{Q}_r(\lambda) : \hat{Z}_r(\mu)) = [\hat{Z}_r(\mu) : \hat{L}_r(\lambda)]$$

for $\mu \in X(T)$. Thus two elements λ, $\mu \in X(T)$ belong to the same block of G_rT if, and only if, there exists a chain $\lambda = {}_1\lambda, \ldots, {}_t\lambda = \mu$ of elements of $X(T)$ such that for each $1 \leq i < t$ we have either $[\hat{Z}_r({}_i\lambda) : \hat{L}_r({}_{i+1}\lambda)] \neq 0$ or $[\hat{Z}_r({}_{i+1}\lambda) : \hat{L}_r({}_i\lambda)] \neq 0$.

1.9 Other quantisations of GL(n, k)

As remarked in Section 1.2, there exist other deformations of the general linear group that give rise to the q-Schur algebra. In particular there is the Manin quantisation (see [37] and [27]), whose representation theory has been extensively developed in [39]. In this section we define a two-parameter quantum group, following [44], which include both the Manin and Dipper–Donkin quantisations as special cases. The main result of this section describes when two such quantisations have isomorphic module categories. We then consider the infinitesimal case, and conclude this chapter with some remarks on the advantages of the respective quantisations. This section is largely based on [25].

We fix $\alpha, \beta \in k\backslash\{0\}$, and define $A_{\alpha,\beta}(n)$ to be the k-algebra generated by the n^2 indeterminates X_{ij}, with $1 \leq i, j \leq n$, subject to the relations

$$\begin{array}{lll} X_{ij}X_{il} & = \alpha X_{il}X_{ij} & \text{for } j > l, \\ X_{ji}X_{li} & = \beta X_{li}X_{ji} & \text{for } j > l, \\ X_{ij}X_{rs} & = \alpha^{-1}\beta X_{rs}X_{ij} & \text{for } i > r \text{ and } j < s, \\ X_{ij}X_{rs} & = (\alpha^{-1} - \beta)X_{is}X_{rj} + X_{rs}X_{ij} & \text{for } i < r \text{ and } j < s. \end{array}$$

Now consider the algebra maps δ and ϵ given on generators by

$$\delta : X_{ij} \longmapsto \sum_{t=1}^{n} X_{it} \otimes X_{tj},$$

$$\epsilon : X_{ij} \longmapsto \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Then it is routine to check that

Proposition 1.9.1 *$A_{\alpha,\beta}(n)$ is a bialgebra with comultiplication δ and counit ϵ.*

We will denote by $\mathrm{M}_{\alpha,\beta}(n,k)$, or just $M^{\alpha,\beta}$, the quantum monoid associated to this bialgebra. Setting the degree of each X_{ij} equal to one, it is clear that the relations above are homogeneous. Hence we can define $A_{\alpha,\beta}(n,d)$ to be the subspace of $A_{\alpha,\beta}(n)$ consisting of those elements of degree d. This is a subcoalgebra of $A_{\alpha,\beta}(n)$.

Next we define a *quantum determinant* in $A_{\alpha,\beta}(n)$. This is the element

$$D_{\alpha,\beta} = \sum_{\pi\in\Sigma_n} (-\alpha)^{-l(\pi)} X_{1,1\pi} X_{2,2\pi} \dots X_{n,n\pi},$$

where Σ_n denotes the symmetric group on n symbols, and l is the standard length function. Just as in the first section we can then localise and consider $A_{\alpha,\beta}(n)(D_{\alpha,\beta}^{-1})$. Once again we have

Theorem 1.9.2 *There exists a k-algebra anti-endomorphism σ which, along with δ and ϵ, endows $A_{\alpha,\beta}(n)(D_{\alpha,\beta}^{-1})$ with a Hopf algebra structure.*

Proof: See [25, Section 1] for further details.

We will denote by $\mathrm{GL}_{\alpha,\beta}(n,k)$ (or just $G^{\alpha,\beta}$) the quantum group associated to this Hopf algebra. As remarked in [44], we recover our original quantum group G by taking $\alpha = 1$ and $\beta = q$. The other special case we shall wish to consider is when $\alpha = \beta = q$; we call this the *Manin quantisation*, and will sometimes denote $G^{q,q}$ by $\mathrm{GL}_q(n,k)$.

Our next result verifies the claim in Section 1.2. Setting $\alpha\beta = q$ we have

Theorem 1.9.3 *There is a k-algebra isomorphism*

$$A_{\alpha,\beta}(n,d)^* \cong S_q(n,d).$$

Proof: See [25, Theorem 5.5]

We are now in a position to state the main result of this section. For $\alpha, \beta, \alpha', \beta' \in k\backslash\{0\}$ with $\alpha\beta = \alpha'\beta'$ we define

$$\xi = \frac{\alpha'}{\alpha} = \frac{\beta'}{\beta}.$$

The generators of $k[M^{\alpha,\beta}]$ and $k[M^{\alpha',\beta'}]$ will be denoted by X_{ij} and X'_{ij} respectively. For sequences $\mathbf{i} = (i_1, \ldots, i_r)$ and $\mathbf{j} = (j_1, \ldots, j_r)$ we write $X_{\mathbf{ij}}$ for $X_{i_1 j_1} \ldots X_{i_r j_r}$. For such an $\mathbf{i}$ we denote the number of inversions in $\mathbf{i}$ by $\tau(\mathbf{i})$. Then we have

Theorem 1.9.4 *With the assumptions above there is a homogeneous coalgebra isomorphism $\phi_\xi : k[M^{\alpha,\beta}] \longrightarrow k[M^{\alpha',\beta'}]$ given by*

$$X_{\mathbf{ij}} \longmapsto \xi^{\tau(\mathbf{i})-\tau(\mathbf{j})} X'_{\mathbf{ij}}$$

which determines a coalgebra isomorphism $\phi_\xi : k[G^{\alpha,\beta}] \longrightarrow k[G^{\alpha',\beta'}]$.

Proof: See [25, Proposition 2.1 and Theorem 2.4].

Corollary 1.9.5 *Under the above assumptions, the module categories* $\mathrm{Mod}(G^{\alpha,\beta})$ *and* $\mathrm{Mod}(G^{\alpha',\beta'})$ *are isomorphic.*

As observed in [25, Remark 2.7], we should note that this is not necessarily an isomorphism of tensor categories.

Remark 1.9.6 We will often wish to use the last result to translate between the Dipper–Donkin and Manin quantisations. By the above we have

$$\mathrm{Mod}(q^2\text{-}\mathrm{GL}(n,k)) \cong \mathrm{Mod}\ \mathrm{GL}_q(n,k).$$

When translating results across that depend on the value of l, it should be noted that these results may change. In particular, if q is a primitive lth root of unity with l even, then q^2 is a primitive $l/2$th root of unity. Thus results that for q-$\mathrm{GL}(n,k)$ depend on l will translate across to $\mathrm{GL}_q(n,k)$ as results that depend either on l, if l odd, or $l/2$ if l even.

Next we show that a Frobenius morphism can be defined in this setting. However, in general it is no longer a morphism from $G^{\alpha,\beta}$ to $\mathrm{GL}(n,k)$. For the rest of this section we assume that $q = \alpha\beta$ is a primitive lth root of unity. Let $\bar{\alpha}$ and $\bar{\beta}$ denote α^{l^2} and β^{l^2} respectively (and note that $\bar{\alpha}\bar{\beta} = 1$). As usual, we denote the generators of $k[M^{\alpha,\beta}]$ by X_{ij}; the generators of $k[M^{\bar{\alpha},\bar{\beta}}]$ will be denoted by x_{ij}. Then

Theorem 1.9.7 *With the assumptions above, there exists a bialgebra monomorphism* $\hat{F} : k[M^{\bar{\alpha},\bar{\beta}}] \longrightarrow k[M^{\alpha,\beta}]$ *sending* x_{ij} *to* X_{ij}^l*. Also,* $\hat{F}(D_{\bar{\alpha},\bar{\beta}}) = D_{\alpha,\beta}^l$*. This extends to a Hopf algebra homomorphism* $k[G^{\bar{\alpha},\bar{\beta}}] \longrightarrow k[G^{\alpha,\beta}]$*.*

Proof: See [25, Theorem 3.1].

This Frobenius map commutes with the earlier coalgebra isomorphisms. That is

Proposition 1.9.8 *With the assumptions above, let* $\alpha', \beta' \in k$ *satisfy* $\alpha'\beta' = \alpha\beta$*. Let* $\bar{\alpha}' = (\alpha')^{l^2}$*,* $\bar{\beta}' = (\beta')^{l^2}$ *and* $\bar{\xi} = \xi^{l^2}$*. Then the following diagram commutes:*

$$\begin{array}{ccc} k[G^{\bar{\alpha},\bar{\beta}}] & \xrightarrow{\hat{F}} & k[G^{\alpha,\beta}] \\ \phi_{\bar{\xi}}\Big\downarrow & & \Big\downarrow\phi_{\xi} \\ k[G^{\bar{\alpha}',\bar{\beta}'}] & \xrightarrow{\hat{F}'} & k[G^{\alpha',\beta'}]. \end{array}$$

Proof: See [25, Proposition 3.6].

If further we assume that $\alpha^l = \beta^l = 1$ then we have

Theorem 1.9.9 *With the assumptions above, the image* $\hat{F}(k[\mathrm{GL}(n,k)])$ *in* $k[G^{\alpha,\beta}]$ *is a factor group satisfying the assumptions of (1.4.2). In this case we can define the first (and higher) Frobenius kernels, and the corresponding Jantzen subgroups, just as in Section 1.3.*

Proof: See [25, Proposition 3.8].

When the hypotheses of the last theorem hold, we will denote by $G_r^{\alpha,\beta}$ (respectively $G_r^{\alpha,\beta}T$) the rth Frobenius kernel (respectively Jantzen subgroup).

Remark 1.9.10 Clearly the quantisation of Dipper and Donkin satisfies the last proposition whenever a Frobenius morphism can be defined. However, in the case of the Manin quantisation, this is not always so. In this case we require α (and hence also q) to be a primitive lth root of unity with l odd in order for the last result to hold. A discussion of the other possible cases can be found in [25, Examples 3.7].

Suppose now that q is an odd root of unity, and that k contains a square root α of q. Then by the last theorem and the remarks above, we can define infinitesimal and Jantzen subgroups of both G and $G^{\alpha,\alpha}$. From the explicit description of the isomorphism in (1.9.4) we obtain immediately

Theorem 1.9.11 *There is a coalgebra isomorphism* $\phi_\xi : k[G_r] \longrightarrow k[G_r^{\alpha,\alpha}]$ *determined by*

$$X_{\mathbf{ij}} \longmapsto \xi^{\tau(\mathbf{i})-\tau(\mathbf{j})} X'_{\mathbf{ij}}$$

and similarly a coalgebra isomorphism $\phi_\xi : k[G_rT] \longrightarrow k[G_r^{\alpha,\alpha}T]$.

Corollary 1.9.12 *In this case the module categories* $\mathrm{Mod}(G_r)$ *and* $\mathrm{Mod}(G_r^{\alpha,\alpha})$ *are isomorphic. A similar result holds for the corresponding Jantzen subgroups.*

There are a number of advantages to the Manin quantisation. Most notably for our purposes, these are precisely the two-parameter groups in which a quantum $\mathrm{SL}(n,k)$ can be constructed (see [25, Remark 1.10]). There are also certain technical simplifications in the theory as compared to the Dipper–Donkin quantisation, for example when discussing contravariant duality or the symmetric powers of the natural module. The representation theory of this quantisation has been extensively developed in [39].

In certain respects however, the Dipper–Donkin quantisation is preferable. First, the statement of many results for the Manin quantisation can be more complicated than for the Dipper–Donkin case, for the reason outlined in (1.9.10). However, the main disadvantage of the Manin quantisation is the necessity to restrict to the case of an odd root of unity in order to develop an infinitesimal theory. Although one can work round this problem, this tends to introduce new complications (see [3] for an example of this). For this reason we will work with the Dipper–Donkin quantisation, occasionally translating results between quantisations using (1.9.5).

Chapter 2

Existence of extensions

In [26], Erdmann has calculated Ext^1_G between Weyl modules for $\mathrm{SL}(2,k)$. In this chapter we generalise this result to solve the corresponding problem for q-$\mathrm{GL}(2,k)$. We also show that our result also holds for the Manin quantisation. To apply the methods of [26], it is necessary to first determine the block structure of q-$\mathrm{GL}(2,k)$. In Chapter 4 we will determine these blocks for general n; however in the case $n = 2$ it is easy to derive the result directly, and as this will be needed for the general case we include it here. The result is derived from the analysis of the subcomodule structure of the symmetric powers in [46].

We also need a quantum analogue of two short exact sequences from [47], which we give in the second section. With these results, the argument now follows much as in [26]; we consider the infinitesimal case, and then use the Lyndon–Hochschild–Serre spectral sequence to obtain the desired result. Finally we show how the result also holds for the Manin quantisation.

It should be noted that the result here uses the classical case, so is not independent of that in [26]. In particular we shall assume that $l > 1$. Our standing assumption that q is a root of unity is no great restriction here, as we already know from [20, 4(8)] that non-trivial extensions do not exist in the non-root of unity case. The only real difference in the arguments used here as compared with those in [26] occurs in (2.3.8) where the original methods do not generalise, so we use a more direct argument. There is also an unfortunate typographical error in the statement of the main result in [26].

2.1 The blocks of q-GL$(2,k)$

The first part of this section depends on the submodule structure of the symmetric powers as described in [46]. We begin by recalling some notation from that paper. Let E be the quantum analogue of the natural module for GL$(2,k)$, with basis $\{e_1, e_2\}$. Given a basis element $e^a = e_1^{a_1} e_2^{a_2} \in \mathrm{S}_q^r(\mathrm{E})$ we write

$$\begin{array}{llll} a_i = a_i^1 l + a_i^0 & \text{with } 0 \le a_i^0 < l & \text{and } a_i^1 = \sum_j a_i^{1,j} p^j & \text{with } 0 \le a_i^{1,j} < p \quad \forall i,j, \\ r = r_1 l + r_0 & \text{with } 0 \le r_0 < l & \text{and } r_1 = \sum_j r_1^j p^j & \text{with } 0 \le r_1^j < p \quad \forall j. \end{array}$$

Set $m = \max\{0, j \mid r_1^j > 0\}$. We define the *carry pattern* $c(e^a) = (c_0(e^a), \ldots, c_m(e^a))$ recursively using

$$\left.\begin{array}{r} a_1^0 + a_2^0 = c_0(e^a) l + r_0 \\ c_{t-1}(e^a) + a_1^{1,t-1} + a_2^{1,t-1} = c_t(e^a) p + r_1^{t-1} \end{array}\right\} \tag{2.1}$$

Let $\mathrm{C}(r) = \{c(e^a) \mid e^a \in \mathrm{S}_q^r(\mathrm{E})\}$. The submodules of $\mathrm{S}_q^r(\mathrm{E})$ correspond to order closed subsets of $\mathrm{C}(r)$, where $c \le c'$ if $c_i \le c'_i$ for all i. The results of [46], along with [21, Lemma 3], give $(\mathrm{c}_0, \ldots, \mathrm{c}_m) \in \mathrm{C}(r)$ if, and only if,

$$\begin{array}{rl} c_0 \in \{0, \ldots, \mathrm{M}\}, & \\ 0 \le c_k \le \sum_{j \ge k} r_1^j p^{j-k} & \text{for } 1 \le k \le m, \\ 0 \le r_1^k + p c_{k+1} - c_k \le 2p - 2 & \text{for } 0 \le k \le m, \end{array} \tag{2.2}$$

where we set $c_{m+1} = 0$ and $\mathrm{M} = \begin{cases} 0 & \text{if } r < l-1, \\ 1 & \text{if } r > l-1 \quad \text{and } r_0 \ne l-1, \\ 0 & \text{otherwise.} \end{cases}$

From (2.1) it is easy to determine the highest weight $a = (a_1, a_2)$ such that $c(e^a) = c$; call this the *highest weight in* c. We obtain

$$\begin{array}{rl} a_1^0 = & \min\{l-1, r_0 + l c_0\}, \\ a_1^{1,t-1} = & \min\{p-1, r_1^{t-1} - c_{t-1} + p c_t\}. \end{array} \tag{2.3}$$

Theorem 2.1.1 *A weight $a = (a_1, a_2)$ is linked to $(r+d, d)$ if, and only if, the following conditions hold.*

i) $a_1 + a_2 = r + 2d$.

ii) $\bar{a} \equiv \pm \bar{r} \pmod{2l}$.

iii) If $\bar{a} \equiv 0 \pmod{l}$ *then* $\bar{a} \equiv \pm l p^t (r_1^t + 1) \pmod{p^{t+1}}$,

where $\bar{a} := a_1 - a_2 + 1$, $\bar{r} := r + 1$ *and* $t := \max\{0, s \mid \bar{r} \equiv 0 \pmod{p^s}\}$.

Proof: The statement of the linkage condition in terms of equivalence classes under the relation generated by

$$\lambda \sim \mu \text{ if } [\nabla(\lambda) : \mathrm{L}(\mu)] \neq 0,$$

implies that i) must hold. Note that i) implies i$'$) $\bar{a} \equiv \bar{r} \pmod 2$. For the necessity of ii) and iii), we show that $[\nabla(r+d,d) : \mathrm{L}(a)] \neq 0$ implies both ii) and iii), as then this must clearly be true for every element of the equivalence class generated by $(r+d,d)$ under $\sim$. Further we may assume that $d=0$ as we can tensor with an appropriate power of the q-determinant to get the general result.

Necessity of ii): Let $c \in \mathrm{C}(r)$, and a be the highest weight in c. We have

$$a_1^0 = \begin{cases} r_0 & \text{if } c_0 = 0, \\ l-1 & \text{if } c_0 = 1. \end{cases}$$

But $a_1 + a_2 = r$ implies $a_2^0 = \begin{cases} 0 & \text{if } c_0 = 0, \\ r_0 + 1 & \text{if } c_0 = 1. \end{cases}$ Hence we have

$$a_1^0 - a_2^0 = \begin{cases} r_0 & \text{if } c_0 = 0, \\ l - r_0 - 2 & \text{if } c_0 = 1. \end{cases} \tag{2.4}$$

If l is odd then (2.4) implies that $\bar{a} \equiv \pm\bar{r} \pmod l$, and this together with i$'$) gives the necessity of ii). If l is even then p is odd (as $(l,p)=1$). Now by (2.1) we have

$$\begin{aligned} a_1 - a_2 + 1 &= a_1^0 - a_2^0 + l\left(\textstyle\sum_{j=0}^m p^j (a_1^{1,j} - a_2^{1,j})\right) + 1 \\ &= a_1^0 - a_2^0 + 1 + l\left(\textstyle\sum_{j=0}^m p^j (c_{j+1}p + r_1^j - c_j - 2a_2^{1,j})\right) \\ &= a_1^0 - a_2^0 + 1 + l\phi, \end{aligned}$$

where $\phi = \sum_{j=0}^m p^j (c_{j+1}p + r_1^j - c_j - 2a_2^{1,j})$. So using (2.4) we obtain

$$\bar{a} = \begin{cases} r_0 + 1 + l\phi & \text{if } c_0 = 0, \\ -(r_0+1) + l(\phi+1) & \text{if } c_0 = 1. \end{cases}$$

Also we have that

$$\bar{r} = \begin{cases} r_0 + 1 + l \pmod{2l} & \text{if } r_1 \text{ odd}, \\ r_0 + 1 \pmod{2l} & \text{if } r_1 \text{ even}. \end{cases}$$

So it is enough to show that ϕ satisfies

$$\phi \equiv \begin{cases} 1 \pmod 2 & \text{if } c_0 + r_1 \text{ odd}, \\ 0 \pmod 2 & \text{if } c_0 + r_1 \text{ even}. \end{cases} \tag{2.5}$$

As we are only interested in ϕ mod 2, and p is odd, we can replace ϕ by $\hat{\phi}$ where

$$\hat{\phi} = \sum_{j=0}^{m}(c_{j+1} + r_1^j - c_j) = \sum_{j=0}^{m} r_1^j + c_{m+1} - c_0 = r_1 - c_0,$$

which satisfies (2.5). So ii) is necessary.

Necessity of iii): If $r_0 = l-1$ then we have $c_0 = 0$. From (2.2) we have $0 \leq p-1+pc_{s+1}-c_s \leq 2p-2$ for all $s \leq t-1$. So by induction we have $c_s = 0$ for all $s \leq t$. Hence for a the highest weight in c we have

$$\begin{array}{lll} a_1^{1,s} & = p-1 & \forall s \leq t-1, \\ a_1^{1,t} & = \min\{p-1, r_1^t + pc_{t+1}\} & = \begin{cases} r_1^t & \text{if } c_{t+1} = 0, \\ p-1 & \text{otherwise.} \end{cases} \end{array} \tag{2.6}$$

Note that this implies that $a_2^{1,s} = 0$ for all $s \leq t-1$. Now $a_1 + a_2 = r$ implies that

$$\begin{aligned} a_1 + a_2 & \equiv r_0 + l(r_1^0 + pr_1^1 + \cdots + p^t r_1^t) && \pmod{p^{t+1}} \\ & \equiv l(1+p-1+p^2-\cdots+p^t-p^{t-1}+p^t r_1^t) - 1 && \pmod{p^{t+1}} \\ & \equiv lp^t(r_1^t+1) - 1 && \pmod{p^{t+1}}. \end{aligned}$$

Similarly we have

$$\begin{aligned} a_1 + a_2 & \equiv a_1^0 + a_2^0 + l(a_1^{1,0} + a_2^{1,0} + \cdots + p^t a_1^{1,t} + p^t a_2^{1,t}) && \pmod{p^{t+1}} \\ & \equiv l(1+p-1+p^2-\cdots+p^t-p^{t-1}+p^t(a_1^{1,t}+a_2^{1,t})) - 1 && \pmod{p^{t+1}} \\ & \equiv lp^t(a_1^{1,t}+a_2^{1,t}+1) - 1 && \pmod{p^{t+1}}. \end{aligned}$$

These give

$$\begin{aligned} lp^t(r_1^t+1) - 1 & \equiv lp^t(a_1^{1,t}+a_2^{1,t}+1) - 1 && \pmod{p^{t+1}} \\ lp^t r_1^t & \equiv lp^t(a_1^{1,t}+a_2^{1,t}) && \pmod{p^{t+1}} \\ r_1^t & \equiv a_1^{1,t}+a_2^{1,t} && \pmod{p}. \end{aligned}$$

Then (2.6) implies that $a_2^{1,t} \equiv \begin{cases} 0 & \pmod{p} \quad \text{if } c_{t+1} = 0, \\ r_1^t + 1 & \pmod{p} \quad \text{if } c_{t+1} \neq 0, \end{cases}$ and hence we get

$$\begin{aligned} a_1 - a_2 & \equiv a_1^0 - a_2^0 + l(a_1^{1,0} - a_2^{1,0} + \cdots + p^t(a_1^{1,t} - a_2^{1,t})) && \pmod{p^{t+1}} \\ & \equiv l(1+p-1+p^2-\cdots+p^t-p^{t-1}+p^t(a_1^{1,t}-a_2^{1,t})) - 1 && \pmod{p^{t+1}} \\ & \equiv lp^t(a_1^{1,t}-a_2^{1,t}+1) - 1 && \pmod{p^{t+1}} \\ & \equiv \begin{cases} +lp^t(r_1^t+1) - 1 & \text{if } c_{t+1} = 0, \\ -lp^t(r_1^t+1) - 1 & \text{if } c_{t+1} \neq 0 \end{cases} && \pmod{p^{t+1}} \end{aligned}$$

as required. So i)–iii) are necessary.

For sufficiency: Consider $\nabla(a,b) \cong \nabla(a-b,0) \otimes (q\text{-det})^b$. If this is not irreducible then its submodule structure is determined by that of $\nabla(a-b,0)$. This must have a composition factor with highest weight (c,d) such that $0 \leq c-d < a-b$. Thus (a,b) is linked to whatever $(c+b, d+b)$ is; so it is enough to consider (c,d) and tensor up with an appropriate power of the q-determinant. Continuing this descent, the sequence must terminate in an irreducible module. Hence it is sufficient to show that there is a unique irreducible $\nabla(a_1, a_2)$ satisfying the conditions. In fact, we need only consider $\nabla(a_1 - a_2, 0)$ with i) replaced by i′), as if this is unique then tensoring up will give the result.

Let $r = a_1 - a_2 = r_0 + l(r_1^0 + \cdots + p^m r_1^m)$. It is necessary to determine which $\mathrm{S}_q^r(\mathrm{E})$ are irreducible. By Steinberg's Tensor Product Theorem, we have

$$\begin{aligned} \dim \mathrm{L}(r,0) &= (r_0+1)(r_1^0+1)\cdots(r_1^m+1) \\ &= 1 + r_0 + (r_0+1)[r_1^0 + (r_1^0+1)r_1^1 + \cdots + \left(\textstyle\prod_{i=0}^{m-1}(r_1^i+1)\right) r_1^m]. \end{aligned}$$

As $\mathrm{soc}\nabla(\lambda) = \mathrm{L}(\lambda)$ we require that $\dim \mathrm{S}_q^r(\mathrm{E}) = \dim \mathrm{L}(r,0) = r+1$. Hence we require that

$$r + 1 = 1 + r_0 + (r_0+1)[r_1^0 + (r_1^0+1)r_1^1 + \cdots + \left(\prod_{i=0}^{m-1}(r_1^i+1)\right) r_1^m].$$

That is

$$r_0 + 1 + l[r_1^0 + \cdots + p^m r_1^m] = 1 + r_0 + (r_0+1)[r_1^0 + \cdots + \left(\prod_{i=0}^{m-1}(r_1^i+1)\right) r_1^m].$$

This holds if, and only if, either $r_1^i = 0$ for all i, or $r_0 + 1 = l$ and $r_1^i + 1 = p$ for $0 \leq i \leq m-1$. Hence $\mathrm{S}_q^r(\mathrm{E})$ is irreducible precisely when $r \leq l-1$ or $r = lp^m(r_1^m+1)-1$. Amongst these r there is a unique one satisfying the required conditions, and so we are done.

We record from the above proof the following fact:

Corollary 2.1.2 *For all $r \geq 0$, we have $\mathrm{S}_q^r(\mathrm{E})$ is irreducible if, and only if, $r \leq l-1$ or $r = lp^m(r_1^m+1)-1$.*

We also use the results of [46] to prove the following lemma, which will be needed later.

Lemma 2.1.3 *If $\lambda_1 + \lambda_2 = 2s$ then we have*

$$\mathrm{Hom}_G(\Delta(s,s), \Delta(\lambda)) \cong \begin{cases} k & \text{if } \lambda_1 - \lambda_2 = 2(lp^m - 1) \text{ or } 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Since $\Delta(\lambda_1,\lambda_2)\cong\nabla^*(-\lambda_2,-\lambda_1)$, this will follow from

$$\nabla(s,s) \text{ occurs in } \mathrm{hd}\nabla(\lambda) \text{ if, and only if, } \lambda_1-\lambda_2=2(lp^m-1) \text{ or } \ 0,$$

once we have shown that $\nabla(\lambda)$ has a simple head. Clearly, it is enough to show this when $\lambda_2=0$, as then the result follows by tensoring up with an appropriate power of the q-determinant. Hence we will work with $\nabla(r,0)$. By the last proposition we have $1\equiv\pm(r+1) \pmod{2l}$; that is $r=2lm$ or $2lm-2$ for some m. We first find a c maximal in C, say $c_{\max}=(c_0,\ldots,c_m)$. From (2.2) we have $c_0\in\{0,1\}$ unless $r<l-1$, in which case we must have $r=0$.

By induction on t we have that if $r\neq 0$ then $c_t\in\{0,1\}$ for all $t\leq m$. This follows as for $1\leq t\leq m$ the first condition of (2.2) is clearly satisfied by 0 and 1, while the second gives $0\leq pc_{t+1}\leq 2p-2+c_t-r_1^t\leq 2p-1$, by induction. Hence $0\leq c_t<2$, as claimed. Suppose $c_t=1$; then $0\leq pc_{t+1}\leq p+(p-1)-r_1^t=p+\epsilon$ with $\epsilon\geq 0$. So $c_t=1$ implies that c_{t+1} can equal 1 (for $t<m$). Hence $c_{\max}$ is unique, and is either 0 or $\mathbf{1}=(1,\ldots,1)$, which implies that $\nabla(r,0)$ has a simple head. The zero case corresponds to $r=0$.

Suppose that $c_{\max}=\mathbf{1}$, and let a be the highest weight in $c_{\max}$. Then (2.3) implies that $a_1^0=l-1$, and $a_1^{1,t}=\begin{cases} p-1 & \text{if } t\leq m-1,\\ r_1^m-1 & \text{if } t=m.\end{cases}$ We require that

$$\begin{aligned} r &= 2a_1=2a_1^0+\textstyle\sum_{t=0}^m 2a_1^{1,t}p^tl\\ &=2l-2+2l\left(\textstyle\sum_{t=0}^{m-1}(p^{t+1}-p^t)+p^m(r_1^m-1)\right)\\ &=2(lr_1^mp^m-1).\end{aligned}$$

Thus $r_0=l-2$, and $\sum_{t=0}^m r_1^tp^t=2r_1^mp^m-1$, which implies that $\sum_{t=0}^{m-1} r_1^tp^t=r_1^mp^m-1$. This forces

$$r_1^t=\begin{cases} p-1 & \text{if } 0\leq t\leq m-1,\\ 1 & \text{if } t=m,\end{cases}$$

which gives $r=2lp^m-2$ as required.

2.2 Two short exact sequences

This section, largely based on results in [47], will produce two short exact sequences of G-modules which are essential to our later results. We first fix some notation that shall be used henceforth in this chapter.

We set $\lambda = (\lambda_1, \lambda_2) = (\mu + \delta, \delta)$, where $0 \leq \mu \leq l - 2$, and put $|\lambda| = \lambda_1 + \lambda_2$. Then $\bar{\mu}$ is defined to be the unique integer such that $\mu + \bar{\mu} = l - 2$. Recall that in this case $\rho = (1, 0)$ and $\varpi = (1, 1)$. Finally, we define $\tilde{\lambda} = (\bar{\mu} + \delta, \delta) + (\mu - l + 1)\varpi = (\lambda_2 - 1, \lambda_1 + 1 - l)$. Note that $\tilde{\tilde{\lambda}} = \lambda - l\varpi$.

Proposition 2.2.1 *i) For $n > 0$ there exists a (non-split) short exact sequence of G-modules:*

$$0 \to \nabla(\lambda) \otimes \nabla(n\rho)^{\mathrm{F}} \to \nabla(\lambda + ln\rho) \to \nabla(\tilde{\lambda} + l\varpi) \otimes \nabla((n-1)\rho)^{\mathrm{F}} \to 0.$$

ii) There is an isomorphism of G-modules:

$$\nabla(ln - 1 + \delta, \delta) \cong \nabla(l - 1 + \delta, \delta) \otimes \nabla((n-1)\rho)^{\mathrm{F}}.$$

Proof: Part i): It is enough to show that we have the short exact sequence:

$$0 \to \nabla(\mu, 0) \otimes \nabla(n\rho)^{\mathrm{F}} \to \nabla(\mu + ln, 0) \to \nabla(l - 1, \mu + 1) \otimes \nabla((n-1)\rho)^{\mathrm{F}} \to 0,$$

as the result follows on tensoring up with an appropriate power of the q-determinant. Now we use the isomorphism noted in [20, 3.7] of $\nabla(\mu, 0)$ with k-span$\{c_{11}^{r_1} c_{12}^{r_2} \mid r_1 + r_2 = \mu\}$. This gives the first injection via the multiplication map.

Now consider

$$\phi: \ \nabla(\mu + l, 0) \otimes \nabla(n - 1, 0)^{\mathrm{F}} \xrightarrow{\mathbf{m}} \nabla(\mu + nl, 0) \xrightarrow{\mathbf{p}} \frac{\nabla(\mu + nl, 0)}{\nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}}},$$

where $\mathbf{m}$ is multiplication of polynomials and $\mathbf{p}$ is the natural projection. We first show that ϕ is surjective. Let $c_{11}^a c_{12}^b + \nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}}$ be a non-zero element of $\nabla(\mu + nl, 0)/(\nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}})$. Suppose $a = a_1 + la_2, \quad b = b_1 + lb_2$, where $0 \leq a_1, b_1 \leq l - 1$. Then $ln + \mu = a + b$ implies that $a_1 + b_1 = \mu$ or $l + \mu$. The former is impossible as then $c_{11}^a c_{12}^b + \nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}} = 0$. Hence $a_1 + b_1 = l + \mu$. Then, under ϕ, the element $c_{11}^{a_1} c_{12}^{b_1} \otimes c_{11}^{la_2} c_{12}^{lb_2} \in \nabla(\mu + l, 0) \otimes \nabla((n-1)\rho)^{\mathrm{F}}$ has image $c_{11}^a c_{12}^b + \nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}}$. Hence ϕ is surjective as claimed.

Clearly $\nabla(\mu, 0) \otimes \nabla(1, 0)^{\mathrm{F}} \otimes \nabla(n - 1, 0)^{\mathrm{F}} \subseteq \ker \phi$. But then

$$\frac{\nabla(\mu + l, 0) \otimes \nabla(n - 1, 0)^{\mathrm{F}}}{\nabla(\mu, 0) \otimes \nabla(1, 0)^{\mathrm{F}} \otimes \nabla(n - 1, 0)^{\mathrm{F}}} \cong \left(\frac{\nabla(\mu + l, 0)}{\nabla(\mu, 0) \otimes \nabla(1, 0)^{\mathrm{F}}} \right) \otimes \nabla(n - 1, 0)^{\mathrm{F}}$$

has dimension $n(\bar{\mu}+1)$. Also

$$\dim \frac{\nabla(\mu+nl,0)}{\nabla(\mu,0)\otimes\nabla(n,0)^{\mathrm{F}}} = n(\bar{\mu}+1).$$

Hence $\ker\phi = \nabla(\mu,0)\otimes\nabla(1,0)^{\mathrm{F}}\otimes\nabla(n-1,0)^{\mathrm{F}}$. So

$$\begin{aligned}
\operatorname{Im}\phi &\cong \frac{\nabla(\mu+nl,0)}{\nabla(\mu,0)\otimes\nabla(n,0)^{\mathrm{F}}} \\
&\cong \frac{\nabla(\mu+l,0)\otimes\nabla(n-1,0)^{\mathrm{F}}}{\nabla(\mu,0)\otimes\nabla(1,0)^{\mathrm{F}}\otimes\nabla(n-1,0)^{\mathrm{F}}} \cong \left(\frac{\nabla(\mu+l,0)}{\nabla(\mu,0)\otimes\nabla(1,0)^{\mathrm{F}}}\right)\otimes\nabla(n-1,0)^{\mathrm{F}}.
\end{aligned}$$

So it remains to show that

$$\frac{\nabla(\mu+l,0)}{\nabla(\mu,0)\otimes\nabla(1,0)^{\mathrm{F}}} \cong \nabla(l-1,\mu+1) \quad \left[\,\cong \nabla(\bar{\mu},0)\otimes q\text{-}\det{}^{\mu+1}\,\right].$$

As the right-hand side is simple, it is enough to show that these have the same character, which is a straightforward calculation.

Part ii): The map is injective as in i), and the result then follows by dimension.

We will need the following properties, shown in [10, 3.3–3.4], of the modules there denoted $Q(\lambda)$. These are certain tilting modules whose restrictions to G_1 are the injective envelopes of the corresponding simples. There $Q(\lambda)$ is defined to be $T(\tilde{\lambda}+l\rho)$, which is the indecomposable tilting module of highest weight $\tilde{\lambda}+l\rho$. Further, from the character formula for the $Q(\lambda)$'s we obtain that $\mathrm{ch}Q(\lambda)=\chi(\lambda)+\chi(\tilde{\lambda}+l\rho)$. From [46] we see that $\mathrm{hd}\nabla(\tilde{\lambda}+l\rho)\cong L(\lambda)$ and $\mathrm{soc}\nabla(\tilde{\lambda}+l\rho)=\mathrm{rad}\nabla(\tilde{\lambda}+l\rho)\cong L(\tilde{\lambda}+l\rho)$. We also have that $\mathrm{soc}Q(\lambda)\cong L(\lambda)$ and $Q(\lambda)^*\cong Q(\lambda)\otimes q\text{-}\det^{-|\lambda|}$. We first prove

Lemma 2.2.2 *For all $n\geq 0$, the module $Q(\lambda)\otimes\nabla(n,0)^{\mathrm{F}}$ has a good filtration.*

Proof: By the above remarks, $Q(\lambda)\cong T((\bar{\mu}+l)\rho+\delta'\varpi)$ for some $\delta'\in\mathbb{Z}$. We may assume that $\delta'=0$, as the general case will follow by tensoring up with an appropriate power of q-det. Now $St_1\otimes\nabla(\bar{\mu}+1,0)$ is a tilting module by [20, 4(3)(i)], and hence

$$St_1\otimes\nabla(\bar{\mu}+1,0)\cong T(l+\bar{\mu},0)\oplus\text{other terms.} \tag{2.7}$$

For $n\geq 0$ consider $\bar{\nabla}(n,0)^F\otimes St_1\otimes\nabla(\bar{\mu}+1,0)$. This is isomorphic to $\nabla(p-1+pn,0)\otimes\nabla(\bar{\mu}+1,0)$, and hence has a good filtration. As any direct summand of a module with a good filtration has a good filtration [20, Appendix A2 Proposition 1(vi)], we see from

(2.7) that $\bar{\nabla}(n,0)^F \otimes T(l+\bar{\mu},0)$ has a good filtration. Now by (1.9.9) along with [39, Remarks before (2.4.1)], we have that $\bar{\nabla}(n,0)^F \otimes T(l+\bar{\mu},0) \cong T(l+\bar{\mu},0) \otimes \bar{\nabla}(n,0)^F$ which gives the result.

With this we can now show

Proposition 2.2.3 *For $n \geq 0$ there exists a non-split short exact sequence of G-modules:*

$$0 \to \nabla(\lambda + ln\rho) \to Q(\lambda) \otimes \nabla(n\rho)^{\mathrm{F}} \to \nabla(\tilde{\lambda} + l(n+1)\rho) \to 0.$$

Proof: Using [10, 3.3(5)] we have that $Q(\lambda) \otimes \nabla(n\rho)^{\mathrm{F}}$ is indecomposable. So, as we have from (1.6.4) that $\mathrm{Ext}^1_G(\nabla(\alpha), \nabla(\beta)) \neq 0$ implies that $\alpha > \beta$, it is enough to prove the above at the level of characters. We use induction on n. The case $n = 0$ is clear from the remarks above, while $n = 1$ follows by direct calculation.

For $n > 1$ recall that

$$\begin{aligned} \mathrm{ch}\nabla(n,0) &= e(n,0) + \cdots + e(0,n) \\ &= e(n,0) + e(0,n) + \mathrm{ch}\nabla(n-2,0)\chi(1,1). \end{aligned}$$

Hence we have

$$\mathrm{ch}\,(Q(\lambda) \otimes \nabla(n,0)^{\mathrm{F}}) = \mathrm{ch}\,(Q(\lambda) \otimes \nabla(n-2,0)^{\mathrm{F}})\,\chi(l,l) + \mathrm{ch}Q(\lambda)(e(ln,0) + e(0,ln))$$

$$= \mathrm{ch}\nabla(\lambda + (n-2)l\rho)\chi(l,l) + \mathrm{ch}\nabla(\tilde{\lambda} + (n-1)l\rho)\chi(l,l) + \mathrm{ch}Q(\lambda)(e(ln,0) + e(0,ln))$$

$$= \sum_{i=0}^{\mu+(n-2)l} e(\mu+\delta+(n-1)l-i, \delta+l+i) + \sum_{i=0}^{2l-2-\mu+(n-2)l} e(ln-1-i+\delta, \mu+1+i+\delta)$$

$$+ \sum_{i=0}^{\mu} (e(\mu+\delta-i+nl, \delta+i) + e(\mu+\delta-i, \delta+i+nl))$$

$$+ \sum_{i=0}^{2l-2-\mu} (e(\delta+(n+1)l-1-i, \mu+\delta+1+i-l) + e(\delta+l-1-i, \mu+\delta+1+i+(n-1)l)).$$

Taking the second and third terms we get $\mathrm{ch}\nabla(\lambda + ln\rho)$, and the rest give $\mathrm{ch}\nabla(\tilde{\lambda} + l(n+1)\rho)$, so the result follows by induction.

After dualising, and tensoring with appropriate powers of the q-determinant, we may rewrite the last two propositions in terms of Δ's as

Proposition 2.2.4 *i) For $n > 0$ there exists a (non-split) short exact sequence of G-modules:*

$$0 \to \Delta((n-1)\rho)^{\mathrm{F}} \otimes \Delta(\tilde{\lambda} + l\varpi) \to \Delta(\lambda + ln\rho) \to \Delta(n\rho)^{\mathrm{F}} \otimes \Delta(\lambda) \to 0.$$

ii) There is an isomorphism of G-modules:

$$\Delta(ln - 1 + \delta, \delta) \cong \Delta((n-1)\rho)^{\mathrm{F}} \otimes \Delta(l - 1 + \delta, \delta).$$

Proposition 2.2.5 *For $n \geq 0$ there exists a non-split short exact sequence of G-modules:*

$$0 \to \Delta(\tilde{\lambda} + l(n+1)\rho) \to \Delta(n\rho)^{\mathrm{F}} \otimes Q(\lambda) \to \Delta(\lambda + ln\rho) \to 0.$$

Corollary 2.2.6 *Considered as G_1-modules, the central term of the above sequence is the projective cover (respectively injective envelope) of the right (respectively left) term.*

Proof: As G_1-modules, the $Q(\lambda)$'s are projective by [10, 3.3(2)], and hence also injective (as $Q(\lambda)^* \cong Q(\lambda) \otimes q\text{-det}^{-|\lambda|}$). Thus $Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}$ is also both projective and injective. To show that $Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}$ is the projective cover (respectively injective envelope) of the appropriate module in the last proposition, it thus suffices to prove:

$$\begin{array}{rll} \text{i)} & \mathrm{hd}_{G_1}\left(Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}\right) & \cong \mathrm{hd}_{G_1}\Delta(\lambda + ln\rho), \\ \text{ii)} & \mathrm{soc}_{G_1}\left(Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}\right) & \cong \mathrm{soc}_{G_1}\Delta(\tilde{\lambda} + l(n+1)\rho). \end{array}$$

In both cases the previous proposition gives one inclusion.

Consider i): As $\Delta(n\rho)^{\mathrm{F}}$ has trivial G_1 action we have

$$\begin{aligned} \mathrm{hd}_{G_1}\left(Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}\right) & \cong \mathrm{hd}_{G_1}\left(Q(\lambda)\right) \otimes \Delta(n\rho)^{\mathrm{F}} \\ & \cong \hat{L}_1(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}} \\ & \cong L_1(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{hd}_{G_1}\Delta(\lambda + ln\rho) & \geq \mathrm{hd}_{G_1}\left(\Delta(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}\right) \qquad \text{(by (1.1)(i))} \\ & \cong \mathrm{hd}_{G_1}\left(\Delta(\lambda)\right) \otimes \Delta(n\rho)^{\mathrm{F}} \\ & \cong L_1(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}. \end{aligned}$$

Consider ii). By a similar argument we have

$$\begin{aligned}\mathrm{soc}_{G_1}(Q(\lambda)\otimes\Delta(n\rho)^{\mathrm{F}}) &\cong \mathrm{soc}_{G_1}(Q(\lambda))\otimes\Delta(n\rho)^{\mathrm{F}}\\ &\cong \hat{L}_1(\lambda)\otimes\Delta(n\rho)^{\mathrm{F}}\\ &\cong L_1(\lambda)\otimes\Delta(n\rho)^{\mathrm{F}},\end{aligned}$$

and

$$\begin{aligned}\mathrm{soc}_{G_1}\Delta(\tilde{\lambda}+l(n+1)\rho) &\cong \mathrm{soc}_{G_1}\Delta((\lambda_2-1,\lambda_1-l+1)+l(n+1)\rho)\\ &\geq \mathrm{soc}_{G_1}(\Delta(\lambda)\otimes\Delta(n\rho)^{\mathrm{F}}) \qquad \text{(by (1.1)(i))}\\ &\cong \mathrm{soc}_{G_1}(\Delta(\lambda))\otimes\Delta(n\rho)^{\mathrm{F}}\\ &\cong L_1(\lambda)\otimes\Delta(n\rho)^{\mathrm{F}}.\end{aligned}$$

These give the reverse inclusions.

2.3 Calculations for G_1

If M is an indecomposable, non-projective G_1-module, we denote the kernel of the projective cover by $\Omega(M)$, and the cokernel of the injective hull by $\Omega^{-1}(M)$. We have $\Omega\Omega^{-1}(M)\cong M\cong\Omega^{-1}\Omega(M)$, and $\mathrm{Ext}^1_{G_1}(A,B)\cong\mathrm{Ext}^1_{G_1}(\Omega^{-1}A,\Omega^{-1}B)$ for arbitrary G_1-modules A,B. From (2.2.5), along with the remark that $\tilde{\tilde{\lambda}}=\lambda-l\varpi$, we can determine $\Omega^n(\Delta(\lambda))$. We obtain

$$\Omega^n(\Delta(\lambda))\cong\begin{cases}\Delta(\lambda-\frac{nl}{2}\varpi+nl\rho) & \text{if } n \text{ even,}\\ \Delta(\tilde{\lambda}-\frac{(n-1)l}{2}\varpi+nl\rho) & \text{if } n \text{ odd.}\end{cases}\tag{2.8}$$

Lemma 2.3.1 *For $m\geq n\geq 0$ we have*

$$\Omega^{-n}\Delta(\tilde{\lambda}+l(m+1)\rho)\cong\begin{cases}\Delta(\tilde{\lambda}+\frac{nl}{2}\varpi+(m+1-n)l\rho) & \text{if } n \text{ even,}\\ \Delta(\lambda+\frac{(n-1)l}{2}\varpi+(m+1-n)l\rho) & \text{if } n \text{ odd.}\end{cases}$$

Proof: We have

$$\Delta(\tilde{\lambda}+l(m+1)\rho)\cong\begin{cases}\Omega^{m+1}\Delta(\lambda+\frac{ml}{2}\varpi) & \text{if } m \text{ even,}\\ \Omega^{m+1}\Delta(\tilde{\lambda}+\frac{(m+1)l}{2}\varpi) & \text{if } m \text{ odd.}\end{cases}$$

So

$$\Omega^{-n}\Delta(\tilde{\lambda}+l(m+1)\rho)\cong\begin{cases}\Omega^{m+1-n}\Delta(\lambda+\frac{ml}{2}\varpi) & \text{if } m \text{ even,}\\ \Omega^{m+1-n}\Delta(\tilde{\lambda}+\frac{(m+1)l}{2}\varpi) & \text{if } m \text{ odd.}\end{cases}$$

The result now follows from (2.8), replacing λ by $\tilde{\lambda}$ for the case m odd.

The rest of this section is devoted to calculating Hom_{G_1} and $\mathrm{Ext}^1_{G_1}$ between various Weyl modules, for use in the next section. We write $\cong_{G_1}$ for an isomorphism of G_1-modules, and use t to denote an integer.

Lemma 2.3.2 *For $n \geq 0$ we have*

$$\mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), \Delta(\tilde{\lambda} + l(n+1)\rho)) \cong \begin{cases} \left(\Delta(n\rho) \otimes q\text{-}\det^{-u}\right)^{\mathrm{F}} & \text{if } t \equiv 0 \pmod{l}, \\ 0 & \text{otherwise,} \end{cases}$$

where $lu = t$.

Proof: As $\Delta(\lambda + t\varpi)$ is simple, and (2.2.5) gives the injective envelopes, we have

$$\begin{aligned} &\mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), \Delta(\tilde{\lambda} + l(n+1)\rho)) \\ &\quad\cong \mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), \Delta(n\rho)^{\mathrm{F}} \otimes Q(\lambda)) \\ &\quad\cong \Delta(n\rho)^{\mathrm{F}} \otimes \mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), Q(\lambda)). \end{aligned}$$

Now $\Delta(\lambda + t\varpi) \cong_{G_1} L_1(\lambda + t\varpi)$, and $\mathrm{soc}_{G_1} Q(\lambda) \cong_{G_1} L_1(\lambda)$. Writing $t = s + lu$ with $0 \leq s < l$, we have

$$L_1(\lambda + t\varpi) \cong_{G_1} L_1(\lambda) \otimes q\text{-}\det^t \cong_{G_1} L_1(\lambda) \otimes q\text{-}\det^s \cong_{G_1} L_1(\lambda + s\varpi).$$

Hence $L_1(\lambda) \cong_{G_1} L_1(\lambda + s\varpi)$ if, and only if, $s = 0$. If $s = 0$ then

$$\begin{aligned} &\Delta(n\rho)^{\mathrm{F}} \otimes \mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), Q(\lambda)) \\ &\quad\cong \Delta(n\rho)^{\mathrm{F}} \otimes \mathrm{Hom}_{G_1}(q\text{-}\det^t \otimes L(\lambda), Q(\lambda)) \\ &\quad\cong \Delta(n\rho)^{\mathrm{F}} \otimes \mathrm{Hom}_{G_1}(L(\lambda), Q(\lambda)) \otimes (q\text{-}\det^{-u})^{\mathrm{F}} \\ &\quad\cong \left(\Delta(n\rho) \otimes q\text{-}\det^{-u}\right)^{\mathrm{F}}, \end{aligned}$$

as required.

Lemma 2.3.3 *For $n \geq 0$ we have*

$$\mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), \Delta(\lambda + ln\rho))$$

$$\cong \begin{cases} (q\text{-}\det^{-u})^{\mathrm{F}} & \text{if } n = 0 \quad \text{and } t \equiv 0 \pmod{l}, \\ (q\text{-}\det^{-v} \otimes \Delta((n-1)\rho))^{\mathrm{F}} & \text{if } n \geq 1, \quad 2\mu = l - 2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l}, \\ 0 & \text{otherwise,} \end{cases}$$

where $lu = t$ and $lv = t - \frac{l}{2}$.

Proof: Suppose $n = 0$, and consider $\mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), \Delta(\lambda))$. Then for this to be non-zero we require $\Delta(\lambda + t\varpi) \cong_{G_1} \mathrm{soc}_{G_1} \Delta(\lambda)$. That is $L_1(\lambda + t\varpi) \cong_{G_1} L_1(\lambda)$. As in the previous lemma, this requires $t \equiv 0 \pmod{l}$, say $t = lu$. The rest follows as in the previous lemma.

Suppose $n \geq 1$. The injective envelope of $\Delta(\lambda + nl\rho)$ is $\Delta((n-1)\rho)^{\mathrm{F}} \otimes Q(\tau)$, where $\lambda = \tilde{\tau}$ by (2.2.5). This implies that $\tau = \tilde{\lambda} + l\varpi$. Then as in the previous lemma we have

$$\begin{aligned} &\mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), \Delta(\lambda + ln\rho)) \\ &\quad \cong \mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), \Delta((n-1)\rho)^{\mathrm{F}} \otimes Q(\tilde{\lambda} + l\varpi)) \\ &\quad \cong \Delta((n-1)\rho)^{\mathrm{F}} \otimes \mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), Q(\tilde{\lambda} + l\varpi)). \end{aligned}$$

As before we require $L_1(\lambda + t\varpi) \cong_{G_1} L_1(\tilde{\lambda} + l\varpi)$. That is $L_1(\mu, 0) \otimes q\text{-}\mathrm{det}^{\lambda_2 + t} \cong_{G_1} L_1(l - 2 - \mu, 0) \otimes q\text{-}\mathrm{det}^{\lambda_1 + 1}$. This holds if, and only if, $l - 2 = 2\mu$ and $\lambda_2 + t \equiv \lambda_1 + 1 \pmod{l}$. When these conditions hold, set $lv = \lambda_2 + t - \lambda_1 - 1 = t - \mu - 1$, and then as before we obtain the required result.

Lemma 2.3.4 *We have*

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda + t\varpi), \Delta(\tilde{\lambda})) \cong \mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda} + t\varpi + l\rho), \Delta(\tilde{\lambda}))$$

$$\cong \begin{cases} \left(q\text{-}\mathrm{det}^{-u} \otimes \Delta^*(\rho)\right)^{\mathrm{F}} & \text{if } t \equiv 0 \pmod{l}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda + t\varpi), \Delta(\lambda)) \cong \mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda} + t\varpi + l\rho), \Delta(\lambda))$$

$$\cong \begin{cases} \left(q\text{-}\mathrm{det}^{-v} \otimes \Delta^*(\rho)\right)^{\mathrm{F}} & \text{if } 2\mu = l - 2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l}, \\ 0 & \text{otherwise,} \end{cases}$$

where $lu = t$ and $lv = t - \frac{l}{2}$.

Proof: Applying $\mathrm{Hom}_{G_1}(-, \Delta(\tau))$ to the sequence in (2.2.5) gives:

$$0 \to \mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), \Delta(\tau)) \to \mathrm{Hom}_{G_1}(Q(\lambda + t\varpi), \Delta(\tau))$$

$$\to \mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda} + l\rho + t\varpi), \Delta(\tau)) \to \mathrm{Ext}^1_{G_1}(\Delta(\lambda + t\varpi), \Delta(\tau)) \to 0.$$

Taking $\tau = \lambda$ or $\tau = \tilde{\lambda}$ we have that the first two terms are isomorphic, and hence the last two are. We have

$$0 \to \Delta(\tilde{\lambda} + (t + l)\varpi) \to \Delta(\tilde{\lambda} + l\rho + t\varpi) \to \Delta(\rho)^{\mathrm{F}} \otimes \Delta(\tilde{\lambda} + t\varpi) \to 0,$$

and this restricts to a Loewy series, as G_1-modules, for $\Delta(\tilde{\lambda} + l\rho + t\varpi)$. Hence

$$\begin{aligned} \mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda} + l\rho + t\varpi), \Delta(\tau)) &\cong \mathrm{Hom}_{G_1}(\Delta(\rho)^{\mathrm{F}} \otimes \Delta(\tilde{\lambda} + t\varpi), \Delta(\tau)) \\ &\cong \mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda} + t\varpi), \Delta(\tau)) \otimes \Delta^*(\rho)^{\mathrm{F}}. \end{aligned}$$

Applying (2.3.3) with $\tau = \tilde{\lambda}$ gives the first result. For the second, take $\tau = \lambda$ and then the right-hand side above becomes

$$\mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda} + t\varpi), \Delta(\lambda)) \otimes \Delta^*(\rho)^{\mathrm{F}} \cong \mathrm{Hom}_{G_1}(L_1(\tilde{\lambda}) \otimes q\text{-}\det{}^t, L_1(\lambda)) \otimes \Delta^*(\rho)^{\mathrm{F}}.$$

For this to be non-zero we must have $\mu = \bar{\mu}$, that is $2\mu = l - 2$, which implies that $L_1(\tilde{\lambda}) \otimes q\text{-}\det^t \cong L_1(\lambda) \otimes q\text{-}\det^{t+l-1-\mu}$ which gives the rest of the condition, and the result.

Lemma 2.3.5 *For $m \geq n \geq 0$ we have*

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + lm\rho))$$
$$\cong \begin{cases} \mathrm{Ext}^1_{G_1}(\Delta(\lambda + \left(t + \frac{nl}{2}\right)\varpi), \Delta(\lambda + \frac{nl}{2}\varpi + (m-n)l\rho)) & \text{if } n \text{ even,} \\ \mathrm{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + \left(t + \frac{(n+1)l}{2}\right)\varpi), \Delta(\tilde{\lambda} + \frac{(n+1)l}{2}\varpi + (m-n)l\rho)) & \text{if } n \text{ odd.} \end{cases}$$

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tilde{\lambda} + lm\rho))$$
$$\cong \begin{cases} \mathrm{Ext}^1_{G_1}(\Delta(\lambda + \left(t + \frac{nl}{2}\right)\varpi), \Delta(\tilde{\lambda} + \frac{nl}{2}\varpi + (m-n)l\rho)) & \text{if } n \text{ even,} \\ \mathrm{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + \left(t + \frac{(n+1)l}{2}\right)\varpi), \Delta(\lambda + \frac{(n-1)l}{2}\varpi + (m-n)l\rho)) & \text{if } n \text{ odd.} \end{cases}$$

Proof: Writing τ for λ or $\tilde{\lambda}$, we have

$$\begin{aligned} &\mathrm{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tau + lm\rho)) \\ &\quad\cong \mathrm{Ext}^1_{G_1}(\Omega^{-n}\Delta(\lambda + ln\rho + t\varpi), \Omega^{-n}\Delta(\tau + lm\rho)) \\ &\cong \begin{cases} \mathrm{Ext}^1_{G_1}(\Delta(\lambda + \left(t + \frac{nl}{2}\right)\varpi), \Omega^{m-n}\Delta(\tau + \frac{ml}{2}\varpi)) & \text{if } m, n \text{ even,} \\ \mathrm{Ext}^1_{G_1}(\Delta(\lambda + \left(t + \frac{nl}{2}\right)\varpi), \Omega^{m-n}\Delta(\tilde{\tau} + \frac{(m+1)l}{2}\varpi)) & \text{if } m \text{ odd}, n \text{ even,} \\ \mathrm{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + \left(t + \frac{(n+1)l}{2}\right)\varpi), \Omega^{m-n}\Delta(\tau + \frac{ml}{2}\varpi)) & \text{if } m \text{ even}, n \text{ odd,} \\ \mathrm{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + \left(t + \frac{(n+1)l}{2}\right)\varpi), \Omega^{m-n}\Delta(\tilde{\tau} + \frac{(m+1)l}{2}\varpi)) & \text{if } m, n \text{ odd,} \end{cases} \end{aligned}$$

using the results of Lemma (2.3.1). The result now follows using (2.8).

Lemma 2.3.6 *For $n \geq 0$ we have*

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda + t\varpi), \Delta(\tilde{\lambda} + ln\rho))$$

$$\cong \begin{cases} \left(q\text{-}\det^{-\alpha} \otimes \Delta^*(\rho)\right)^{\mathrm{F}} & \text{if } n = 0 \quad \text{and } t \equiv 0 \pmod{l}, \\ \left(q\text{-}\det^{-\alpha}\right)^{\mathrm{F}} & \text{if } n = 1 \quad \text{and } t \equiv 0 \pmod{l}, \\ \left(q\text{-}\det^{-\beta} \otimes \Delta((n-2)\rho)\right)^{\mathrm{F}} & \text{if } n \geq 2, \quad 2\mu = l - 2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l}, \\ 0 & \text{otherwise,} \end{cases}$$

where $l\alpha = t$, $l\beta = t - \frac{l}{2}$.

Proof: The case $n=0$ is done in (2.3.4). For $n \geq 1$ apply $\mathrm{Hom}_{G_1}(\Delta(\lambda+t\varpi), -)$ to

$$0 \to \Delta(\tilde{\lambda}+ln\rho) \to \Delta((n-1)\rho)^{\mathrm{F}} \otimes Q(\lambda) \to \Delta(\lambda+l(n-1)\rho) \to 0$$

to obtain

$$0 \to \mathrm{Hom}_{G_1}(\Delta(\lambda+t\varpi), \Delta(\tilde{\lambda}+ln\rho)) \to \mathrm{Hom}_{G_1}(\Delta(\lambda+t\varpi), \Delta((n-1)\rho)^{\mathrm{F}} \otimes Q(\lambda))$$

$$\to \mathrm{Hom}_{G_1}(\Delta(\lambda+t\varpi), \Delta(\lambda+l(n-1)\rho)) \to \mathrm{Ext}^1_{G_1}(\Delta(\lambda+t\varpi), \Delta(\tilde{\lambda}+ln\rho)) \to 0.$$

As in earlier lemmas, the first two terms are isomorphic. Hence the next two are, and the result follows from (2.3.3).

Lemma 2.3.7 *For $n \geq 0$ we have*

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda+t\varpi), \Delta(\lambda+ln\rho))$$
$$\cong \begin{cases} \left(q\text{-}\det^{-\alpha} \otimes \Delta^*(\rho)\right)^{\mathrm{F}} & \text{if } n=0, \quad 2\mu = l-2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l}, \\ \left(q\text{-}\det^{-\alpha}\right)^{\mathrm{F}} & \text{if } n=1, \quad 2\mu = l-2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l}, \\ \left(q\text{-}\det^{-\gamma} \otimes \Delta((n-2)\rho)\right)^{\mathrm{F}} & \text{if } n \geq 2 \quad \text{and } t \equiv 0 \pmod{l}, \\ 0 & \text{otherwise,} \end{cases}$$

where $l\alpha = t - \frac{l}{2}$, $l\gamma = t - l$.

Proof: The case $n=0$ is done in (2.3.4). For $n \geq 1$ apply $\mathrm{Hom}_{G_1}(\Delta(\lambda+t\varpi), -)$ to

$$0 \to \Delta(\lambda+lnp) \to \Delta((n-1)\rho)^{\mathrm{F}} \otimes Q(\tilde{\lambda}+l\varpi) \to \Delta(\tilde{\lambda}+l\varpi+l(n-1)\rho) \to 0.$$

As in the previous lemma, the first two terms are isomorphic, and hence the next two are also; that is

$$\mathrm{Hom}_{G_1}(\Delta(\lambda+t\varpi), \Delta(\tilde{\lambda}+l\varpi+l(n-1)\rho)) \cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda+t\varpi), \Delta(\lambda+ln\rho)).$$

For the case $n \geq 2$ write $\lambda' = \lambda + l\varpi$ and $t' = t - l$. Then the left-hand side equals $\mathrm{Hom}_{G_1}(\Delta(\lambda'+t'\varpi), \Delta(\tilde{\lambda}'+l(n-1)\rho))$, and the result follows from (2.3.2). For the case $n=1$ consider $\mathrm{Hom}_{G_1}(\Delta(\lambda+t\varpi), \Delta(\tilde{\lambda}+l\varpi))$. This is clearly zero unless $\mu = \bar{\mu}$, in which case it is isomorphic to $\mathrm{Hom}_{G_1}(\Delta(\lambda+t\varpi), \Delta(\lambda+\frac{l}{2}\varpi))$, when the result follows from (2.3.3).

For the next two lemmas, it is necessary to restrict to a specific value of t. However, as this condition will always hold in the cases of interest, this is of no great consequence.

Lemma 2.3.8 *For $m \geq n \geq 0$ and $t = \frac{l}{2}(m-n)$ we have*

$$\begin{aligned} &\mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tilde{\lambda} + l(m+1)\rho)) \\ &\cong \begin{cases} \left(q\text{-}\det^{-u} \otimes \Delta(m\rho) \otimes \Delta^*(n\rho)\right)^{\mathrm{F}} & \text{if } t \equiv 0 \pmod{l}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $lu = t$.

Proof: Applying $\mathrm{Hom}_{G_1}(-, \Delta(\tilde{\lambda} + l(m+1)\rho))$ to (2.2.4(i)) we obtain

$$\begin{aligned} 0 \;\rightarrow\; & \mathrm{Hom}_{G_1}(\Delta(n\rho)^{\mathrm{F}} \otimes \Delta(\lambda + t\varpi), \Delta(\tilde{\lambda} + l(m+1)\rho)) \\ \rightarrow\; & \mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tilde{\lambda} + l(m+1)\rho)) \\ \rightarrow\; & \mathrm{Hom}_{G_1}(\Delta((n-1)\rho)^{\mathrm{F}} \otimes \Delta(\tilde{\lambda} + (l+t)\varpi), \Delta(\tilde{\lambda} + l(m+1)\rho)) \\ \rightarrow\; & \mathrm{Ext}^1_{G_1}(\Delta(n\rho)^{\mathrm{F}} \otimes \Delta(\lambda + t\varpi), \Delta(\tilde{\lambda} + l(m+1)\rho)) \\ \rightarrow\; & \mathrm{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tilde{\lambda} + l(m+1)\rho)). \end{aligned}$$

We claim that the first two terms are isomorphic. With this we are done, as the first term is isomorphic to $\mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), \Delta(\tilde{\lambda} + l(m+1)\rho)) \otimes \Delta^*(n\rho)^{\mathrm{F}}$ and hence the result follows from (2.3.2).

Proof of the claim: Consider the third term. Setting $\lambda' = \tilde{\lambda}$ and $t' = t + l$, this is isomorphic to $\mathrm{Hom}_{G_1}(\Delta(\lambda' + t'\varpi), \Delta(\lambda' + l(m+1)\rho)) \otimes \Delta^*((n-1)\rho)^{\mathrm{F}}$. By (2.3.3), this is zero unless $2\mu' = l - 2$ and $t' \equiv \frac{l}{2} \pmod{l}$, that is $2\mu = l-2$ and $t \equiv \frac{l}{2} \pmod{l}$. Further, if it is non-zero then it has dimension $(m+1)n$. If this term is zero we are done, so we may assume that $2\mu = l-2$, $t \equiv \frac{l}{2} \pmod{l}$. Hence $m-n$ is odd, and so $m \geq 1$.

Term four is isomorphic to $\mathrm{Ext}^1_{G_1}(\Delta(\lambda + t\varpi), \Delta(\tilde{\lambda} + l(m+1)\rho)) \otimes \Delta^*(n\rho)^{\mathrm{F}}$, which is isomorphic to $(q\text{-}\det^{-\beta} \otimes \Delta((m-1)\rho)) \otimes \Delta^*(n\rho)^{\mathrm{F}}$ by (2.3.6). By (2.3.5) term five is isomorphic to

$$\begin{cases} \mathrm{Ext}^1_{G_1}(\Delta(\lambda + (t + \frac{nl}{2})\varpi), \Delta(\tilde{\lambda} + \frac{nl}{2}\varpi + (m+1-n)l\rho)) & \text{if } n \text{ even}, \\ \mathrm{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + (t + \frac{(n+1)l}{2})\varpi), \Delta(\lambda + \frac{(n-1)l}{2}\varpi + (m+1-n)l\rho)) & \text{if } n \text{ odd}. \end{cases}$$

For appropriate λ''s, both cases are isomorphic to

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda' + t\varpi), \Delta(\tilde{\lambda}' + (m+1-n)l\rho)) \cong (q\text{-}\det^{-\beta} \otimes \Delta((m-n-1)\rho))^{\mathrm{F}}$$

by (2.3.6), as $m+1-n \geq 2$ (since $m-n$ is odd). So the fourth and fifth terms have dimension $m(n+1)$ and $m-n$ respectively. Thus the dimension of the fourth term is the sum of the dimensions of the terms on either side; hence the map into it must be injective. This implies that the first two terms are isomorphic as required.

Lemma 2.3.9 *For $m > n \geq 0$ and $t = \frac{l}{2}(m-n)$ we have*

$$\begin{aligned}&\mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + lm\rho))\\ &\cong \begin{cases} \left(q\text{-det}^{-v} \otimes \Delta((m-1)\rho) \otimes \Delta^*(n\rho)\right)^{\mathrm{F}} & \text{if } 2\mu = l-2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l}, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

where $lv = t - \frac{l}{2}$.

Proof: If $2\mu = l-2$ then

$$\begin{aligned}&\mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + lm\rho))\\ &\quad\cong \mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tilde{\lambda} + \tfrac{l}{2}\varpi + lm\rho))\\ &\quad\cong \mathrm{Hom}_{G_1}(\Delta(\lambda' + ln\rho + \left(t - \tfrac{l}{2}\right)\varpi), \Delta(\tilde{\lambda}' + lm\rho)),\end{aligned}$$

where $\lambda' = \lambda + \frac{l}{2}\varpi$, and the result follows from the previous lemma. So we may assume that $\mu \neq \bar{\mu}$. Applying $\mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), -)$ to (2.2.5) we obtain

$$\begin{aligned}0 &\to \mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tilde{\lambda} + l(m+1)\rho))\\ &\to \mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(m\rho)^{\mathrm{F}} \otimes Q(\lambda))\\ &\to \mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + lm\rho))\\ &\to \mathrm{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tilde{\lambda} + l(m+1)\rho)) \to 0.\end{aligned}$$

As $\mu \neq \bar{\mu}$, any map $\Delta(\lambda + ln\rho + t\varpi) \to \Delta(m\rho)^{\mathrm{F}} \otimes Q(\lambda)$ has image in the socle. Hence the first two terms are isomorphic, and so the next two are also. By (2.3.5) we then have

$$\begin{aligned}&\mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + lm\rho))\\ &\cong \begin{cases} \mathrm{Ext}^1_{G_1}(\Delta(\lambda + \left(t + \frac{ln}{2}\right)\varpi), \Delta(\tilde{\lambda} + \frac{ln}{2}\varpi + (m+1-n)l\rho)) & \text{if } n \text{ even,} \\ \mathrm{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + \left(t + \frac{l(n+1)}{2}\right)\varpi), \Delta(\lambda + \frac{l(n-1)}{2}\varpi + (m+1-n)l\rho)) & \text{if } n \text{ odd.} \end{cases}\end{aligned}$$

Let $\lambda' = \lambda + \frac{nl}{2}\varpi$ (respectively $\widetilde{(\lambda + \frac{(n+1)l}{2}\varpi)}$) for n even (respectively n odd). Then in both cases this is isomorphic to $\mathrm{Ext}^1_{G_1}(\Delta(\lambda' + t\varpi), \Delta(\tilde{\lambda}' + l(m+1-n)\rho))$. Repeating the argument above, with $n = 0$, $m = m-n$, this is isomorphic to $\mathrm{Hom}_{G_1}(\Delta(\lambda' + t\varpi), \Delta(\lambda' + l(m-n)\rho))$, and now (as $\mu \neq \bar{\mu}$) the result follows from (2.3.3).

Lemma 2.3.10 *For $m, n \geq 1$ we have*

$$\begin{aligned}&\mathrm{Hom}_{G_1}(\Delta(ln - 1 + \delta + t, \delta + t), \Delta(lm - 1 + \delta, \delta))\\ &\cong \begin{cases} \left(q\text{-det}^{-u} \otimes \Delta((m-1)\rho) \otimes \Delta^*((n-1)\rho)\right)^{\mathrm{F}} & \text{if } t \equiv 0 \pmod{l}, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

where $lu = t$.

Proof: By (2.2.4)(ii) applied twice we have

$$\begin{aligned}
&\mathrm{Hom}_{G_1}(\Delta(ln-1+\delta+t,\delta+t),\Delta(lm-1+\delta,\delta))\\
&\cong \mathrm{Hom}_{G_1}(\Delta((n-1)\rho)^{\mathrm{F}}\otimes\Delta(l-1+\delta+t,\delta+t),\Delta((m-1)\rho)^{\mathrm{F}}\otimes\Delta(l-1+\delta,\delta))\\
&\cong \Delta((m-1)\rho)^{\mathrm{F}}\otimes\mathrm{Hom}_{G_1}(\Delta(l-1+\delta+t,\delta+t),\Delta(l-1+\delta,\delta))\otimes\Delta^*((n-1)\rho)^{\mathrm{F}}.
\end{aligned}$$

Let $\tau=(l-1+\delta,\delta)$; then

$$\begin{aligned}
&\mathrm{Hom}_{G_1}(\Delta((l-1)\rho+(\delta+t)\varpi),\Delta(l-1+\delta,\delta))\\
&\quad=\mathrm{Hom}_{G_1}(\Delta(\tau+t\varpi),\Delta(\tau))\cong\mathrm{Hom}_{G_1}(L_1(\tau+t\varpi),L_1(\tau)),
\end{aligned}$$

and the result now clearly follows.

2.4 Ext^1_G for Weyl Modules

In this section we calculate $\mathrm{Ext}^1_G(\Delta,\Delta')$ for all possible Δ,Δ''s. This uses the results of the previous section, along with the Lyndon–Hochschild–Serre spectral sequence (see (1.4.2)), which gives rise to the five term exact sequence

$$0\to H^1(\bar{G},V^{G_1})\to H^1(G,V)\to H^1(G_1,V)^{\bar{G}}\to H^2(\bar{G},V^{G_1})\to H^2(G,V)$$

which will form the basis of the calculations in this section. From the isomorphism between $\mathrm{GL}(2,k)$ and $\bar{G}$ from Section 1.3 it follows that

$$H^i(\bar{G},V^{\mathrm{F}})\cong H^i(\mathrm{GL}(2,k),V).$$

This will allow us to use the existing result in [26] for the classical case.

Lemma 2.4.1 *For all λ, λ' such that $0\le\mu$, $\mu'\le l-1$ we have*

$$\mathrm{Ext}^1_G(\Delta(\lambda),\Delta(\lambda'))=0.$$

Proof: This is clear, as for all λ' such that $0\le\mu'\le l-1$ we have

$$\Delta(\lambda')\cong L(\lambda')\cong\nabla(\lambda'),$$

and by [20, 4(2)] we have that

$$\mathrm{Ext}^1_G(\Delta(\lambda),\nabla(\lambda'))=0.$$

In the rest of this section we will frequently make use of the fact that $\mathrm{Ext}^1_{\mathrm{GL}_2}$ can be easily determined from $\mathrm{Ext}^1_{\mathrm{SL}_2}$. To be more precise, $\mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta(\alpha), \Delta(\beta)) = \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(\alpha_1 - \alpha_2), \Delta(\beta_1 - \beta_2))$ provided that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$; else it is zero.

Lemma 2.4.2 *For n, $m > 0$ we have*

$$\mathrm{Ext}^1_G(\Delta(ln - 1 + \delta + t, \delta + t), \Delta(lm - 1 + \delta, \delta))$$
$$\cong \begin{cases} \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(n-1), \Delta(m-1)) & \text{if } m - n \text{ even and } t = \frac{l}{2}(m-n), \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We may assume that $1 \leq n < m$. Set $V = \Delta(lm - 1 + \delta, \delta) \otimes \Delta^*(ln - 1 + \delta + t, \delta + t)$. Then we have

$$0 \to H^1(\bar{G}, V^{G_1}) \to H^1(G, V) \to H^1(G_1, V)^{\bar{G}}.$$

The third term is isomorphic to $\mathrm{Ext}^1_{G_1}(\Delta(ln - 1 + \delta + t, \delta + t), \Delta(lm - 1 + \delta, \delta))^{\bar{G}}$ which equals zero by (2.2.4(ii)) applied twice and (2.4.1). Hence the first two terms must be isomorphic. Now, by (2.3.10)

$$\begin{aligned} V^{G_1} &\cong \mathrm{Hom}_{G_1}(\Delta(ln - 1 + \delta + t, \delta + t), \Delta(lm - 1 + \delta, \delta)) \\ &\cong \begin{cases} (\Delta((m-1)\rho) \otimes \Delta^*((n-1)\rho))^{\mathrm{F}} \otimes q\text{-}\det^t & \text{if } m - n \text{ even,} \\ 0 & \text{otherwise} \end{cases} \\ &\cong \begin{cases} (\Delta(m - 1 - t', -t') \otimes \Delta^*((n-1)\rho))^{\mathrm{F}} & \text{if } \quad m - n \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $lt' = t$. So

$$\begin{aligned} &\mathrm{Ext}^1_G(\Delta(ln - 1 + \delta + t, \delta + t), \Delta(lm - 1 + \delta, \delta)) \\ &\quad \cong \begin{cases} \mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta((n-1)\rho), \Delta((m-1)\rho - t'\varpi)) & \text{if } m - n \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

which, by the remark above, implies the result.

Lemma 2.4.3 *For $0 \leq n < m$ we have*

$$\mathrm{Ext}^1_G(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tilde{\lambda} + lm\rho))$$

$$\cong \begin{cases} k & \text{if } m - n = 2p^\alpha, \quad \alpha \geq 0, \quad 2\mu = l - 2 \\ & \qquad \text{and } t = \frac{l}{2}(m - n - 1), \\ k & \text{if } m - n = 1 \text{ and } t = \frac{l}{2}(m - n - 1), \\ \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(n), \Delta(m-1)) & \text{if } m - n \text{ odd, } m - n \neq 1 \text{ and } t = \frac{l}{2}(m - n - 1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof: First note that in the first three cases t is an integer, as required. Let $V=\Delta(\tilde{\lambda}+lm\rho)\otimes\Delta^*(\lambda+ln\rho+t\varpi)$. Now

$$\begin{aligned} V^{G_1} &\cong \mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t\varpi),\Delta(\tilde{\lambda}+lm\rho)) \\ &\cong \begin{cases} (q\text{-det}^{-u}\otimes\Delta((m-1)\rho)\otimes\Delta^*(n\rho))^{\mathrm{F}} & \text{if } m-n \text{ odd,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $lu=t$, by (2.3.8). By (2.3.5) we have

$$\begin{aligned} H^1(G_1,V) &\cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda+ln\rho+t\varpi),\Delta(\tilde{\lambda}+lm\rho)) \\ &\cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda'+t\varpi),\Delta(\tilde{\lambda'}+(m-n)l\rho)) \end{aligned}$$

where $\lambda'=\begin{cases} \lambda+\frac{nl}{2}\varpi & \text{if } n \text{ even,} \\ \tilde{\lambda}+\frac{(n+1)l}{2}\varpi & \text{if } n \text{ odd.} \end{cases}$ Now by (2.3.6) this is isomorphic to

$$\begin{cases} k & \text{if } m-n=1, \\ (q\text{-det}^{-\beta}\otimes\Delta((m-n-2)\rho))^{\mathrm{F}} & \text{if } m-n\geq 2, \quad 2\mu=l-2 \quad \text{and } m-n \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta=\frac{1}{2}(m-n-2)$.

Consider the five term exact sequence. If $m-n$ is even then the first and fourth terms are zero by above. Hence

$$\begin{aligned} H^1(G,V) &\cong H^1(G_1,V)^{\bar{G}} \\ &\cong \begin{cases} \left(\Delta\left(\frac{m-n-2}{2}(1,-1)\right)^{\mathrm{F}}\right)^{\bar{G}} & \text{if } 2\mu=l-2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now the first case is isomorphic to $\mathrm{Hom}_{\mathrm{GL}_2}\left(\Delta(0),\Delta\left(\frac{m-n-2}{2}(1,-1)\right)\right)$ which, by (2.1.3) with $l=1$, is isomorphic to

$$\begin{cases} k & \text{if } m-n-2=2(p^{\alpha}-1), \quad \alpha\geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

which gives the result for $m-n$ even. If $m-n$ is odd, and $m-n\neq 1$, then $H^1(G_1,V)$ (and hence $H^1(G_1,V)^{\bar{G}}$)$=0$. Hence

$$\begin{aligned} H^1(G,V) &\cong H^1(\bar{G},V^{G_1}) \\ &\cong \mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta(n\rho-u\varpi),\Delta((m-1)\rho)) \\ &\cong \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(n),\Delta(m-1)). \end{aligned}$$

If $m=n+1$ then $V^{G_1}\cong(\Delta(n\rho)\otimes\Delta^*(n\rho))^{\mathrm{F}}$. Now for $i>0$,

$$H^i(\bar{G},V^{G_1})\cong\mathrm{Ext}^i_{\mathrm{GL}_2}(\Delta(n\rho),\Delta(n\rho))=0.$$

Thus $H^1(G,V)\cong H^1(G_1,V)^{\bar{G}}\cong k$, and we are done.

Lemma 2.4.4 *For $0 \leq n < m$ we have*

$$\begin{aligned}
&\mathrm{Ext}^1_G(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + lm\rho)) \\
&\cong \begin{cases} k & \text{if } m-n = 2p^\alpha, \quad \alpha \geq 0 \quad \text{and } t = \frac{l}{2}(m-n), \\ k & \text{if } m-n = 1,\ 2\mu = l-2 \text{ and } t = \frac{l}{2}(m-n), \\ \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(n), \Delta(m-1)) & \text{if } m-n \text{ odd},\ m-n \neq 1,\ 2\mu = l-2 \\ & \qquad \text{and } t = \frac{l}{2}(m-n), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof: Again note that in the first three cases t is an integer as required. Let $V = \Delta(\lambda + lm\rho) \otimes \Delta^*(\lambda + ln\rho + t\varpi)$. Now

$$\begin{aligned}
V^{G_1} &\cong \mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + lm\rho)) \\
&\cong \begin{cases} (q\text{-det}^{-v} \otimes \Delta((m-1)\rho) \otimes \Delta^*(n\rho))^{\mathrm{F}} & \text{if } 2\mu = l-2 \quad \text{and } m-n \text{ odd}, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

by (2.3.9). First consider the case when this is zero. Then by the five term exact sequence we must have $H^1(G, V) \cong H^1(G_1, V)^{\bar{G}}$.

$$\begin{aligned}
H^1(G_1, V) &\cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + lm\rho)) \\
&\cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda' + \tfrac{l}{2}(m-n)\varpi), \Delta(\lambda' + (m-n)l\rho)),
\end{aligned}$$

where $\lambda' = \lambda + \frac{nl}{2}\varpi$ (respectively $\lambda' = \tilde{\lambda} + \frac{(n+1)l}{2}\varpi$) for n even (respectively n odd), by (2.3.5). This, by (2.3.7), is isomorphic to

$$\begin{cases} \left(q\text{-det}^{-\alpha}\right)^{\mathrm{F}} & \text{if } m = n+1 \quad \text{and } t \equiv \mu' + 1 \pmod{l}, \\ (q\text{-det}^{-\beta} \otimes \Delta((m-n-2)\rho))^{\mathrm{F}} & \text{if } m-n \geq 2 \quad \text{and } t \equiv 0 \pmod{l}, \\ 0 & \text{otherwise,} \end{cases}$$

where $l\alpha = t - \mu' - 1$ and $l\beta = t - l$. But $m = n+1$ implies that $t = \frac{l}{2}$. So $t \equiv \mu' + 1 \pmod{l}$ implies that $\mu' = \bar{\mu}'$, that is $\mu = \bar{\mu}$, so the first case is impossible.

Thus for $\mu \neq \bar{\mu}$ or $m-n$ even we have

$$H^1(G_1, V) \cong \begin{cases} \Delta\left(\frac{m-n-2}{2}(1,-1)\right)^{\mathrm{F}} & \text{if } m-n \geq 2 \quad \text{and } t \equiv 0 \pmod{l}, \\ 0 & \text{otherwise.} \end{cases}$$

In the zero case we are done; if non-zero then

$$\begin{aligned}
H^1(G_1, V)^{\bar{G}} &\cong H^0\left(\bar{G}, \Delta\left(\tfrac{m-n-2}{2}(1,-1)\right)^{\mathrm{F}}\right) \\
&\cong H^0\left(\mathrm{GL}(2,k), \Delta\left(\tfrac{m-n-2}{2}(1,-1)\right)\right) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_2}\left(\Delta(0), \Delta\left(\tfrac{m-n-2}{2}(1,-1)\right)\right) \\
&\cong \begin{cases} k & \text{if } m-n-2 = 2(p^\alpha - 1), \quad \alpha \geq 0, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

by (2.1.3), with $l=1$. Now if $\mu=\bar{\mu}$ and $m-n$ odd then we have

$$V^{G_1} \cong \left(q\text{-det}^{-u} \otimes \Delta((m-1)\rho) \otimes \Delta^*(n\rho)\right)^{\mathrm{F}},$$

where $lu = t - \frac{l}{2}$, by our earlier calculation. In this case

$$\begin{aligned} H^1(G_1, V) &\cong \operatorname{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + lm\rho)) \\ &\cong \begin{cases} (q\text{-det}^{-\beta})^{\mathrm{F}} & \text{if } m = n+1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $l\beta = t - \frac{l}{2}$ by (2.3.5) and (2.3.7). So if $m \neq n+1$ then from the five term exact sequence we have

$$\begin{aligned} H^1(G, V) &\cong H^1(\bar{G}, V^{G_1}) \\ &\cong H^1(\mathrm{GL}(2,k), \Delta((m-1)\rho - u\varpi) \otimes \Delta^*(n\rho)) \\ &\cong \operatorname{Ext}^1_{\mathrm{GL}_2}(\Delta(n\rho), \Delta((m-1)\rho - u\varpi)) \\ &\cong \operatorname{Ext}^1_{\mathrm{SL}_2}(\Delta(n), \Delta(m-1)). \end{aligned}$$

If $m = n+1$ then $V^{G_1} \cong (\Delta(n\rho) \otimes \Delta^*(n\rho))^{\mathrm{F}}$. Hence, for $i \geq 1$,

$$H^i(\mathrm{GL}(2,k), V^{G_1}) \cong \operatorname{Ext}^i_{\mathrm{GL}_2}(\Delta(n\rho), \Delta(n\rho)) = 0,$$

and so $H^1(G, V) \cong H^1(G_1, V)^{\bar{G}} \cong k^{\bar{G}} = k$, and this completes the proof.

By (1.6.4), and the characterisation of blocks calculated earlier, we see that these lemmas have exhausted all possible cases where a non-trivial extension could exist. Thus these results, in conjunction with the results of [26], complete the calculation. The final result of this section merely combines these into a more manageable form.

Suppose that $l=1$. Then for an integer a with $0 \leq a \leq p-1$ we define $\hat{a}$ by $a + \hat{a} = p-1$. If $r = \sum_{i\geq 0} r_i p^i$ with $0 \leq r_i \leq p-1$ then, as in [26], we define

$$\Psi(r)(=\Psi_p(r)) = \begin{array}{c} \left\{\sum_{i=0}^{u-1} \hat{r}_i p^i + p^{u+a} : \hat{r}_u \neq 0, a \geq 1, u \geq 0\right\} \\ \bigcup \left\{\sum_{i=0}^{u} \hat{r}_i p^i : \hat{r}_u \neq 0, u \geq 0\right\}. \end{array} \tag{2.9}$$

Now suppose that $l \geq 1$. Then if $r = r_{-1} + l\sum_{i\geq 0} r_i p^i$ with $0 \leq r_i \leq p-1$, for $i \geq 0$ and $0 \leq r_{-1} \leq l-1$, we define $\hat{r}_i$ as before for $i \geq 0$, while $\hat{r}_{-1}$ is defined by $r_{-1} + \hat{r}_{-1} = l-1$. With this we can now define a quantum version of the above set by

$$\tilde{\Psi}(r)(=\tilde{\Psi}_{l,p}(r)) = \begin{array}{c} \left\{\sum_{i=-1}^{u-1} \hat{r}_i \theta(i) + lp^{u+a} : \hat{r}_u \neq 0, a \geq 1, u \geq -1\right\} \\ \bigcup \left\{\sum_{i=-1}^{u} \hat{r}_i \theta(i) : \hat{r}_u \neq 0, u \geq -1\right\} \end{array}$$

where $\theta(i) = \begin{cases} lp^i & \text{if } i \geq 0, \\ 1 & \text{if } i = -1. \end{cases}$ We can now state the main result. Note that we now drop our long-standing restriction on λ.

Theorem 2.4.5 *Let $\lambda = (r+\delta, \delta)$ and $\tau = (s+\delta', \delta')$. Then*

$$\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\tau)) \cong \begin{cases} k & \text{if } r+2\delta = s+2\delta', \text{ and } s = r+2e \text{ with } e \in \tilde{\Psi}(r), \\ 0 & \text{otherwise.} \end{cases}$$

Proof: It is clear that we require $r+2\delta = s+2\delta'$ by consideration of blocks. So we only need consider the cases that arise from (2.4.1–2.4.4). We consider when each of these could give a non-zero $\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\tau))$ in turn.

First, suppose that $r = l-1+ln, s = l-1+lm$. By (2.4.2) and [26] we must have $s = r+2dl$ with $d \in \Psi(n)$. Second, suppose that $r = \mu + ln$ and $s = \mu + lm$. By (2.4.4) and [26] we must have

$$s = \begin{cases} r + 2lp^a & \text{if } a \geq 0, \\ r + l & \text{if } \mu = \bar{\mu}, \\ r + l(2d+1) & \text{if } \mu = \bar{\mu} \quad \text{and } d \in \Psi(n). \end{cases}$$

Finally, suppose that $r = \mu + ln$ and $s = \bar{\mu} + lm$. By (2.4.3) and [26] we must have

$$s = \begin{cases} r + 2lp^a & \text{if } \mu = \bar{\mu} \quad \text{and } a \geq 0, \\ r + l + \bar{\mu} - \mu, & \\ r + l(2d+1) + \bar{\mu} - \mu & \text{if } d \in \Psi(n). \end{cases}$$

Further, if r, s satisfy any of the above conditions then $\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\tau))$ is non-zero. Thus, if we allow $\mu = l-1$, we can state the above results as $\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\tau))$ is non-zero if, and only if,

$$s = \begin{cases} r + 2dl & \text{if } \mu = l-1 \quad \text{and } d \in \Psi(n), \\ r + 2p^a l & \text{if } \mu \neq l-1 \quad \text{and } a \geq 0, \\ r + 2\bar{\mu} + 2 & \text{if } \mu \neq l-1, \\ r + 2\bar{\mu} + 2 + 2ld & \text{if } \mu \neq l-1 \quad \text{and } d \in \Psi(n). \end{cases}$$

So in the form of the statement of the theorem we have that $\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\tau))$ is non-zero if, and only if,

$$e = \begin{cases} ld & \text{if } \mu = l-1 \quad \text{and } d \in \Psi(n), \\ lp^a & \text{if } \mu \neq l-1 \quad \text{and } a \geq 0, \\ \bar{\mu} + 1 & \text{if } \mu \neq l-1, \\ \bar{\mu} + 1 + ld & \text{if } \mu \neq l-1 \quad \text{and } d \in \Psi(n). \end{cases}$$

It is now straightforward to see that these give rise to the required result.

2.5 The Manin quantisation

As noted in the first chapter, most of the theory we have developed also holds for the Manin quantisation. In particular there is an analogue of the Borel subgroup, so we can again consider the modules induced up from the one-dimensional Borel modules (see [39, 8.3]). As before, the non-zero induced modules correspond to the dominant weights, and can again be described explicitly (see [39, (8.6.1)]). Again, we can define the Weyl modules as duals of appropriate induced modules (see [39, (8.10.1–2)]). Thus one can ask if a result corresponding to that in the last section holds.

Corollary 2.5.1 *The previous theorem also holds for $GL_q(2, k)$, when q is a primitive lth root of unity (with the appropriate modifications — see 1.9.6).*

Proof: By the explicit description of the coalgebra isomorphism in (1.9.4), along with the universal property of the induced modules, the module category isomorphism (1.9.5) takes Weyl modules to Weyl modules of the same weight. The result is now immediate from the previous theorem and the isomorphism in (1.9.5).

Note that this result holds even for l not an odd root of unity, despite the lack in this case of a satisfactory infinitesimal theory. This is just one of a number of results we shall give whose proofs would, without recourse to the Dipper–Donkin quantisation, only hold in the odd l case.

Chapter 3

Constructing extensions

By the results of the preceding chapter, if a non-split extension exists between two induced modules for q-GL$(2,k)$, then it is unique (up to isomorphism). In this chapter we consider the structure of such extensions. It will prove convenient to divide these into two classes: the elementary and non-elementary extensions. After a brief section in which we determine the simplest non-elementary extensions, we restrict our attention to the elementary ones. For these, we give a conjecture as to their structure, and verify this in some easy cases. Finally we conclude with a necessary condition for an extension to satisfy the conjecture. To make the various results more transparent, we will restrict attention throughout to the classical case.

3.1 Elementary and non-elementary extensions

As noted above, we now restrict our attention to the classical case, and work with SL$(2,k)$ or GL$(2,k)$. We use the usual notation in these cases (see for example [26]). Recall that, from [26], the extensions in these cases are parameterised by a certain set, $\Psi(r)$ (see (2.9)). In the following it will be convenient to divide this into two subsets; we shall call extensions corresponding to elements in

$$\left\{\sum_{i=0}^{u}\hat{r}_i p^i : \hat{r}_u \neq 0, u \geq 0\right\}$$

elementary extensions, and the remainder *non-elementary extensions*.

In considering such extensions, the following easy generalisation of [12, Section 2, Lemma] will prove useful. For convenience we state the quantum version, which will also be used later.

Lemma 3.1.1 *Let V, $Z \in \mathrm{mod}\ G$ be such that $res_{G_r} V$ is absolutely indecomposable, G_r acts trivially on Z, and Z is absolutely indecomposable as a $\bar{G}^r$-module. Then $V \otimes Z$ is an absolutely indecomposable G-module.*

Proof: This follows by the same arguments as in [10, 3.3(5)] (or [12, Section 2, Lemma] in the classical case).

We conclude this section with a proposition taking care of the simplest non-elementary case.

Proposition 3.1.2 *For all $n \geq 0$, $m \geq 1$ and $r \geq 0$ we have*

$$\nabla(r + p^m, 0) \otimes \nabla(n, 0)^{F^m} \to \nabla(r + (n+1)p^m, 0) \to 0$$

and for $n = 1$, $r \not\equiv p - 1 \pmod p$ this can be extended to a non-split short exact sequence

$$0 \to \nabla(r + p^m, p^m) \to \nabla(r + p^m, 0) \otimes \nabla(1, 0)^{F^m} \to \nabla(r + 2p^m, 0) \to 0.$$

Proof: We begin by constructing the desired surjection. First note that $\nabla(r + p^m, 0) \otimes \nabla(n, 0)^{F^m}$ and $\nabla(r + (n+1)p^m, 0)$ have bases

$$\{e_1^a e_2^{r+p^m-a} \otimes g_1^i g_2^{n-i} \mid 0 \leq a \leq r + p^m, \quad 0 \leq i \leq n\}$$

and

$$\{e_1^a e_2^{r+(n+1)p^m-a} \mid 0 \leq a \leq r + (n+1)p^m\}$$

respectively.

Consider $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2, k)$. The action of $\mathrm{GL}(2, k)$ on these bases is given by

$$g(e_1^a e_2^{r+p^m-a} \otimes g_1^i g_2^{n-i}) = (\alpha e_1 + \gamma e_2)^a (\beta e_1 + \delta e_2)^{r+p^m-a} \otimes (\alpha^{p^m} g_1 + \gamma^{p^m} g_2)^i (\beta^{p^m} g_1 + \delta^{p^m} g_2)^{n-i}$$

and

$$g(e_1^a e_2^{r+(n+1)p^m-a}) = (\alpha e_1 + \gamma e_2)^a (\beta e_1 + \delta e_2)^{r+(n+1)p^m-a}$$

respectively. The required map, ϕ, is defined on basis elements by

$$\phi(e_1^a e_2^{r+p^m-a} \otimes g_1^i g_2^{n-i}) = e_1^{a+ip^m} e_2^{r+p^m+(n-i)p^m-a}.$$

It is enough to show that this is a module homomorphism, as it is clearly a surjection. This, however, is a routine calculation.

Now take $n = 1$. Recall that $\nabla(r + p^m, p^m)$ has basis $\{e_1^a e_2^{r-a} \mid 0 \leq a \leq r\}$. Also, $g \in \mathrm{GL}(2, k)$ acts by $g(e_1^a e_2^{r-a}) = (\alpha e_1 + \gamma e_2)^a (\beta e_1 + \delta e_2)^{r-a} (\det g)^{p^m}$. Set

$$\psi(e_1^a e_2^{r-a}) = e_1^{a+p^m} e_2^{r-a} \otimes g_1^0 g_2^1 - e_1^a e_2^{r+p^m-a} \otimes g_1^1 g_2^0.$$

Another routine calculation gives that ψ is a module homomorphism. Clearly ψ is an injection, and $\mathrm{Im}\psi \subseteq \ker\phi$. Hence by dimensions the required sequence exists.

To show that the sequence does not split, we note that for $r \not\equiv p-1 \pmod{p}$, [47, 7.1.4 Corollary] gives that the restriction of $\nabla(r + p^m, 0)$ to G_1 is indecomposable. Hence by the last lemma we obtain the required result.

This proposition gives the extension corresponding to the non-elementary case $d = p^m$.

3.2 A conjecture on elementary extensions

In this section we restrict our attention to the elementary extensions. In certain special cases these can be described, and we also give a conjecture as to the general result in terms of so-called elementary generators. Finally we give a sufficient condition for an extension to be an elementary generator.

Given $R = \sum_{i \geq 0} r_i p^i \in \mathbb{N}$, with $0 \leq r_i \leq p-1$ for all i and $r_u \neq p-1$, we set $R(u) = \sum_{i=0}^{u} r_i p^i$. We define $\hat{R}(u)$ by $R(u) + \hat{R}(u) = p^{u+1} - 1$. Then there exists, by the main result in [26], a non-split short exact sequence

$$0 \to \nabla(R(u)) \to E(R(u)) \to \nabla(\hat{R}(u) + p^{u+1} - 1) \to 0,$$

with $\dim E(R(u)) = 2p^{u+1}$. With this we can now formulate

Conjecture 3.2.1 *For all $n \geq 0$, $u \geq 0$ and $R(u)$ as above there is a non-split short exact sequence*

$$0 \to \nabla(R(u) + np^{u+1}) \to E(R(u)) \otimes \nabla(n)^{F^{u+1}} \to \nabla(\hat{R}(u) + (n+1)p^{u+1} - 1) \to 0.$$

If this holds, then all the elementary extensions can be obtained from the $E(R(u))$ as above — that is, it is enough to determine for each u a finite family of modules, each of dimension $2p^{u+1}$. We say that $E(R(u))$ is an *elementary generator* if for all $n \geq 0$ there is a (non-split) sequence as above.

A simple modification of the proof of [47, 7.1.1 Lemma] gives that the conjecture is at least true on the level of characters:

Proposition 3.2.2 *For all $n \geq 0$, $u \geq 0$ and $R(u)$ as above we have*

$$\mathrm{ch}\left(E(R(u)) \otimes \nabla(n)^{F^{u+1}}\right) = \mathrm{ch}\nabla(R(u) + np^{u+1}) + \mathrm{ch}\nabla(\hat{R}(u) - 1 + (n+1)p^{u+1}).$$

Proof: Compare with [47, 7.1.1 Lemma], or (2.2.3).

While we have no general proof of the conjecture, there do at least exist some families of elementary generators. Indeed, the proof of the main theorem in [26] has already used one such family:

Proposition 3.2.3 *All the $E(R(0))$ with $R(0)$ as above are elementary generators.*

Proof: For all $R(0)$, the $P_{R(0)}$ defined in [47, 5.1] are elementary generators by [47, 7.1.2 Proposition].

We can also construct new examples of elementary generators from old by the following proposition, which we can apply to the family above.

Proposition 3.2.4 *For all $u \geq 0$ and $R(u)$ as above, if $E(R(u))$ is an elementary generator then so is $E((p^d - 1 + p^d R(u))(u+d))$.*

Proof: Set $R'(u+d) = p^d - 1 + p^d R(u)$. Then we have $R'(u+d) = \sum_{i=o}^{u+d} r'_i p^i$ with $0 \leq r'_i \leq p-1$ for all i and $r'_{u+d} \neq p-1$. So define $\hat{R}'(u+d)$ as before, and observe that $\hat{R}'(u+d) = p^d \hat{R}(u)$. Now, by [34, II 3.19],

$$St_d \otimes \nabla(S)^{F^d} \cong \nabla(p^d - 1 + Sp^d),$$

where St_d is the dth Steinberg module. Applying F^d to the defining sequence for $E(R(u))$, and then tensoring up with St_d, we obtain

$$0 \to St_d \otimes \nabla(R(u))^{F^d} \to St_d \otimes E(R(u))^{F^d} \to St_d \otimes \nabla(\hat{R}(u) + p^{u+1} - 1)^{F^d} \to 0,$$

which, as St_d is indecomposable as a G_d-module, is non-split by (3.1.1). Hence $E(R'(u+d)) \cong St_d \otimes E(R(u))^{F^d}$. It now follows that for all $n \geq 0$, the short exact sequence

$$0 \to \nabla(R'(u+d) + np^{u+d+1}) \to E(R'(u+d)) \otimes \nabla(n)^{F^{u+d+1}}$$

$$\to \nabla(\hat{R}'(u+d) + (n+1)p^{u+d+1} - 1) \to 0.$$

exists, as it may be obtained from the corresponding one for $E(R(u)) \otimes \nabla(n)^{F^u}$ by again applying F^d and tensoring up with St_d, and is non-split just as above.

Corollary 3.2.5 *For all $u \geq 0$ and $0 \leq r_u < p-1$, the module $E((p^u - 1 + r_u p^u)(u))$ is an elementary generator.*

Proof: This follows immediately from the previous two propositions.

The rest of this section will be devoted to giving a sufficient condition for $E(R(u))$ to be an elementary generator. We begin with an easy G_1 calculation, for which we adopt the notational conventions of [26]. In particular, we fix integers $i, j \geq 0$ such that $i + j = p - 2$.

Lemma 3.2.6 *For all $n \geq 0$ we have*

$$\mathrm{Hom}_{G_1}(\Delta(i), \nabla(pn+i)) \cong \nabla(n)^F.$$

If further $p > 2$ then

$$\mathrm{Hom}_{G_1}(\Delta(i), \nabla(pn+j)) = 0.$$

Proof: This follows just as in [26, (2.2)(3)(a) and (b)], but using the duals of the short exact sequences considered there. (Compare also with (2.3.2) and (2.3.3).)

Lemma 3.2.7 *For all $n \geq 1$, $u \geq 0$ and $R(u)$ as above we have*

$$\textit{i)}\ \mathrm{soc}(\nabla(R(u)) \otimes \nabla(n)^{F^{u+1}}) = L(R(u) + np^{u+1});$$

$$\textit{ii)}\ \mathrm{soc}(\nabla(\hat{R}(u) - 1 + p^{u+1}) \otimes \nabla(n)^{F^{u+1}}) \leq \bigoplus_{i=0}^{1} L(\hat{R}(u) - 1 + (n + (-1)^i)p^{u+1}).$$

Proof: i) Let $s = s_0 + p^{u+1}s_1$ where $0 \leq s_0 \leq p^{u+1} - 1$. Then

$$\begin{aligned}
&\mathrm{Hom}_G(L(s), \nabla(R) \otimes \nabla(n)^{F^{u+1}}) \\
&\cong \mathrm{Hom}_{G/G_{u+1}}(L(s_1)^{F^{u+1}}, \mathrm{Hom}_{G_{u+1}}(L(s_0), \nabla(R) \otimes \nabla(n)^{F^{u+1}})) \\
&\cong \mathrm{Hom}_{G/G_{u+1}}(L(s_1)^{F^{u+1}}, \mathrm{Hom}_{G_{u+1}}(L(s_0), \nabla(R)) \otimes \nabla(n)^{F^{u+1}}) \\
&\cong \begin{cases} \mathrm{Hom}_{G/G_{u+1}}(L(s_1)^{F^{u+1}}, \nabla(n)^{F^{u+1}}) & \text{if } s_0 = R, \\ 0 & \text{otherwise} \end{cases} \\
&\cong \begin{cases} \mathrm{Hom}_G(L(s_1), \nabla(n)) & \text{if } s_0 = R, \\ 0 & \text{otherwise} \end{cases} \\
&\cong \begin{cases} k & \text{if } s = R + np^{u+1}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence the first part follows.

ii) For the second part we will consider

$$\mathrm{Hom}_G(L(s), \nabla(r + p^t) \otimes \nabla(n)^{F^m}),$$

where $r = \sum_{i=0}^{m-1} r_i p^i$ and $s = \sum_{i=0}^{m-1} s_i p^i + s_m p^m$ with $0 \leq r_i, s_i \leq p - 1$ for all $i \leq t - 1$. First we define

$$r^a = \sum_{i=a}^{m-1} r_i p^{i-a}, \quad s^a = \sum_{i=a}^{m-1} s_i p^{i-a} + s_m p^{m-a}.$$

Then for $i \leq m - 1$ we have

$$\begin{aligned}
&\mathrm{Hom}_G(L(s^i), \nabla(r^i + p^{m-i}) \otimes \nabla(n)^{\mathrm{F}^{m-i}}) \\
&\cong \mathrm{Hom}_{G/G_1}(L(s^{i+1})^{\mathrm{F}}, \mathrm{Hom}_{G_1}(L(s_i), \nabla(r^i + p^{m-i}) \otimes \nabla(n)^{\mathrm{F}^{m-i}})) \\
&\cong \mathrm{Hom}_{G/G_1}(L(s^{i+1})^{\mathrm{F}}, \mathrm{Hom}_{G_1}(L(s_i), \nabla(r^i + p^{m-i})) \otimes \nabla(n)^{\mathrm{F}^{m-i}}) \\
&\cong \begin{cases} \mathrm{Hom}_{G/G_1}(L(s^{i+1})^{\mathrm{F}}, \nabla(r^{i+1} + p^{m-i-1})^{\mathrm{F}} \otimes \nabla(n)^{\mathrm{F}^{m-i}}) & \text{if } s_i = r_i, \\ 0 & \text{otherwise} \end{cases} \\
&\cong \begin{cases} \mathrm{Hom}_G(L(s^{i+1}), \nabla(r^{i+1} + p^{m-i-1}) \otimes \nabla(n)^{\mathrm{F}^{m-i-1}}) & \text{if } s_i = r_i, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

using the previous lemma for the penultimate step.

So by induction we have

$$\begin{aligned}
&\mathrm{Hom}_G(L(s), \nabla(r + p^m) \otimes \nabla(n)^{F^m}) \\
&\cong \begin{cases} \mathrm{Hom}_G(L(s^m), \nabla(r^m + 1) \otimes \nabla(n)) & \text{if } s_i = r_i \text{ for } 0 \leq i < m, \\ 0 & \text{otherwise} \end{cases} \\
&\cong \begin{cases} \mathrm{Hom}_G(L(s^m), \nabla(1) \otimes \nabla(n)) & \text{if } r = s - s_m p^m, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Now $\nabla(1) \otimes \nabla(n)$ has a good filtration (by [15, Proposition 4.2.2]) with terms $\nabla(n \pm 1)$. Hence for the above to be non-zero must have $s_m = n \pm 1$. Taking $m = u+1$ and $r = \hat{R}(u) - 1$ gives the desired result.

Consider the short exact sequence defining $E(R(u))$. Tensoring this with $\nabla(n)^{F^{u+1}}$ we obtain an exact sequence

$$E(R(u)) \otimes \nabla(n)^{F^{u+1}} \to \nabla(\hat{R}(u) - 1 + p^{u+1}) \otimes \nabla(n)^{F^{u+1}} \to 0.$$

Combining this with the surjection from (3.1.2) (taking $r = \hat{R}(u) - 1$ and $m = u+1$) we obtain a short exact sequence

$$0 \to X(R(u), n) \to E(R(u)) \otimes \nabla(n)^{F^{u+1}} \to \nabla(\hat{R}(u) - 1 + (n+1)p^{u+1}) \to 0,$$

where $X(R(u), n)$ is the kernel of the surjection. By the character lemma, we have that $\mathrm{ch}X(R(u), n) = \mathrm{ch}\nabla(R + np^{u+1})$. Now ∇'s have the usual universal property (as noted in Section 1.6), so to show that $X(R(u), n) \cong \nabla(R + np^{u+1})$ it is enough to show that $\mathrm{soc}X(R(u), n) \cong L(R + np^{u+1})$.

Lemma 3.2.8 *For all $n \geq 0$, $u \geq 0$ and $R(u)$ as above we have*

$$L(R(u) + np^{u+1}) \leq \mathrm{soc}(X(R(u), n)) \leq L(R(u) + np^{u+1}) \oplus L(\hat{R}(u) - 1 + (n-1)p^{u+1}).$$

Proof: From the definition of $X(R(u), n)$, along with the previous lemma, we have

$$\begin{aligned} \mathrm{soc}X(R(u), n) &\leq \mathrm{soc}(E(R(u)) \otimes \nabla(n)^{F^{u+1}}) \\ &\leq \mathrm{soc}(\nabla(R(u)) \otimes \nabla(n)^{F^{u+1}}) + \mathrm{soc}(\nabla(\hat{R}(u) - 1 + p^{u+1}) \otimes \nabla(n)^{F^{u+1}}) \\ &\subseteq L(R(u) + np^{u+1}) \bigoplus \oplus_{i=0}^{1} L(\hat{R}(u) - 1 + (n + (-1)^i)p^{u+1}). \end{aligned}$$

As $\nabla(R(u)) \otimes \nabla(n)^{F^{u+1}} \leq X(R(u), n)$, we have $L(R(u) + np^{u+1}) \leq \mathrm{soc}(X(R(u), n))$ by the previous lemma. From the character of $X(R(u), n)$ we see that $L(\hat{R}(u) - 1 + (n+1)p^{u+1}) \not\leq X(R(u), n)$, and the result now follows.

We can now give a sufficient condition for $E(R(u))$ to be an elementary generator.

Proposition 3.2.9 *For all $u \geq 0$, if $E(R(u))$ has a simple socle then it is an elementary generator.*

Proof: By the previous lemma, we have that $\mathrm{soc}(E(R(u))) \cong L(R(u))$. Let $s = s_0 + s_1 p^{u+1}$ with $0 \leq s_0 < p^{u+1}$. Now

$$\mathrm{Hom}_G(L(s), E(R(u))) \cong \mathrm{Hom}_{G/G_{u+1}}(L(s_1)^{F^{u+1}}, \mathrm{Hom}_{G_{u+1}}(L(s_0), E(R(u)))).$$

If $\mathrm{Hom}_{G_{u+1}}(L(s_0), E(R(u))) \neq 0$ then it is isomorphic to $V^{F^{u+1}}$ for some V. In this case $\mathrm{Hom}_G(L(s), E(R(u))) \cong \mathrm{Hom}_G(L(s_1), V) \neq 0$ for some s_1. By the above remarks, $\mathrm{Hom}_G(L(s), E(R(u))) \neq 0$ implies that $s = R(u)$, and in particular that $s_0 = R$. So $\mathrm{Hom}_{G_{u+1}}(L(s_0), E(R(u))) \neq 0$ if, and only if, $s_0 = R(u)$. Consider

$$\mathrm{Hom}_G(L(s), E(R(u)) \otimes \nabla(n)^{F^{u+1}})$$

$$\cong \mathrm{Hom}_{G/G_{u+1}}(L(s_1)^{F^{u+1}}, \mathrm{Hom}_{G_{u+1}}(L(s_0), E(R(u))) \otimes \nabla(n)^{F^{u+1}}).$$

By the last remark this is zero unless $s \equiv R(u) \pmod{p^{u+1}}$. We already have an upper bound for $\mathrm{soc}(E(R(u)) \otimes \nabla(n)^{F^{u+1}})$ from the proof of the previous lemma, and combining these results gives $\mathrm{soc}(E(R(u)) \otimes \nabla(n)^{F^{u+1}}) \cong L(R(u) + np^{u+1})$. This implies that $X(R(u), n)$ has the desired socle, and that the sequence does not split, as required.

Chapter 4

The blocks of the q-Schur algebras

The main result in this chapter is a description of the blocks of the q-Schur algebra. We have already considered the case $n = 2$ in Chapter 2, and the result has been proved for $d \leq n$ in [32, 4.24]. Our argument will proceed just as in the proof of the classical result in [18].

To do this however, we will first need to prove some quantum analogues of various standard classical group results. These are of independent interest, and we devote the first section to a brief review of them. The main result there is the Strong Linkage Principle, and the proof of this is ultimately just that of the classical result given in [1]. Our arguments merely verify that the proofs given for the Manin quantisation (for l odd) in [39, Chapter 10] also hold for the Dipper-Donkin quantisation (without restriction on l). The usual category isomorphism (1.9.5) will then give the corresponding results for the Manin quantisation without restriction to odd l.

The second section is devoted to showing that the various technical results used to determine the blocks for the classical case also hold in our setting. With these, the proof of the main result is identical to that in [18]. Finally we conclude with a simple corollary of this result, which gives the blocks of the quantum group.

4.1 The strong linkage principle

In this section we will prove the strong linkage principle for q-$\mathrm{GL}(n,k)$. This is proved for the Manin quantisation (for q a primitive lth root of unity with l odd) in [39, (10.3.5)] and hence, by the isomorphism of module categories in (1.9.5), for our chosen quantisation (for odd l). However we will show that the restriction on l is unnecessary in both cases.

Just as in [39], we first define a strong linkage relation on $X(T)$ with respect to the dot action of the affine Weyl group. In particular, a weight λ is *strongly linked to* μ, written $\lambda \uparrow \mu$, if $\lambda = \mu$ or there exists a finite sequence of weights $\mu = {}_0\mu, {}_1\mu, \ldots, {}_t\mu = \lambda$ such that for $i = 0, \ldots, t-1$,

$$_{i+1}\mu = s_{\alpha_i} \cdot {}_i\mu + m_i l \alpha_i$$

for α_i a positive root and m_i a non-negative integer with $\langle {}_i\mu + \rho, \alpha_i \rangle \geq m_i l$. The main result in this section is

Theorem 4.1.1 (The strong linkage principle) *Let $\lambda \in X(T)^+$ and $\mu \in X(T)$ with $\mu + \rho \in X(T)^+$. If $L(\lambda)$ is a composition factor of $H^r(w.\mu)$ for some $w \in W$ and $r \in \mathbb{N}$ then $\lambda \uparrow \mu$.*

Proof: As noted in [39], the result follows just as in [1] provided certain preliminary results hold. We merely verify that each of these results holds just as for the Manin quantisation. The result will follow from Serre duality, the Borel–Weil–Bott theorem for small dominant weights, and two technical lemmas, [39, (10.2.1–2)]. As the proof of [39, (10.2.1)] given there is valid for all l, the isomorphism in (1.9.5) gives the result for our quantisation. With this, and [20, Lemma 3.2], we obtain [39, (10.2.2)] just as in [39]. So we will be done if we can prove the two theorems below.

A dominant weight λ is called *small* if either $\lambda = \sum_{i=1}^n r_i \varpi_i$ with $\sum_{i=1}^{n-1} r_i \leq l + 1 - n$, or λ is a minimal dominant weight. We have the following result (compare with [20, Theorem 3.9]).

Theorem 4.1.2 (Borel–Weil–Bott) *Let $\lambda \in X(T)$ be a small dominant weight. Then*

$$H^r(w.\lambda) = \begin{cases} \nabla(\lambda) & \text{if } r = l(w), \\ 0 & \text{otherwise.} \end{cases}$$

Proof: As we have the Grothendieck vanishing theorem (see [20, Proposition 3.10]), this follows just as in [39, (10.2.3)], noting that the other results used there have already been verified above.

So it remains to prove that Serre duality holds. We define $N = \frac{n}{2}(n-1)$.

Theorem 4.1.3 (Serre duality) *For any finite dimensional rational B-module V,*

$$H^{N-r}(-2\bar{\rho} \otimes V^*) \cong (H^r(V))^*.$$

Proof: This will follow just as in [39, (10.3.4)] once we have shown that [39, (10.1.1), (10.3.1–3) and the definition of pairing before (10.3.4)] all hold in this case, as we have the generalised tensor identities (see 1.4.1). Now [39, (10.3.1)] and [39, (10.3.2)] clearly hold here which, together with the last theorem, gives the desired pairing. So it remains to check [39, (10.1.1) and (10.3.3)].

We first observe that analogous results to those in [39, Sections 7.3–4] hold here, by similar arguments. Also, the results of [39, Sections 9.1–3] all hold in this setting with the same proofs, along with the usual isomorphism (1.9.5) which allows us to quote the required results from [39, Chapter 8]. With these preliminaries, plus the generalised tensor identities, the proof of [39, (10.1.1)] also holds in this setting.

So it just remains to verify [39, (10.3.3)]. The results in [39, (9.1.1),(9.5.1),(9.6.1) and (9.6.2–4)] all follow just as there (as they do not require l to be odd), replacing reference to [39, (3.5.1)] by [5, 1.1.5 Corollary]. Then we obtain [39, (9.6.5)] by the arguments given there. The proof of [39, (10.3.3)] now also follows just as in [39].

This concludes the proof of the strong linkage principle. Note that, via our usual isomorphism (1.9.5) this also gives the result for the Manin quantisation without restriction on l (with the usual modifications — see (1.9.6)).

4.2 The blocks of the q-Schur algebra

The main result in this section is a determination of the block structure of the q-Schur algebra, and hence of q-GL(n,k). The blocks of $S_q(n,d)$ have already been determined by James and Mathas [32, 4.24] in the case $d \leq n$. We first consider the blocks of G, using an easy argument based on the strong linkage principle and the following lemma.

Lemma 4.2.1 *For any dominant weight λ and $r \geq 1$,*

$$St_r \otimes \bar{\nabla}(\lambda)^{F^r} \cong \nabla((lp^{r-1}-1)\rho + lp^{r-1}\lambda). \tag{4.1}$$

Proof: Set $\lambda' = (lp^{r-1}-1)\rho + lp^{r-1}\lambda$. As both sides of (4.1) have the same character, by the universal property of ∇'s it is enough to show that

$$\mathrm{soc}(St_r \otimes \bar{\nabla}(\lambda)^{F^r}) \cong L(\lambda').$$

For $0 \leq \alpha < lp^{r-1}$, consider

$$\mathrm{Hom}_G(L(\alpha) \otimes \bar{L}(\beta)^{F^r}, St_r \otimes \bar{\nabla}(\lambda)^{F^r}) \cong \mathrm{Hom}_{G_r}(L(\alpha) \otimes \bar{L}(\beta)^{F^r}, St_r \otimes \bar{\nabla}(\lambda)^{F^r})^{\bar{G}^r}.$$

As $\bar{L}(\beta)^{F^r}$ and $\bar{\nabla}(\lambda)^{F^r}$ are both trivial as G_r-modules, the terms in the right-hand side are isomorphic to direct sums of $L(\alpha)$'s and St_r's. But St_r and $L(\alpha)$ are simple as G_r-modules, so for a non-zero homomorphism to exist we must have $L(\alpha) \cong St_r$.

By Schur's lemma, $\mathrm{Hom}_{G_r}(St_r, St_r) = k$, and we have an injection

$$\mathrm{Hom}_k(\bar{L}(\beta)^{F^r}, \bar{\nabla}(\lambda)^{F^r}) \longrightarrow \mathrm{Hom}_{G_r}(St_r \otimes \bar{L}(\beta)^{F^r}, St_r \otimes \bar{\nabla}(\lambda)^{F^r}),$$

taking θ to $1 \otimes \theta$. By dimensions this is an isomorphism. Hence

$$\begin{aligned}
\mathrm{Hom}_G(St_r \otimes \bar{L}(\beta)^{F^r}, St_r \otimes \bar{\nabla}(\lambda)^{F^r}) &\cong \mathrm{Hom}_k(\bar{L}(\beta)^{F^r}, \bar{\nabla}(\lambda)^{F^r})^{\bar{G}^r} \\
&\cong \mathrm{Hom}_{\bar{G}^r}(\bar{L}(\beta)^{F^r}, \bar{\nabla}(\lambda)^{F^r}) \\
&\cong \mathrm{Hom}_{\mathrm{GL}_n}(\bar{L}(\beta), \bar{\nabla}(\lambda)) \\
&\cong \begin{cases} k & \text{if } \beta = \lambda, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence $\mathrm{soc}(St_r \otimes \bar{\nabla}(\lambda)^{F^r})$ consists of copies of $St_r \otimes \bar{L}(\lambda) \cong L(\lambda')$. But $\dim \bar{\nabla}(\lambda)_\lambda = 1$ and all other weights are less than λ, so only one such copy can occur, which gives the result.

Recall the θ notation from Section 2.4. For $\lambda \in X(T)$, not equal to $-\rho$, we define $m(\lambda)$ to be the least positive integer such that there exists an $\alpha \in \Phi^+$ with $\langle \lambda + \rho, \alpha \rangle \notin \theta(m(\lambda))\mathbb{Z}$. Our first partial result on the blocks of G is

Proposition 4.2.2 *If $\lambda \in X(T)^+$, then $(W.\lambda + \theta(m(\lambda))\mathbb{Z}\Phi) \cap X(T)^+$ is a union of blocks for G.*

Proof: First consider the case $m(\lambda) = 0$. It is enough to check that if τ, ν are dominant, with $\tau \in W.\lambda + l\mathbb{Z}\Phi$ and $L(\nu)$ a composition factor of $\nabla(\lambda)$, then $\nu \in W.\lambda + l\mathbb{Z}\Phi$. But this is an easy consequence of the strong linkage principle (4.1.1).

Now suppose that λ is any dominant weight, with $m(\lambda) = m$. Again, it is enough to show that if τ, ν are dominant, with $\tau \in W.\lambda + lp^{m-1}\mathbb{Z}\Phi$ and $L(\nu)$ a composition factor of $\nabla(\tau)$ then $\nu \in W.\lambda + lp^{m-1}\mathbb{Z}\Phi$. We first note that we have

$$\tau = (lp^{m-1} - 1)\rho + a\varpi + lp^{m-1}\tau',$$

with $\tau' \in X^+$ and $0 \leq a < lp^{m-1}$ (c.f. the definition of *normal form* in [18, Section 1]). So by the preceding lemma, $\nabla(\tau - a\varpi) \cong St_m \otimes \bar{\nabla}(\tau')^{F^m}$, and hence $\nabla(\tau) \cong (q\text{-det})^a \otimes St_m \otimes \bar{\nabla}(\tau')^{F^m}$ (as both sides have a simple socle and the same character). It is easy to see, by Steinberg's Tensor Product Theorem, that any composition factor of this module is of the form $L(\nu) \cong (q\text{-det})^a \otimes St_m \otimes \bar{L}(\nu')^{F^m}$, and (decomposing λ in the same way as for τ) that $\tau' \in W.\lambda' + p\mathbb{Z}\Phi$. So it is enough to show that $\nu' \in W.\lambda' + p\mathbb{Z}\Phi$. But this follows from the Strong Linkage Principle for $\mathrm{GL}(n,k)$ ([1, Theorem 1]).

We will show later that the sets described in the theorem are in fact precisely the blocks of G, which will follow from our description of the blocks of the q-Schur algebra. It should also be noted that in [45] Thams has already determined the blocks of the quantum enveloping algebra (from which the blocks of G could be derived) but under the additional assumptions that l is odd and greater than the Coxeter number for Φ.

Most of the rest of this section is taken up with proving

Theorem 4.2.3 *For any $\lambda \in \Lambda^+(n,d)$, the $S_q(n,d)$-block containing λ is*

$$(W.\lambda + \theta(m(\lambda))\mathbb{Z}\Phi) \cap \Lambda^+(n,d).$$

In what follows, it will be convenient to call a weight λ *primitive* if $m(\lambda) = 0$. By the last result, it makes sense to define a *primitive block* as one consisting of primitive elements. We first deal with the non-primitive blocks, as for these the result can be easily deduced from the classical case.

Proposition 4.2.4 *For $d \geq 0$, $m \geq 0$, $0 \leq a < \theta(m)$ and $\mathcal{B}$ a block of $S(n,d)$, the set*

$$\mathcal{B}^\dagger = \{(\theta(m) - 1)\rho + a\varpi + \theta(m)\mu : \mu \in \mathcal{B}\}$$

is a block of $S_q(n,e)$, where $e = (\theta(m) - 1)|\rho| + na + \theta(m)d$.

Proof: Define $\Phi : \text{mod } \mathrm{GL}(n,k) \longrightarrow \text{mod } G$ by

$$\begin{array}{ll} \Phi(\bar{V}) & = (q\text{-det})^a \otimes St_{m+1} \otimes \bar{V}^{F^{m+1}}, \\ \Phi(\theta) & = 1 \otimes 1 \otimes \theta. \end{array}$$

Now $(q\text{-det})^a \otimes St_{m+1} \cong \nabla(\sigma)$ where $\sigma = (\theta(m)-1)\rho + a\varpi$, as both sides have a simple socle and the same character. The result now follows just as in [18, Section 4, Theorem], noting that if $\bar{V}$ is indecomposable then so is $\nabla(\sigma) \otimes \bar{V}^{F^{m+1}}$ by (3.1.1).

With the above proposition, the main theorem now follows for λ non-primitive from the description of the blocks for $\mathrm{S}(n,d)$ given in [18, Section 4, Corollary].

So it remains to prove the theorem when λ is a primitive weight. We first note that the remarks in [18, page 405] concerning p-cores all hold when p is replaced by l, and so (given our partial result on the blocks of G) we have the following result.

Lemma 4.2.5 *For primitive dominant weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$, if λ and μ belong to the same G-block then there exists a $\pi \in \Sigma_n$ such that*

$$\lambda_i - i \equiv \mu_{\pi(i)} - \pi(i) \pmod{l}$$

for all $1 \leq i \leq n$. If further λ, $\mu \in \Lambda^+(n,d)$ then they have the same l-core.

The remainder of this section is essentially devoted to verifying that the proof give in the classical case for primitive blocks in [18, Section 3, Theorem] holds (with the obvious modifications) in this setting. Examination of the proof given there gives that this will be the case provided [18, Section 3 (1–6), Section 1 (5,8) and Section 2 (3, Proposition)] all hold.

The six results in Section 3 are all straightforward. A quantum Carter's criterion (1) is proved in [32, 4.15], whilst the criterion for λ to be a p-core in (2) can be seen from [31, 2.7.40] not to require p prime (provided we replace "coprime to p" by "not divisible by p"). The result on core classes (3) follows directly from the previous lemma and proposition, whereas (4) is immediate. The final two results, both consequences of James' results on row and column removal and decomposition numbers, hold with the same proofs as given, but replacing references to [14, Theorems 1 and 2] by [10, 4.2(9) and 4.2(15)] respectively.

The next result is an analogue of [18, Section 1(5)].

Proposition 4.2.6 *Let $\lambda \in X(T)^+$. Suppose that λ is primitive and $\nabla(\lambda)$ is irreducible. Then we have $\langle \lambda, \alpha \rangle < l$ for all $\alpha \in \Pi$.*

Proof: Consider the Manin quantisation. Now Steinberg's tensor product theorem holds for l odd by [39, (9.4.1)], and for l even by [3, Theorem, and concluding remarks] (but replacing l in this case by $l/2$ as remarked in (1.9.6)). Hence it also holds for $\mathrm{SL}_q(n,k)$. Now [30, Section 2.5 Theorem] clearly holds with l or $l/2$ replacing p, and hence we have [30, Section 2.5 Corollary], possibly with l modified. The result now follows just as in [18] (possibly with modified l). Again, tensoring with $\det_q$ will not affect reducibility, giving the result for $\mathrm{GL}_q(n,k)$. The usual category isomorphism (1.9.5) now gives the result for our quantisation, and corrects any modifications to l introduced during the Manin stage.

To start the induction off in the proof of the main theorem, we need to check some small cases. It will be convenient for this to define, as in [34, II 5.7], an Euler characteristic for any given finite dimensional B-module M by

$$\chi(M) := \sum_{i \geq 0} (-1)^i \mathrm{ch} H^i(M).$$

As usual, we write $\chi(\lambda)$ for $\chi(k_\lambda)$, and then Kempf's vanishing theorem (1.6.2) gives

$$\chi(\lambda) = \mathrm{ch}\nabla(\lambda) \quad \text{for all } \lambda \in X(T)^+,$$

so this notation is compatible with that in Section 1.6. Just as in the classical case we have

Lemma 4.2.7 *i) The characters $\mathrm{ch}L(\lambda)$ with $\lambda \in X(T)^+$ form a basis of $\mathbb{Z}[X]^W$.*

ii) For all $\lambda \in X$ and $\sum_\mu a(\mu)e(\mu) \in \mathbb{Z}[X]^W$,

$$\chi(\lambda) \sum_\mu a(\mu)e(\mu) = \sum_\mu a(\mu)\chi(\lambda + \mu).$$

iii) For all $w \in W$ and $\lambda \in X$ we have $\chi(w.\lambda) = (\mathrm{sgn}\ w)\chi(\lambda)$.

Proof: The first two parts follow just as in [34, II 5.8 Lemma], using [20, Theorem 2.10 and Lemma 3.1], while the last part follows just as in [34, II 5.9] from [20, Lemmas 2.12 and 3.1].

We are now able to check the necessary small cases, corresponding to those in [18, Section 1(8)].

Proposition 4.2.8 *Let $n = 1$,2 or 3, and $\lambda \in X(T)^+$ be primitive. Then*

i) the module $\nabla(\lambda)$ is irreducible if, and only if, λ is minimal in its block;

ii) each primitive G-block contains a unique minimal element.

Proof: The case $n = 1$ is clear, while $n = 2$ follows from (2.1.1) and (2.1.2). Consider $\lambda = (A, B, C)$ primitive. Without loss of generality we may assume that $C = 0$ (as tensoring up with an appropriate power of q-det will give the general case in what follows). By (4.2.6), if λ is primitive and $\nabla(\lambda)$ is irreducible then $0 \leq A-B, B \leq l-1$ and $(A - B, B) \neq (l-1, l-1)$. Suppose $(A - B) + B = l + m$ with $m > 0$, and $A-B, B < l-1$ (this cannot arise when $l = 2$). Then $A-B = l-1-a$ and $B = l-1-b$ with a, $b > 0$ and $a+b < l-2$. Consider $\nabla(l-2-b, l-1-a-b, 0)$. This is minimal in its block, and hence simple, so by Steinberg's Tensor Product Theorem we see that $\bar{\nabla}(1, 0, 0)^F \otimes \nabla(l-2-b, l-1-a-b, 0)$ is also simple, isomorphic to $L(\tau)$ for some τ.

Now, using the previous proposition, we have

$$\begin{aligned} \mathrm{ch}L(\tau) &= (e(l,0,0) + e(0,l,0) + e(0,0,l))\chi(l-2-b, l-1-a-b, 0) \\ &= \chi(\tau) + \chi(l-2-b, 2l-1-a-b, 0) + \chi(l-2-b, l-1-a-b, l) \\ &= \chi(\tau) - \chi(2l-2-a-b, l-1-b, 0) + \chi(l-2, l-1-b, l-a-b). \end{aligned}$$

After rearranging, and noting that the central term on the right is just $\mathrm{ch}\nabla(A, B, C)$, we see that

$$\mathrm{ch}\nabla(\tau) = \mathrm{ch}L(\tau) + \mathrm{ch}\nabla(\lambda) - \mathrm{ch}\nabla(l-2, l-1-b, l-a-b),$$

which implies that $\nabla(\lambda)$ is not simple. After tensoring with q-det we see that the primitive weights λ with $\nabla(\lambda)$ simple are a subset of

$$\begin{gathered} \{(A, B, C) \in X(T)^+ : 0 \leq A - C \leq l-2\} \\ \cup\{(l-1+a+C, a+C, C), (l-1+a+C, l-1+C, C) : 0 \leq a \leq l-2\}. \end{gathered}$$

But all these elements are minimal in their corresponding blocks, and as any minimal element must be simple this gives the result.

Proposition 4.2.9 *If λ, $\mu \in \Lambda^+(n, d)$ belong to the same block of $S_q(n, d)$ then they are in the same block of $S_q(m, d)$ for all $m \geq n$.*

Proof: This is clear from [10, 4.2(6)].

As this last result corresponds to [18, Section 2(3)], it just remains to check the proposition in [18, Section 2]. To verify this we need to consider the Schur functor $f = f_{n,d} : \text{mod } S_q(n,d) \longrightarrow \text{mod } \mathcal{H}_q(d)$, defined when $d \leq n$. This is analogous to the usual Schur functor defined in [29, Chapter 6], and its basic properties are outlined in [10, Section 2.1]. For the rest of this section we assume that $d \leq n$.

We say that $\lambda \in \Lambda^+(n,d)$ is *column l-regular* if $\lambda_i - \lambda_{i+1} < l$ for all $1 \leq i \leq n$, and denote the set of these by $\Lambda^+(n,d)_{\text{col}}$. Further, λ is called *row l-regular* if there does not exist an i with $0 \leq i \leq n-l$ such that $\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+l} > 0$. The set of row l-regular elements in $\Lambda^+(n,d)$ is denoted $\Lambda^+(n,d)_{\text{row}}$. With this notation we have the following result from [10, 4.4(4)(ii)].

Lemma 4.2.10 *Suppose that $d \leq n$. Then $\{fL(\lambda) : \lambda \in \Lambda^+(n,d)_{\text{col}}\}$ is a complete set of inequivalent irreducible $\mathcal{H}_q(d)$-modules.*

As noted in [10, 4.3(10)(ii)], there is a bijection $i : \Lambda^+(n,d)_{\text{row}} \longrightarrow \Lambda^+(n,d)_{\text{col}}$ such that, for $\lambda \in \Lambda^+(n,d)_{\text{row}}$, we have $I_S(\lambda) \cong T(i(\lambda))$, the indecomposable tilting module of highest weight $i(\lambda)$. This, along with (1.8.2), gives

Lemma 4.2.11 *Suppose $d \leq n$ and $\lambda \in \Lambda^+(n,d)_{\text{row}}$. Then $i(\lambda)$ is the unique highest element in the set $\mathcal{D}(\lambda) = \{\mu \in \Lambda^+(n,d) : [\nabla(\mu) : L(\lambda)] \neq 0\}$, and further $[\nabla(i(\lambda)) : L(\lambda)] = 1$.*

We also have a notion of contravariant duality (see [10, Remarks before 4.1d]), and this, combined with the results above, allows us to prove an analogue of [18, 2(5)]. For we have that $E^{\otimes d}$ is injective by [10, 2.1(8)], and the rest of part i) follows just as in the classical case, using [10, 4.3(9)] instead of [29, (6.4b)]. Part ii) follows from the arguments above and [10, 4.3(10)(i)]. Finally part iii) follows much as in the original case, but replacing reduction to characteristic zero by reduction to the case q a non-root of unity, and then using that the corresponding q-Schur algebra is semi-simple (see [10, 4.3(7)(i)]).

We are now almost in a position to verify the proof of the desired proposition. The one outstanding fact needed is an analogue of [29, (6.4c) Theorem] giving a basis of $f\nabla(\lambda)$. But, using the identification given in [10, 4.5h] of $f\nabla(\lambda)$ with the Specht module of Dipper and James, this follows from [8, 8.1], as noted in [24, Remark after Theorem 1.5]. The required proposition now follows just as in [18], which then gives the main result.

Finally in this section, we use the above result to determine precisely the blocks of G.

Theorem 4.2.12 *For $\lambda \in X(T)^+$, the G-block containing λ is*

$$(W.\lambda + \theta(m(\lambda))\mathbb{Z}\Phi) \cap X(T)^+.$$

Proof: Clearly, by (4.2.2), it is enough to show that any two elements of the above set are in the same block. So suppose that τ, $\mu \in (W.\lambda + \theta(m(\lambda))\mathbb{Z}\Phi) \cap X(T)^+$. If these lie in $\Lambda^+(n,d)$ for some d then we are done, as they are then in the same $S_q(n,d)$ block, and the result follows from [20, 4(5)]. Otherwise there exists an e such that $\tau' = \tau + e\varpi$ and $\mu' = \mu + e\varpi$ lie in $\Lambda^+(n,d)$ for some d. As these then lie in the same block of G by the above argument, there exists a sequence of weights, say $\tau' = {}_1\tau', \ldots, {}_t\tau' = \mu'$, with $[\nabla({}_i\tau') : L({}_{i+1}\tau')] \neq 0$ or $[\nabla({}_{i+1}\tau') : L({}_i\tau')] \neq 0$ for $1 \leq i < t$. Setting ${}_i\tau = {}_i\tau' - e\varpi$, we note that $\nabla({}_i\tau') \cong \nabla({}_i\tau) \otimes (q\text{-det})^e$ and $L({}_i\tau') \cong L({}_i\tau) \otimes (q\text{-det})^e$. So the sequence $\tau = {}_1\tau, \ldots, {}_t\tau = \mu$ satisfies the conditions of (1.8.1), and hence the two weights lie in the same G-block.

Chapter 5

Infinitesimal q-Schur algebras

In this section we introduce the infinitesimal q-Schur algebras — quantum analogues of the infinitesimal Schur algebras introduced in [22]. These allow us to combine both the infinitesimal and polynomial theories introduced earlier. The first section is devoted to the construction of these algebras, and a classification of their simple modules.

Next we define two truncation functors, following [16]. The main result of this section shows that they are equal, at least when $n = 2$. This result will be used in the following chapter to determine the blocks of these algebras. After considering two bases for the q-Schur algebra, and their restriction to the infinitesimal case, the chapter concludes with a brief survey of induction for infinitesimal monoids.

5.1 Definition and basic properties

In this first section we will define the infinitesimal q-Schur algebras, and develop some of their basic representation theory. Most of this follows closely the development of the corresponding classical theory in [22, Sections 1–3].

Recall that in Section 1.1 we defined M to be the object with coordinate algebra $k[M] = A_q(n)$. Let J_r be the ideal in $A_q(n)$ generated by all $c_{ij}^{lp^{r-1}}$ for $1 \leq i \neq j \leq n$. This is in fact a coideal — $\epsilon(J_r) = 0$ is clear, while $\delta(c_{ij}^{lp^{r-1}}) = \sum_{k=1}^{n} c_{ik}^{lp^{r-1}} \otimes c_{kj}^{lp^{r-1}}$ by [25, 3.1] and the centrality of c_{ij}^{l} (see [5, 1.3.2]). Thus $A_q(n)/J_r$ is also a bialgebra, and gives rise to a quantum monoid which we denote by $M_r D$.

Just as in the classical case we have the following commutative diagram:

$$\begin{array}{ccccc} k[M] & \longrightarrow & k[M_rD] & \longrightarrow & k[D] \\ \downarrow & & \downarrow & & \downarrow \\ k[G] & \longrightarrow & k[G_rT] & \longrightarrow & k[T]. \end{array}$$

Indeed, $k[M_rD]$ is the subbialgebra of $k[G_rT]$ generated by the c_{ij}, and $k[G_rT]$ is the localisation of $k[M_rD]$ at the quantum determinant. Thus $k[M_rD]$ is the polynomial part of $k[G_rT]$. We call objects in $\mathrm{Mod}_{k[M_rD]}(G_rT)$ *polynomial G_rT modules.*

Now $A_q(n)/J_r$ is graded; in particular we have $A_q(n)/J_r = \bigoplus_{d\geq 0} A_q(n,d)_r$, where $A_q(n,d)_r$ is the subspace consisting of the homogeneous polynomials of degree d in the c_{ij}. This subspace is clearly also a subcoalgebra of $A_q(n)/J_r$, for all d. Hence we may define the *infinitesimal q-Schur algebra* $S_q(n,d)_r = A_q(n,d)_r^*$. We will say that objects in $\mathrm{Mod}_{A_q(n,d)_r}(G_rT)$ are *homogeneous of degree d.*

As $A_q(n,d)_r$ is a quotient of $A_q(n,d)$, we have that $S_q(n,d)_r$ is a subalgebra of $S_q(n,d)$. In fact

$$S_q(n,d)_1 \subseteq \cdots \subseteq S_q(n,d)_r \subseteq \cdots \subseteq S_q(n,d)$$

and $S_q(n,d) = \lim_{r\to\infty} S_q(n,d)_r$ for each n, d.

Remark 5.1.1 Before proceeding further, we note that for l odd a similar theory can be developed for the Manin quantisation, and gives rise to the same algebra by (1.9.11). Indeed, in certain respects this is easier to work with (see for example (5.3.8)).

The first result of this section is an infinitesimal analogue of (1.6.5).

Proposition 5.1.2 *i) The category of polynomial G_rT-modules is equivalent to the category of M_rD-modules.*
ii) Every polynomial G_rT-module V has a direct sum decomposition $V = \bigoplus_{d\geq 0} V_d$ where V_d is homogeneous of degree d.
iii) The category of finite-dimensional $S_q(n,d)_r$-modules is equivalent to the category of homogeneous polynomial G_rT-modules of degree d.

Proof: This follows just as in the ordinary case (see [28, Section 1.6] and [29, pages 3–11]) as noted in [22, 2.1 Proposition].

We next classify the simple M_rD-modules, and hence the simple polynomial G_rT-modules. The result also classifies the simple $S_q(n,d)_r$-modules. First we fix some notation. Corresponding to the character group for T, we have the *character monoid* $P(D)$, which is isomorphic to $\mathbb{N}^n$. This consists of the *polynomial weights*. We set $P(D)^+ = P(D) \cap X(T)^+ = \{\lambda \in \mathbb{N}^n : \lambda_1 \geq \cdots \geq \lambda_n\}$, and elements of this set are called *dominant polynomial weights*. As our definition of lp^{r-1}-restricted weights, $X_r(T)$, is more restricted than classically (we impose the extra restriction that $0 \leq \lambda_n \leq lp^{r-1} - 1$), it coincides with the definition of *lp^{r-1}-restricted polynomial weights* $P_r(D)$ in [22]. The notation we use for this set will depend on the context in which it arises.

Any $\lambda \in X(T)$ can be uniquely written in the form $\lambda = \lambda' + lp^{r-1}\lambda''$, with $\lambda' \in P_r(D)$ and $\lambda'' \in X(T)$. By the remarks in Section 1.7, this corresponds to the tensor decomposition

$$\hat{L}_r(\lambda) \cong L(\lambda') \otimes lp^{r-1}\lambda''.$$

We now obtain

Theorem 5.1.3 *Let V be a simple G_rT-module with all its weights polynomial. Then V is of the form $L(\lambda') \otimes lp^{r-1}\lambda''$ with $\lambda' \in P_r(D)$ and $\lambda'' \in P(D)$.*

Proof: This follows just as in the classical case (see [22, 3.2 Theorem]), using the fact that the character of a G-module is invariant under the Weyl group (see [20, Lemma 3.1(v)]).

By (1.7.1) we have the following corollary, as in [22].

Corollary 5.1.4 *A complete set of non-isomorphic simple modules in* $\mathrm{Mod}(M_rD)$ *is given by $\{\hat{L}_r(\lambda) : \lambda \in \Gamma_r(D)\}$, where $\Gamma_r(D) = P_r(D) + lp^{r-1}P(D)$.*

From this it follows that every simple M_rD-module has a unique tensor product decomposition of the form

$$\hat{L}_r(\lambda) \cong L(\lambda') \otimes lp^{r-1}\lambda'',$$

for $\lambda' \in P_r(D)$ and $\lambda'' \in P(D)$. Further, if we set $\Gamma_r^d(D) = \{\lambda \in \Gamma_r(D) : |\lambda| = d\}$, then it is clear that the set of simple $S_q(n,d)_r$-modules is in one-to-one correspondence with $\Gamma_r^d(D)$. Henceforth, we will denote by $\hat{L}_r(\lambda)$ both the simple M_rD- and G_rT-modules corresponding to $\lambda \in \Gamma_r(D)$.

5.2 Truncation functors

Our first remarks are quite general, and based on [16, Section 1]. Let C be any coalgebra, and B be a subcoalgebra of C. For a C-comodule V, we denote by V_B the unique maximal subcomodule of V belonging to B. Now this may be regarded as a B-comodule, and the corresponding functor $\mathcal{F}_B : V \longrightarrow V_B$ from $\mathrm{Mod}(C)$ to $\mathrm{Mod}(B)$ is left exact and takes injectives to injectives. In what follows we will sometimes wish to consider $\mathrm{Inf}_B^C \mathcal{F}_B$, that is consider V_B as a C-comodule.

Let π be a set of simple C-comodules. For V a C-comodule, set $\mathcal{O}_\pi(V)$ equal to the unique maximal C-subcomodule all of whose composition factors lie in π. Set $C(\pi) = \mathcal{O}_\pi(C)$. In [16], $C(\pi)$ is shown to be a subcoalgebra of C, and it is also shown that $\mathcal{O}_\pi(V) = \mathcal{F}_{C(\pi)}(V)$. We regard $\mathcal{O}_\pi$ as a functor from $\mathrm{Mod}(C)$ to $\mathrm{Mod}(C)$ (taking morphisms to their restrictions).

Now consider the case when $C = k[G_rT]$ and $B = k[M_rD]$. If V is a G_rT-module, one can view $\mathcal{F}_{M_rD}(V)$ as the maximal G_rT-submodule of V that lifts to an M_rD-module. Similarly, for $\pi = \{\hat{L}_r(\lambda) : \lambda \in \Gamma_r(D)\}$, we may regard $\mathcal{O}_\pi(V)$ as the maximal G_rT-submodule of V, all of whose composition factors lie in π.

Most of this section is devoted to considering the following conjecture.

Conjecture 5.2.1 *We have an equivalence of functors between $\mathcal{F}_{M_rD}$ and $\mathcal{O}_\pi$; that is for all G_rT-modules V, we have*

$$\mathrm{Inf}_{M_rD}^{G_rT} \mathcal{F}_{M_rD}(V) \cong \mathcal{O}_\pi(V).$$

If this holds, then any G_rT-module, all of whose composition factors lift to M_rD, will itself lift. The conjecture has been proved in the case $q = 1$ by Jantzen [35]. Unfortunately, we are not able to generalise this proof to the quantum case, as it relies on an action of the symmetric group on the coordinate algebra (which does not exist in our setting). However, similar methods will at least give the result in the case $n = 2$.

We begin with a result relating the injective modules for M_rD and G_rT. For each $\lambda \in \Gamma_r(D)$, we denote the injective hull of $\hat{L}_r(\lambda)$ in $\mathrm{Mod}(M_rD)$ by $\hat{I}_r(\lambda)$. By (5.1.4) we have

$$\mathrm{soc}_{M_rD} \mathcal{F}_{M_rD}(V) \cong \mathcal{F}_{M_rD}(\mathrm{soc}_{G_rT} V),$$

as M_rD-modules, for every G_rT-module V.

Proposition 5.2.2 *For $\lambda \in \Gamma_r(D)$ we have $\mathcal{F}_{M_rD}(\hat{Q}_r(\lambda)) \cong \hat{I}_r(\lambda)$.*

Proof: This is immediate as $\mathcal{F}_{M_rD}$ takes injectives to injectives, and $\hat{Q}_r(\lambda)$ has the appropriate simple socle.

Returning to our conjecture, we first note that we have an inclusion $k[M_rD] \subseteq \mathcal{O}_\pi(k[G_rT])$. Equivalence will follow if we can show this is in fact an equality, by the following lemma.

Lemma 5.2.3 *With $\pi = \{\hat{L}_r(\lambda) : \lambda \in \Gamma_r(D)\}$, the following are equivalent:*

i) $\mathcal{O}_\pi$ is equivalent to $\mathcal{F}_{M_rD}$;

ii) $\mathcal{O}_\pi(k[G_rT]) \cong k[M_rD]$;

iii) for all d, if π_d is the set of simple $S_q(n,d)_r$-modules then $\mathcal{O}_{\pi_d}(k[G_rT]) \cong A_q(n,d)_r$;

iv) $\mathcal{O}_\pi(\hat{Q}_r(\lambda)) \cong \hat{I}_r(\lambda)$ for all $\lambda \in \Gamma_r(D)$.

Proof: The equivalence of i) and ii) is clear, as every G_rT-module embeds into a direct sum of copies of $k[G_rT]$, by [39, 2.4.4]. The equivalence of ii) and iii) is also immediate. For the equivalence of ii) and iv) we use that

$$k[M_rD] = \bigoplus_{\lambda \in \Gamma_r(D)} [\dim \hat{L}_r(\lambda)] \hat{I}_r(\lambda).$$

This follows (as $k[M_rD]$ is injective [39, 2.8.2(1)] and $\mathrm{Mod}(M_rD)$ has enough injectives [39, 2.8.1]) by the usual arguments (see [34, I.3.14–17]). From (5.1.4) and the definition of $\mathcal{O}_\pi$, we see that $\mathcal{O}_\pi(\hat{Q}_r(\lambda)) \neq 0$ if, and only if, $\lambda \in \Gamma_r(D)$. There is a similar decomposition to that of $k[M_rD]$ above for $k[G_rT]$, so applying $\mathcal{O}_\pi$ to each side gives

$$\mathcal{O}_\pi(k[G_rT]) = \bigoplus_{\lambda \in \Gamma_r(D)} [\dim \hat{L}_r(\lambda)] \mathcal{O}_\pi(\hat{Q}_r(\lambda)).$$

As $\hat{I}_r(\lambda) \cong \mathcal{F}_{M_rD}(\hat{Q}_r(\lambda)) \subseteq \mathcal{O}_\pi(\hat{Q}_r(\lambda))$, the result now follows.

The following pair of lemmas will allow us to prove the result for the case $n = 2$. The former is a modification of the main lemma used by Jantzen in his proof for the classical case.

In order to be able to state the first result we need another description of $k[G_rT]$. By [25, 3.1], we have that

$$d_q^{lp^{r-1}} = c_{11}^{lp^{r-1}} c_{22}^{lp^{r-1}} \cdots c_{nn}^{lp^{r-1}},$$

and hence for all i,

$$c_{ii}^{-1} = c_{11}^{lp^{r-1}} \cdots c_{i-1i-1}^{lp^{r-1}} c_{ii}^{lp^{r-1}-1} c_{i+1i+1}^{lp^{r-1}} \cdots c_{nn}^{lp^{r-1}} d_q^{-lp^{r-1}}.$$

On the other hand,

$$d_q^{-1} = d_q^{lp^{r-1}-1} c_{11}^{-lp^{r-1}} \cdots c_{nn}^{-lp^{r-1}}.$$

Hence

$$k[G_rT] = k[c_{ij}, c_{ii}^{-1} : 1 \leq i, j \leq n] / \langle c_{ij}^{lp^{r-1}} : i \neq j \rangle,$$

with the usual relations.

Lemma 5.2.4 *Let V be a $k[G_rT]$-module with all weights polynomial. Then the coefficient space of V lies in*

$$k[c_{ij}, c_{tt}^{-1} : 1 \leq i, j \leq n, 1 \leq t \leq n-1] / \langle c_{ij}^{lp^{r-1}} : i \neq j \rangle.$$

Proof: Consider the natural map

$$\phi : k[G_rT] \longrightarrow k[B_rT] \otimes k[T] \otimes k[B_r^+T].$$

This is injective (by standard arguments — compare with [39, (8.1.1) Theorem]), and writing c_{ij} for the generators of all four quantum groups we see that

$$\phi(c_{ij}) = \sum_{t \leq i,j} c_{it} \otimes c_{tt} \otimes c_{tj}. \tag{5.1}$$

In particular, the only case in which any of the middle factors can contain a c_{nn} is when $i = j = n$.

Now take a basis of weight vectors for V, say $\{v_i : 1 \leq i \leq t\}$, with the corresponding set of coefficient functions $\{f_{ij}\}$. By assumption, the f_{ii} are polynomial for all i. As V is a comodule, we have that

$$(\mathrm{id} \otimes \delta)\delta(f_{ij}) = \sum_{s,t} f_{is} \otimes f_{st} \otimes f_{tj},$$

and as $\epsilon(f_{ij}) = \delta_{ij}$ this implies that

$$\phi(f_{ij}) = \sum_t \bar{f}_{it} \otimes \bar{f}_{tt} \otimes \bar{f}_{tj},$$

where the bars denote the appropriate restrictions. Thus we see that

$$\phi(f_{ij}) \subseteq k[B_rT] \otimes k[D] \otimes k[B_r^+T]. \tag{5.2}$$

Suppose now that there exists some f_{ij} involving c_{nn}^{-1}. We have that $f_{ij} = d_q^{-tlp^{r-1}}a$ with $a \in k[M_rD]$, and hence

$$\begin{aligned} \phi(f_{ij}) &= \phi(d_q^{-tlp^{r-1}})\phi(a) \\ &= \bar{d}_q^{-tlp^{r-1}} \otimes \bar{d}_q^{-tlp^{r-1}} \otimes \bar{d}_q^{-tlp^{r-1}}\phi(a). \end{aligned}$$

Writing ϕ_2 for the projection of ϕ onto the central factor of the tensor product we thus have that

$$\phi_2(f_{ij}) = c_{11}^{-tlp^{r-1}} \cdots c_{nn}^{-tlp^{r-1}}\phi_2(a).$$

By assumption, $a = c_{nn}^{tlp^{r-1}}b + e$, where e is non-zero and no term of e contains $c_{nn}^{tlp^{r-1}}$. So, again by [25, 3.1],

$$\phi(a) = (c_{nn}^{tlp^{r-1}} \otimes c_{nn}^{tlp^{r-1}} \otimes c_{nn}^{tlp^{r-1}})\phi(b) + \phi(e),$$

with $\phi(e)$ non-zero by the injectivity of ϕ. However, by (5.1), no term of $\phi_2(e)$ contains $c_{nn}^{tlp^{r-1}}$, and so $\phi_2(a) \notin c_{nn}^{tlp^{r-1}}k[D]$. Thus $\phi_2(f_{ij}) \notin k[D]$, which contradicts (5.2).

In the classical case, Jantzen now uses the invariance of the coefficient space under the action of the symmetric group to obtain the desired result. This action does not exist for non-trivial q, but we can at least prove the result for the case $n = 2$.

Lemma 5.2.5 *Let V be a $k[G_rT]$-module with all weights polynomial. Then the coefficient space of V lies in*

$$k[c_{ij}, c_{tt}^{-1} : 1 \le i, j \le n, 2 \le t \le n-1]/\langle c_{ij}^{lp^{r-1}} : i \ne j \rangle.$$

Proof: Consider $k[q\text{-GL}(n,k)]$ with the usual generators, and $k[q^{-1}\text{-GL}(n,k)]$ with generators x_{ij} and $d_{q^{-1}}$. We define a map

$$\phi : k[q\text{-M}(n,k)] \longrightarrow k[q^{-1}\text{-M}(n,k)]$$

by $\phi(c_{ij}) = x_{n+1-i,n+1-j}$. It is easy to check that this is a well-defined bialgebra homomorphism, and that it extends to a map of the corresponding quantum groups. Furthermore, it is also clear that it restricts to a map between the corresponding Jantzen subgroups, and so induces a map Φ from $\mathrm{Mod}(q\text{-}G_rT)$ to $\mathrm{Mod}(q^{-1}\text{-}G_rT)$. If V is a q-G_rT-module with polynomial weights, and its coefficient space contains terms involving c_{11}^{-1}, then $\Phi(V)$ is a q^{-1}-G_rT-module with polynomial weights whose coefficient space contains terms involving x_{nn}^{-1}. This gives a contradiction, as the previous lemma also holds for $\mathrm{Mod}(q^{-1}\text{-}G_rT)$.

5.3 Two bases for the q-Schur algebra

In this section we describe two bases of the q-Schur algebra, and consider how they give rise to bases for the infinitesimal case. The first of these, due to Dipper and Donkin, is easily obtained as the dual basis of a basis for $A_q(n)$. This will be used to determine the dimensions of the infinitesimal q-Schur algebras, and to construct an infinitesimal version of the Schur functor.

The second basis, due to Dipper and James, arises from regarding the q-Schur algebra as the endomorphism ring of some action of the Hecke algebra. We will relate these two bases, so as to be able to translate results expressed in terms of the latter to the former. In particular, this will allow us to describe an algebra anti-automorphism on $S_q(n,d)$, and show that it restricts to an anti-automorphism of $S_q(n,d)_r$. Hence we will be able to define a theory of contravariant duality in this setting.

We first construct the Dipper–Donkin basis. Let $I(n,d)$ denote the set of maps $\{1,\ldots,d\} \to \{1,\ldots,n\}$, and write $\mathbf{i} = (i_1,\ldots,i_d)$ for an element of this set. Then $I_0^2(n,d)$ is defined to be the subset of $I(n,d) \times I(n,d)$ consisting of all elements $(\mathbf{i},\mathbf{j})$ such that $i_1 \leq \cdots \leq i_d$ and $j_k \leq j_{k+1}$ if $i_k = i_{k+1}$. Now, by [5, 1.1.5], we have that $A_q(n,d)$ has a basis

$$\{c_{\mathbf{i},\mathbf{j}} : (\mathbf{i},\mathbf{j}) \in I_0^2(n,d)\},$$

where $c_{\mathbf{i},\mathbf{j}} = c_{i_1j_1}c_{i_2j_2}\ldots c_{i_dj_d}$. We shall also denote $c_{\mathbf{i},\mathbf{j}}$ by $c^\beta = \prod_{1\leq i,j\leq n} c_{ij}^{\beta_{ij}}$, where $\beta \in \mathcal{M}(n,d) = \{(\beta_{ij}) \mid \sum_{i,j}\beta_{ij} = d \text{ and } \beta_{ij} \geq 0\ \forall i,j\}$ satisfies $\beta_{ij} = \#\{r \mid c_{i_rj_r} = c_{ij}\}$, and the product is in lexicographic order. So our basis of $A_q(n,d)$ can be written

$$\{c^\beta : \beta \in \mathcal{M}(n,d)\}.$$

Now let $\mathcal{M}(n,d)_r$ denote $\{\beta \in \mathcal{M}(n,d) \mid \beta_{ij} < lp^{r-1} \text{ for all } 1 \le i \ne j \le n\}$. Then a basis of $A_q(n,d)_r$ is given by

$$\{c^\beta : \beta \in \mathcal{M}(n,d)_r\}.$$

Given the bijection between $\mathcal{M}(n,d)$ and $I_0^2(n,d)$ just described, we will let $I_0^2(n,d)_r$ be the subset of $I_0^2(n,d)$ corresponding to $\mathcal{M}(n,d)_r$.

For $(\mathbf{i},\mathbf{j}) \in I_0^2(n,d)$, let $\xi_{\mathbf{i},\mathbf{j}}$ denote the element of $S_q(n,d)$ such that for all $(\mathbf{k},\mathbf{l}) \in I_0^2(n,d)$,

$$\xi_{\mathbf{i},\mathbf{j}}(c_{\mathbf{k},\mathbf{l}}) = \begin{cases} 1 & \text{if } (\mathbf{i},\mathbf{j}) = (\mathbf{k},\mathbf{l}), \\ 0 & \text{otherwise.} \end{cases}$$

More generally, for $\mathbf{i},\mathbf{j} \in I(n,d)$, we write $\xi_{\mathbf{i},\mathbf{j}}$ for the element of $S_q(n,d)$ dual to $c_{\mathbf{i},\mathbf{j}}$. After the above discussion, the following proposition is clear.

Proposition 5.3.1 *The set $\{\xi_{\mathbf{i},\mathbf{j}} : (\mathbf{i},\mathbf{j}) \in I_0^2(n,d)\}$ is a basis of $S_q(n,d)$, dual to that of $A_q(n,d)$ above. Furthermore, the set $\{\xi_{\mathbf{i},\mathbf{j}} : (\mathbf{i},\mathbf{j}) \in I_0^2(n,d)_r\}$ is a basis of $S_q(n,d)_r$, dual to that of $A_q(n,d)_r$.*

Having constructed our basis, we can now determine the dimension of the infinitesimal q-Schur algebras.

Theorem 5.3.2 *For all $n \ge 1$, $d \ge 0$ and $r \ge 0$, we have*

$$\dim_k S_q(n,d)_r = \sum_{s \ge 0} (-1)^s \binom{n^2-n}{s} \binom{n^2+d-slp^{r-1}-1}{n^2-1}.$$

Proof: This follows just as in the classical case (see [22, 2.2 Theorem]), given the last proposition.

For $\mathbf{i} \in I(n,d)$, we let $\mathrm{ct}(\mathbf{i}) = (\alpha_1, \dots, \alpha_n)$ denote the *content* of $\mathbf{i}$, where $\alpha_j = \#\{l \in [1,d] \mid j = i_l\}$. The set of contents is in one-to-one correspondence with the set of Σ_n-orbits for the left action of Σ_n on $I(n,d)$ given by $\pi\mathbf{i} = (\pi(i_1), \dots, \pi(i_d))$. We identify this set with $\Lambda(n,d)$.

For all $\mathbf{i}$ such that $i_1 \le \cdots \le i_d$, we write ξ_λ for $\xi_{\mathbf{i},\mathbf{i}}$, when $\mathbf{i}$ has content $\lambda \in \Lambda(n,d)$. As noted in [38, pgs 168 and 183], the ξ_λ, for $\lambda \in \Lambda(n,d)$, form a set of mutually orthogonal idempotents in $S_q(n,d)$, whose sum is the identity element. Clearly ξ_λ

lies in $S_q(n,d)_r$ for all $r \geq 0$ and $\lambda \in \Lambda(n,d)$, so this idempotent decomposition also holds in $S_q(n,d)_r$.

We now define an infinitesimal version of the Schur functor in [10, Section 2.1]. This is done for the classical case in [22, Section 5]. Following [29, Section 6.2], let S be any k-algebra containing a non-zero idempotent e. Then eSe is a subalgebra of S with unit e, and we may define a functor $f(e) : \mathrm{mod}(S) \longrightarrow \mathrm{mod}(eSe)$ as follows. For $V \in \mathrm{mod}(S)$, $f(e)V$ is the eSe-module eV. If ϕ is a morphism in $\mathrm{mod}(S)$, then $f(e)\phi$ is its restriction. It is well-known that $f(e)$ is exact.

Take $S = S_q(n,d)_r$ and $e = \xi_\lambda$, and suppose that $d \leq n$. Then on taking $\lambda = \varpi = (1^d)$, represented by $\mathbf{u} = (1,2,\ldots,d) \in I(n,d)$, we get an infinitesimal version of the Schur functor.

Proposition 5.3.3 *With $d \leq n$, the algebra $\xi_\varpi S_q(n,d)_r \xi_\varpi$ is isomorphic to the Hecke algebra $\mathcal{H}_q = \mathcal{H}_q(\Sigma_d)$.*

Proof: Clearly $\xi_\varpi S_q(n,d)_r \xi_\varpi$ is a subalgebra of $\xi_\varpi S_q(n,d) \xi_\varpi$. By [10, 2.1(6)], we have that $\xi_\varpi S_q(n,d) \xi_\varpi$ has k-basis $\{\xi_{\mathbf{u},\mathbf{u}\pi} : \pi \in \Sigma_d\}$, and by [10, 2.1(10)′] is isomorphic to $\mathcal{H}_q$. So the result will follow if we can show that our subalgebra is actually the whole of $\xi_\varpi S_q(n,d)\xi_\varpi$. But for $\pi \in \Sigma_d$ we have that $\xi_{\mathbf{u},\mathbf{u}\pi} \in S_q(n,d)_r$ for all $r \geq 1$, as the corresponding c^β in $A_q(n,d)_r$ has no $\beta_{i,j}$ greater than 1. So the result now follows.

Hence we may identify $\mathrm{mod}(\xi_\varpi S_q(n,d)_r \xi_\varpi)$ and $\mathrm{mod}(\mathcal{H}_q)$. With this identification we obtain the *infinitesimal Schur functor*

$$f_r : \mathrm{mod}(S_q(n,d)_r) \longrightarrow \mathrm{mod}(\mathcal{H}_q),$$

by defining $f_r = f(\xi_\varpi)$.

Next we turn our attention to the Dipper–James basis for the q-Schur algebra. In order to describe this we need to introduce some more notation. For $\lambda \in \Lambda(n,d)$, we define the *Young subgroup* Σ_λ of Σ_d to consist of those permutations of $\{1,2,\ldots,d\}$ which leave invariant the sets $\{1,2,\ldots,\lambda_1\}$, $\{\lambda_1+1, \lambda_1+2, \ldots, \lambda_1+\lambda_2\}$, $\ldots$, $\{\lambda_1 + \ldots + \lambda_{n-1}+1, \ldots, d\}$. In each coset of Σ_λ in Σ_d, there is a unique element of minimal length, called the *distinguished coset representative.* The set of distinguished coset representatives for right (respectively left) cosets will be denoted $\mathcal{D}_\lambda$ (respectively $\mathcal{D}_\lambda^{-1}$). Given λ and $\mu \in \Lambda(n,d)$, we define $\mathcal{D}_{\lambda\mu}$ to equal $\mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$, the set of distinguished $\Sigma_\lambda - \Sigma_\mu$ double coset representatives.

We are now in a position to describe the Dipper–James basis. However, we will introduce this merely as a set of formal symbols — the interested reader should refer to [8, Section 1] or [38, Section 6.6] for details. As we are about to give an explicit isomorphism relating our two bases, this will suffice for our purposes.

Proposition 5.3.4 *The set* $\{\phi^e_{\lambda\mu} \mid \lambda, \mu \in \Lambda(n,d), e \in \mathcal{D}_{\lambda\mu}\}$ *forms a basis for* $S_q(n,d)$*, where the* $\phi^e_{\lambda\mu}$ *are certain elements of* $\mathrm{End}_{\mathcal{H}_q}(E^{\otimes d})$.

Proof: As noted in [8], this is just [6, 3.4 Lemma], though expressed there rather differently.

These two bases are related by

Theorem 5.3.5 *The isomorphism given in (1.2.1) maps the Dipper–Donkin basis elements to the Dipper–James basis elements above. Furthermore, we may give an explicit description of this correspondence.*

Proof: We will just give the explicit description of the map here. A full proof of the result can be found in [5, 3.2.5 Theorem] or [38, Theorem 6.6.6]. Let $(\mathbf{i},\mathbf{j}) \in I^2_0(n,d)$ be such that $\mathrm{ct}(\mathbf{i}) = \lambda$ and $\mathrm{ct}(\mathbf{j}) = \mu$. Define $\mathbf{u}_\mu$ to be the element of $I(n,d)$ with content μ, whose components are in increasing order. Then we may define $\tilde{e}$ to be the element satisfying

$$\Sigma_\mu \tilde{e} = \{\sigma \in \Sigma_d \mid \mathbf{u}_\mu \sigma = \mathbf{j}\}.$$

Take e to be the distinguished double coset representative in $\Sigma_\lambda \tilde{e} \Sigma_\mu$. Then the desired map takes $\xi_{\mathbf{i},\mathbf{j}}$ to $\phi^e_{\lambda\mu}$.

Our main reason for introducing the Dipper–James basis here is so as to be able to describe an algebra anti-automorphism J of $S_q(n,d)$. Given such an anti-automorphism, we can construct the *contravariant dual* of a (left) $S_q(n,d)$-module V, by giving the dual space V^* the structure of a left $S_q(n,d)$-module via the action

$$(s.f)(v) = f(J(s)v),$$

for all $s \in S_q(n,d)_r$, $f \in V^*$ and $v \in V$.

Theorem 5.3.6 *There exists an algebra anti-automorphism J of $S_q(n,d)$ satisfying*

$$J(\phi^{e}_{\lambda\mu}) = \phi^{e^{-1}}_{\mu\lambda}.$$

Proof: See [8, 1.11 Theorem].

Our next result will allow us to define contravariant duals in the category of $S_q(n,d)_r$-modules. It will be convenient to define for $\lambda \in \Lambda(n,d)$ the element $\mathbf{u}_\lambda = (1,\ldots,1,2,\ldots,2,3,\ldots)$, consisting of λ_1 "1"s, λ_2 "2"s etc.

Corollary 5.3.7 *The map J above restricts to an anti-automorphism of $S_q(n,d)_r$.*

Proof: It is enough to show that the image of any basis element for $S_q(n,d)_r$ is another such basis element. So suppose $(\mathbf{i},\mathbf{j}) \in I^2_0(n,d)_r$, and consider the element $\xi_{\mathbf{i},\mathbf{j}}$. This is the element of $S_q(n,d)_r$ dual to $c^\beta = c_{\mathbf{i},\mathbf{j}}$, which satisfies $\beta_{st} < lp^{r-1}$ for all $1 \leq s \neq t \leq n$. Now under the isomorphism of (5.3.5), $\xi_{\mathbf{i},\mathbf{j}}$ corresponds to $\phi^e_{\lambda\mu}$, where $\mathbf{i}$ has content λ, $\mathbf{j}$ has content μ, and e satisfies $\mathbf{u}_\mu e = \mathbf{j}$. Under J, this is mapped to the element $\phi^{e^{-1}}_{\mu\lambda}$, which corresponds to the element $\xi_{\mathbf{i}',\mathbf{j}'}$. Here $\mathbf{i}' = \mathbf{u}_\mu$ and $\mathbf{j}' = \mathbf{u}_\lambda e^{-1}$. From this we see that $(\mathbf{i}'e, \mathbf{j}'e) = (\mathbf{j},\mathbf{i})$. Letting $c^\alpha = c_{\mathbf{i}'e,\mathbf{j}'e}$, we have $\alpha_{st} < lp^{r-1}$, as $\alpha_{st} = \beta_{ts}$ (here we are abusing notation as c^α is not being expanded in lexicographic order, but as we are about to correct for this it does not matter). Defining $c^{\beta'} = c_{\mathbf{i}',\mathbf{j}'}$, we see that (as β'_{st} corresponds to $\alpha_{(se)(te)}$ and these are now in lexicographic order) the corresponding element $\xi_{\mathbf{i}',\mathbf{j}'}$ lies in $S_q(n,d)_r$, as required.

Remark 5.3.8 If we are prepared to restrict to the case of l being an odd root of unity, then using the Manin quantisation greatly simplifies the above discussion. In this case we have a coalgebra anti-automorphism (see [39, (3.7.1)]) sending X_{ij} to X_{ji}, and the above corollary becomes trivial.

5.4 Induced modules

In this section we define analogues for the monoid case of the induced modules $\hat{Z}_r(\lambda)$ of Section 1.7. Using truncation functors, we are then able to deduce some basic properties of these induced modules. By considering the contravariant duals of certain modules we then prove an infinitesimal version of Kostka duality.

Recall that we have defined the monoid corresponding to the lower triangular matrices, so we may define $k[L_rD] = k[q\text{-L}(n,k)]/J'_r$, where $J'_r = J_r \cap k[q\text{-L}(n,k)]$. Hence for $\lambda \in P(D)$, we can consider the induced module $\hat{A}_r(\lambda) = \mathrm{ind}_{L_rD}^{M_rD} k_\lambda$. Here k_λ denotes the one-dimensional D-module of weight λ, which can be regarded as a module for L in the usual way.

Proposition 5.4.1 *Let $\lambda \in P(D)$. Then*

i) $\hat{A}_r(\lambda) = 0$ unless $\lambda \in \Gamma_r(D)$;

ii) if $\lambda \in \Gamma_r(D)$, then $\hat{A}_r(\lambda) \cong \mathcal{F}_{M_rD}(\hat{Z}_r(\lambda))$.

Proof: Let $\lambda \in P(D)$. There exists an embedding $\hat{A}_r(\lambda) \longrightarrow k[G_rT]$, the composition of the natural inclusion of $\hat{A}_r(\lambda)$ in $k[M_rD]$ with the injection $\iota : k[M_rD] \longrightarrow k[G_rT]$. Consider induction from L_rD to M_rD. We have the obvious map $\hat{\phi} : k[M_rD] \longrightarrow k[L_rD]$ and, by definition,

$$\hat{A}_r(\lambda) = \{f \in |\lambda| \otimes k[M_rD] \mid f = e_\lambda \otimes g \text{ and } \tau(e_\lambda) \otimes g = \sum_i e_\lambda \otimes \hat{\phi}(g'_i) \otimes g''_i\},$$

where $\delta(g) = \sum_i g'_i \otimes g''_i$ and e_λ is a basis element for λ. Now $\tau(e_\lambda) = e_\lambda \otimes c_{11}^{\lambda_1} \dots c_{nn}^{\lambda_n}$, so

$$\hat{A}_r(\lambda) \cong \{g \in k[M_rD] \mid c_{11}^{\lambda_1} \dots c_{nn}^{\lambda_n} \otimes g = \sum_i \hat{\phi}(g'_i) \otimes g''_i\}.$$

Similarly,

$$\hat{Z}_r(\lambda) \cong \{g \in k[G_rT] \mid c_{11}^{\lambda_1} \dots c_{nn}^{\lambda_n} \otimes g = \sum_i \hat{\psi}(g'_i) \otimes g''_i\},$$

where $\hat{\psi} : k[G_rT] \longrightarrow k[B_rT]$ is the obvious map and $\delta'(g) = \sum_i g'_i \otimes g''_i$. Clearly $\hat{\psi}\iota = \hat{\phi}$, and $\delta'\iota = \delta$, so by the embedding above we have that if $f \in k[M_rD]$ lies in $\hat{A}_r(\lambda)$, then f lies in $\hat{Z}_r(\lambda)$. Hence $\hat{A}_r(\lambda)$ injects into $\hat{Z}_r(\lambda)$. The proof now proceeds just as in the classical case (see [22, 5.1 Proposition]).

The following corollary is now an immediate consequence of the result above, along with the known structure of $\hat{Z}_r(\lambda)$ and the classification in (5.1.4).

Corollary 5.4.2 *Let $\lambda \in \Gamma_r(D)$.*

i) Let $\hat{A}_r(\lambda) = \sum_{\mu \in P(D)} \hat{A}_r(\lambda)^\mu$ be a D-weight space decomposition. Then we have $\dim \hat{A}_r(\lambda)^\lambda = 1$ and $\dim \hat{A}_r(\lambda)^\mu \neq 0$ implies that $\mu \leq \lambda$ for all $\mu \in P(D)$.

ii) The module $\hat{A}_r(\lambda)$ has simple socle $\hat{L}_r(\lambda)$.

Proof: See [10, 3.1(13)(i) and (20)(ii)].

Consider the symmetric powers $S_q^d(E)$ of the natural module for M. Choosing a basis $\{X_1, \ldots, X_n\}$ for E, we have by [5, 2.1.9 Theorem] the isomorphism $S_q(E) = \bigoplus_{d\geq 0} S_q^d(E) \cong k[X_1, \cdots, X_n]$. The image of $S_q^d(E)$ in $k[X_1, \cdots, X_n]/\langle X_j^{lp^{r-1}} : j \neq i\rangle$ will be denoted by $S_q^d(E)_i$.

Lemma 5.4.3 *The ideal $\langle X_j^{lp^{r-1}} : j \neq i\rangle$ is an M_rD-submodule of $k[X_1, \cdots, X_n]$, and hence $S_q^d(E)_i$ is an M_rD-module.*

Proof: We use the explicit description of the structure map τ of $S_q^r(E)$ as a $k[G]$-modules given in [46], and note that this gives the structure map as an M_rD-module by composing with the usual quotient map. By [46, Lemma 2.3],

$$\tau(X_j^{lp^{r-1}}) = \sum_{\mathbf{b}} X_1^{b_1} \ldots X_n^{b_n} \otimes d_b c_{1j}^{b_1} \ldots c_{nj}^{b_n}, \tag{5.3}$$

where

$$d_b = \begin{bmatrix} lp^{r-1} \\ b_1 \end{bmatrix} \begin{bmatrix} lp^{r-1} - b_1 \\ b_2 \end{bmatrix} \cdots \begin{bmatrix} lp^{r-1} - b_1 - b_2 - \ldots - b_{n-1} \\ b_n \end{bmatrix},$$

and the sum ranges over all n-tuples $\mathbf{b}$ such that $\sum_{i=1}^n b_i = lp^{r-1}$. Here $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$ is the *Gaussian polynomial* (see for example [36, IV.2]). By [46, Lemma 2.1], if $s = s' + ls''$ and $t = t' + lt''$, with $0 \leq s', t' \leq l - 1$, then

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{pmatrix} s'' \\ t'' \end{pmatrix} \begin{bmatrix} s' \\ t' \end{bmatrix},$$

where $\left(\begin{smallmatrix} s'' \\ t'' \end{smallmatrix}\right)$ is the usual binomial coefficient, and

$$\begin{bmatrix} s' \\ t' \end{bmatrix} \neq 0 \qquad \text{implies that} \qquad s' \geq t'. \tag{5.4}$$

Consider the term in (5.3) corresponding to some fixed $\mathbf{b}$, and assume that $d_b \neq 0$. Now $lp^{r-1} \equiv 0 \pmod{l}$, and so (5.4) implies that $b_1 \equiv 0 \pmod{l}$. By induction we see that the same is true for all the b_i. Hence

$$d_b = \begin{pmatrix} p^{r-1} \\ \bar{b}_1 \end{pmatrix} \begin{pmatrix} p^{r-1} - \bar{b}_1 \\ \bar{b}_2 \end{pmatrix} \cdots \begin{pmatrix} p^{r-1} - \bar{b}_1 - \bar{b}_2 - \ldots - \bar{b}_{n-1} \\ \bar{b}_n \end{pmatrix},$$

where for all t we have $l\bar{b}_t = b_t$. As char $k = p$, we must have (for d_q non-zero) that $\bar{b}_1 = 0$ or p^{r-1}. Hence as $\sum \bar{b}_t = p^{r-1}$, we have by induction that precisely one of the $\bar{b}_t$ is non-zero, and equals p^{r-1}. Thus the only non-zero terms in (5.3) correspond to $\mathbf{b}$ of the form $(0, \ldots, 0, lp^{r-1}, 0, \ldots, 0)$. Suppose the non-zero term occurs in the tth place. Then the corresponding term equals $X_t^{lp^{r-1}} \otimes d_b c_{tj}^{lp^{r-1}}$ and, by the defining relations for $k[M_rD]$, this equals zero unless $t = j$. So (5.3) above reduces to

$$\tau(X_j^{lp^{r-1}}) = X_j^{lp^{r-1}} \otimes dc_{jj}^{lp^{r-1}}, \tag{5.5}$$

and the result follows.

With the last result, we can now give an explicit description of certain induced modules.

Proposition 5.4.4 *Let $\lambda \in \Lambda(n,d)$. Then*

i) $\mathrm{ind}_D^{M_rD}\lambda \cong S_q^{\lambda_1}(E)_1 \otimes S_q^{\lambda_2}(E)_2 \otimes \cdots S_q^{\lambda_n}(E)_n$*;*

ii) $S_q(n,d)_r$ *is isomorphic to the contravariant dual of* $A_q(n,d)_r$*;*

iii) $S_q(n,d)_r\xi_\lambda$ *is isomorphic to the contravariant dual of* $\mathrm{ind}_D^{M_rD}\lambda$*.*

Proof: (i) From the definition of induction we have

$$\mathrm{ind}_D^{M_rD}(\lambda) = \{f \in |\lambda| \otimes k[M_rD] \mid f = e_\lambda \otimes g \text{ and } \tau(e_\lambda) \otimes g = \sum_i e_\lambda \otimes \hat{\phi}(g_i') \otimes g_i''\},$$

where τ is the structure map for the D-module λ, $\hat{\phi} : k[M_rD] \longrightarrow k[D]$ is the natural map, and $\delta(g) = \sum_i g_i' \otimes g_i''$. Now $\tau(e_\lambda) = e_\lambda \otimes c_{11}^{\lambda_1} c_{22}^{\lambda_2} \ldots c_{nn}^{\lambda_n}$, so for $f \in \mathrm{ind}_D^{M_rD}(\lambda)$ we must have $\delta(g) = c_{11}^{\lambda_1} c_{22}^{\lambda_2} \ldots c_{nn}^{\lambda_n} \otimes g$ + terms of the form $h \otimes h'$, where h contains some c_{ij} with $i \neq j$. So $\mathrm{ind}_D^{M_rD}(\lambda)$ corresponds to the space of D-weight vectors of weight λ in $k[M_rD]$ (regarded as a right module). Clearly, as a right module, the D-weight vectors in $A_q(n,d)_r$ of weight λ are the span of those monomials c^β whose $\beta \in \mathcal{M}(n,d)_r$ satisfy

$$\sum_{j=1}^{n} \beta_{ij} = \lambda_i \qquad \text{for } i = 1, \cdots n.$$

It is now routine to check that the map sending c^β to

$$X_1^{\beta_{11}} \ldots X_n^{\beta_{1n}} \otimes X_1^{\beta_{21}} \ldots X_n^{\beta_{2n}} \otimes \cdots \otimes X_1^{\beta_{n1}} \ldots X_n^{\beta_{nn}}$$

gives the required isomorphism (using [20, Remark 3.7] for each factor, and noting that the result is clear for $\lambda \in X_r(T)$, and follows for general λ from (5.5)).

(ii) We define a bilinear form

$$(-,-) : S_q(n,d)_r \times A_q(n,d)_r \longrightarrow k$$

by $(\xi, c) = J(\xi)(c)$, where J is the algebra anti-automorphism of the previous section. As J permutes the basis elements, it is clear that $(-,-)$ is non-singular. Now, just as in the classical case, $A_q(n,d)_r$ is an $S_q(n,d)_r$-bimodule (see [29, pg 35]), and as there it is easy to verify that the form is contravariant. The result now follows from [38, 3.4.5].

(iii) By [38, remarks following 6.6.4], we see that $S_q(n,d)_r\xi_\lambda$ is spanned by the elements $\xi_{\mathbf{j},\mathbf{k}}\xi_{\mathbf{i},\mathbf{i}}$, where $\mathrm{ct}\mathbf{k} = \mathrm{ct}\mathbf{i} = \lambda$ and $(\mathbf{j},\mathbf{k}), (\mathbf{i},\mathbf{i}) \in I_0^2(n,d)_r$. The elements $\xi_{\mathbf{i},\mathbf{i}}$ and $\xi_{\mathbf{j},\mathbf{k}}$ correspond, under our usual isomorphism, to the Dipper–James basis elements $\phi^1_{\lambda\lambda}$ and $\phi^e_{\mu\lambda}$ (for some μ, e) respectively.

Consider the restriction of the form in (ii) to $S_q(n,d)_r\xi_\lambda \times \mathrm{ind}_D^{M_rD}\lambda$. As noted in [38, remarks after 6.6.4], we have that $\phi^e_{\mu\lambda}\phi^1_{\lambda\lambda}$ equals some linear combination of terms of the form $\phi^f_{\mu\lambda}$, for appropriate f's. Hence $(\phi^e_{\mu\lambda}\phi^1_{\lambda\lambda}, v)$ equals some linear combination of terms of the form $\phi^{f^{-1}}_{\lambda\mu}(v)$, by the explicit description of J in (5.3.6). Now by (i), the v range over the span of all $c_{\mathbf{i},\mathbf{j}}$ with $\mathrm{ct}\mathbf{i} = \lambda$. Further, the $\phi^{g^{-1}}_{\theta\tau}$ form a basis for $S_q(n,d)_r$ as θ, τ, g run over appropriate sets, and $\phi^{g^{-1}}_{\theta\tau}(c_{\mathbf{i},\mathbf{j}}) = 0$ if $\mathrm{wt}\mathbf{i} \neq \theta$. Hence there must exist some $v \in \mathrm{ind}_D^{M_rD}\lambda$ for which $(\phi^e_{\mu\lambda}\phi^1_{\lambda\lambda}, v) \neq 0$, as otherwise the unrestricted form is singular, contradicting (ii). Thus the restricted form is also non-singular, which by [38, 3.4.5] gives the result.

For $\mu \in \Gamma_r(D)$, let $\hat{P}_r(\mu)$ denote the projective cover of $\hat{L}_r(\mu)$ in $\mathrm{Mod}(M_rD)$. If P is a projective module, and $P(S)$ is the projective cover of a simple module S, then we denote by $[P : P(S)]$ the multiplicity of $P(S)$ as a summand of P. We use a similar notation for injectives. Then we have the following infinitesimal analogue of Kostka duality.

Proposition 5.4.5 *For $\lambda \in \Lambda(n,d)$, we have the equalities*

$$[\mathrm{ind}_D^{M_rD}\lambda : \hat{I}_r(\mu)] = \dim_k \hat{L}_r(\mu)^\lambda = [S_q(n,d)_r\xi_\lambda : \hat{P}_r(\mu)],$$

for all $\mu \in \Gamma_r(D)$.

Proof: This follows just as in the classical case (see [22, Theorem 5.4]).

Hence computing the multiplicities of the indecomposable summands of either $\mathrm{ind}_D^{M_rD}\lambda$ or $S_q(n,d)_r\xi_\lambda$, for all $\lambda \in P(D)$, would determine the characters of the simple M_rD-modules.

Chapter 6

The blocks of the infinitesimal Schur algebras

To conclude, we turn our attention to the blocks of the infinitesimal Schur and q-Schur algebras. In the classical case these have been calculated for $n = 2$ and $r = 1$ in [23]. Although we cannot yet give a complete description, we are at least able to completely determine the blocks in the case $n = 2$ for arbitrary r. The general strategy is to start from the known blocks of the Schur (and q-Schur) algebras, along with those of the Jantzen subgroups. Then we proceed by increasing induction on d, and descending induction on r.

The first section reviews the classical results that we shall need, and then proceeds to calculate the blocks for $n = 2$ in the classical case. The argument given there will also hold for the quantum case, so the concluding section merely checks that the necessary classical results carry across to this setting. It should be noted that we are able to translate the methods to the quantum case only because of the restriction on n, as the proof makes frequent use of the equivalence of the two truncation functors from the last chapter.

6.1 Infinitesimal blocks

In this section we begin to determine the blocks of the infinitesimal Schur algebras. This will use the description of the blocks of G_rT implicit in [33]. We will denote the block of G_r containing λ by $\mathcal{B}_r(\lambda)$, and the block of G_rT containing λ by $\hat{\mathcal{B}}_r(\lambda)$. Blocks will be identified with subsets of $\mathbb{Z}^n$ in the usual way, thus allowing us to consider the intersection of blocks for different categories of modules.

We begin by recalling various results from [34]. Define m $(=m(\lambda))$ to be the least integer such that there exists an $\alpha \in \Phi^+$ with $\langle \lambda+\rho, \alpha\rangle \notin \mathbb{Z}p^m$ (compare with the definition in Chapter 4). Then, by [34, II 9.19(1)], we have

$$\mathcal{B}_r(\lambda) = W.\lambda + p^m\mathbb{Z}\Phi + p^r X(T). \tag{6.1}$$

By [34, II 9.16 Lemma (a)] we also have that

$$\hat{\mathcal{B}}_r(\lambda) \subseteq W.\lambda + p\mathbb{Z}\Phi. \tag{6.2}$$

We can relate the blocks of G_r and G_rT using

$$\mathrm{Ext}^1_{G_r}(L_r(\lambda), L_r(\mu)) = \bigoplus_{\nu\in X(T)} \mathrm{Ext}^1_{G_rT}(\hat{L}_r(\lambda+p^r\nu), \hat{L}_r(\mu)) \tag{6.3}$$

(see [34, II 9.16(3)]). This, along with (6.1) and (6.2), gives that $\hat{\mathcal{B}}_r(\lambda) \subseteq \mathcal{B}_r(\lambda)$, and hence

$$\hat{\mathcal{B}}_r(\lambda) \subseteq W.\lambda + p^{\min(m,r)}\mathbb{Z}\Phi. \tag{6.4}$$

Proposition 6.1.1 *For all $r > 0$ and $\lambda \in X(T)$, we have*

$$\hat{\mathcal{B}}_r(\lambda) = \begin{cases} W.\lambda + p^m\mathbb{Z}\Phi & \text{if } m \leq r, \\ \{\lambda\} & \text{if } m > r, \end{cases}$$

where m is defined as above.

Proof: We first consider the case $m > r$. By [34, II 11.8], we have that for all $\mu \in W.\lambda + p^r\mathbb{Z}\Phi$, the module $\hat{Z}_r(\lambda)$ is simple. So the result in this case follows from the usual characterisation of blocks (see [34, II 11.4]). Now suppose that $m \leq r$, and $\mu \in W.\lambda + p^m\mathbb{Z}\Phi$. Then λ and μ are in the same G_r block, and so there exists a sequence $\lambda = {}_0\lambda, {}_1\lambda, \ldots, {}_t\lambda = \mu$ such that $\mathrm{Ext}^1_{G_r}(L_r({}_i\lambda), L_r({}_{i+1}\lambda)) \neq 0$. So by (6.3) there exist ${}_0\nu, \ldots, {}_{t-1}\nu \in X(T)$ such that

$$\mathrm{Ext}^1_{G_rT}(\hat{L}_r({}_i\lambda + p^r{}_i\nu), \hat{L}_r({}_{i+1}\lambda)) \neq 0.$$

Thus ${}_i\lambda + p^r{}_i\nu$ is in the same G_rT block as ${}_{i+1}\lambda$ for $0 \leq i \leq t-1$. As $\hat{L}_r(\tau + p^r\nu) \cong \hat{L}_r(\tau) \otimes p^r\nu$ by [34, II 9.5 Proposition], this implies that ${}_0\lambda + p^r({}_0\nu + \ldots + {}_{t-1}\nu)$ is in the same G_rT block as ${}_t\lambda = \mu$. So we will be done if we can show that λ is in the same G_rT block as ${}_0\lambda + p^r({}_0\nu + \ldots + {}_{t-1}\nu)$. But $\mu \in W.\lambda + p^m\mathbb{Z}\Phi$ implies that ${}_0\lambda + p^r({}_0\nu + \ldots + {}_{t-1}\nu) \in W.\lambda + p^m\mathbb{Z}\Phi$ by (6.4), and hence that $p^r({}_0\nu + \ldots + {}_{t-1}\nu) \in p^m\mathbb{Z}\Phi$. The result now follows by repeated use of the short exact sequence in [33, Section 5.5 before (2)].

For the polynomial case we will need the following lemma, which will enable us to proceed by induction on r.

Lemma 6.1.2 *For all $\lambda' \in P_r(D)$ and $\lambda'' \in P(D)$, we have*

$$\mathrm{res}_{M_rD}\hat{I}_{r+1}(\lambda' + p^r\lambda'') \leq \bigoplus_{\nu} \hat{I}_r(\lambda' + p^r\nu),$$

where the sum runs over the set of polynomial weights of $\hat{Q}_1(\lambda'')$, counted with multiplicities.

Proof: We first note that for any $G_{r+1}T$-module X, it is clear that

$$\mathrm{res}_{M_rD}\mathcal{F}_{M_{r+1}D}(X) \leq \mathcal{F}_{M_rD}\mathrm{res}_{M_{r+1}D}(X).$$

We also have, from [34, II 11.15 Lemma], that

$$\hat{Q}_{r+1}(\lambda' + p^r\lambda'') \cong_{G_rT} \hat{Q}_r(\lambda') \otimes \hat{Q}_1(\lambda'')^{F^r},$$

which implies, by [34, II 11.3 (2)], that

$$\hat{Q}_{r+1}(\lambda' + p^r\lambda'') \cong_{G_rT} \bigoplus_{\nu} \hat{Q}_r(\lambda' + p^r\nu),$$

where the sum runs over the set of weights of $\hat{Q}_1(\lambda'')$. The result nows follows from [22, 4.1 Proposition], which gives that $\mathcal{F}_{M_rD}(\hat{Q}_r(\lambda)) \cong \hat{I}_r(\lambda))$.

We will denote the block of $S(n,d)$ containing λ by $\mathcal{B}^d(\lambda)$ and the corresponding block of $S(n,d)_r$ by $\mathcal{B}^d_r(\lambda)$. We also use the notation from [22, Section 3] for various subsets of $X(T)$. We first note that, by [18, Theorem], we have

$$\mathcal{B}^d(\lambda) = (W.\lambda + p^m\mathbb{Z}\Phi) \cap \Lambda^+(n,d). \tag{6.5}$$

The main conjecture of this section is

Conjecture 6.1.3 *For all $r > 0$ and $\lambda \in \Gamma_r^d(D)$, we have*

$$\mathcal{B}_r^d(\lambda) = \hat{\mathcal{B}}_r(\lambda) \cap \Gamma_r^d(D).$$

This is already known to hold in the case $n = 2$ and $r = 1$, as shown in [23]. As a first step we can at least prove one of the inclusions.

Proposition 6.1.4 *For all $r > 0$ and $\lambda \in \Gamma_r^d(D)$ we have*

$$\mathcal{B}_r^d(\lambda) \subseteq \hat{\mathcal{B}}_r(\lambda) \cap \Gamma_r^d(D).$$

Proof: To show that our block is contained in this intersection, we first note that by [22, 4.1 Proposition] we have that $\mathcal{F}_{M_rD}(\hat{Q}_r(\lambda)) \cong \hat{I}_r(\lambda)$. But then if $\hat{L}_r(\mu)$ is a composition factor of $\hat{I}_r(\lambda)$, it is also one of $\hat{Q}_r(\lambda)$, and so the result now follows.

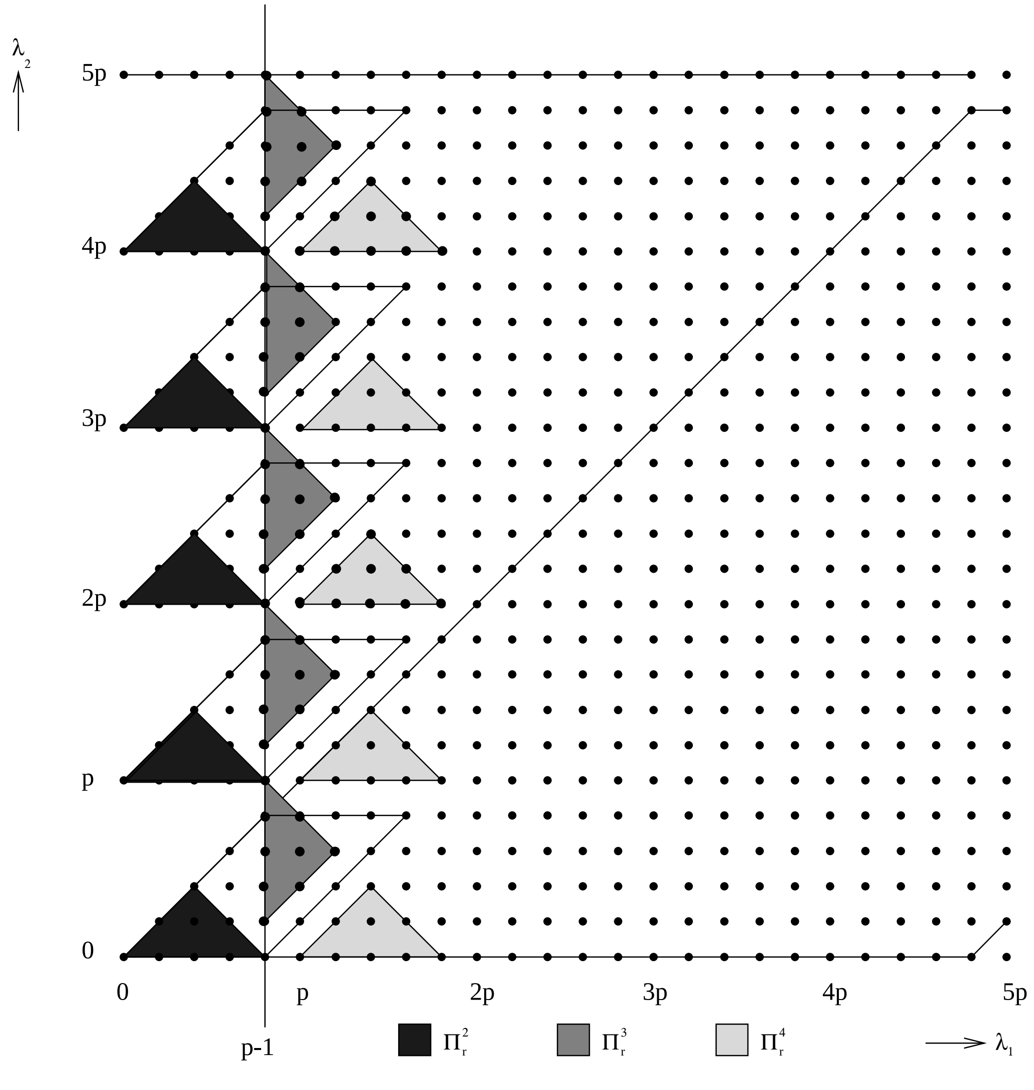

Figure 1: The case n=2, p=5, and r=1.

For convenience we will set $\mathcal{C}_r^d(\lambda) = \hat{\mathcal{B}}_r(\lambda) \cap \Gamma_r^d(D)$ (and we hope ultimately to show that this equals $\mathcal{B}_r^d(\lambda)$). The rest of this section will be devoted to proving the conjecture for the case $n = 2$. In this case there is one simple root $\alpha = (1, -1)$. For the rest of this section we will write $\lambda \sim \mu$ if λ and μ are linked as M_rD-weights.

We will also need to define various regions of the plane, for which the reader may find it helpful to refer to Figure 1. We first set

$$\Pi_r^1 = \{\lambda \in P(D) | \lambda_1 \geq p^r - 1\}.$$

Writing $\lambda \in \Gamma_r(D)$ in the form $\lambda = \lambda' + p^r\lambda''$ with $\lambda' \in P_r(D)$ and $\lambda'' \in P(D)$ we also define

$$\Pi_r^2 = \{\lambda \in \Gamma_r(D) | \lambda_1' + \lambda_2' \leq p^r - 1 \text{ and } \lambda_1'' = 0\}.$$

Then our main result is

Theorem 6.1.5 *For $n = 2$ and $d \geq 0$ we have that, for all $\lambda \in \Gamma_r^d(D)$,*

$$\mathcal{B}_r^d(\lambda) = \mathcal{C}_r^d(\lambda).$$

The rest of this section is devoted to proving this result.

We will fix d and assume that we have proved the result for all $d' < d$. We first note that for $r >> 0$, we have $S(n, d)_r = S(n, d)$ (see [22, Section 2.3 Remark (2)]), so we will proceed by descending induction on r. So assume the result holds for $r+1$, that $d \geq p^r$ (as otherwise we are done by (6.5)) and that $m \leq r$ (as otherwise the result is clear from (6.1.4)). We first show

Lemma 6.1.6 *All weights in the set $\Pi_r^2 \cap \mathcal{C}_r^d(\lambda)$ are linked.*

Proof: Suppose λ and μ lie in this set. Then (using the usual notation) $\lambda'' = \mu''$. Now λ' is linked to μ' as these both have weight $d' < p^r$, for which the result is known from (6.5). So there is a chain of weights $\lambda' = {}_0\lambda', \ldots, {}_t\lambda' = \mu'$ in $\Pi_r^2 \cap \mathcal{B}_r^{d'}(\lambda')$ such that, for each i, we have $\mathrm{Ext}_{M_rD}(\hat{L}_r({}_i\lambda'), \hat{L}_r({}_{i+1}\lambda')) \neq 0$ or $\mathrm{Ext}_{M_rD}(\hat{L}_r({}_{i+1}\lambda'), \hat{L}_r({}_i\lambda')) \neq 0$. Now as tensoring up with a one-dimensional module does not cause an extension to split, we get, in the category of G_rT-modules, a chain of non-trivial extensions by tensoring up with $p^r\lambda''$. But as these are all M_rD-modules by restriction, the equivalence of $\mathcal{F}$ and $\mathcal{O}_\pi$ (see [35]) gives that this is still a chain of non-trivial extensions for M_rD (see [22, Section 6.2, Remark]). The result now follows, as $\hat{L}_r({}_i\lambda) \otimes p^r\lambda'' \cong \hat{L}_r({}_i\lambda + p^r\lambda'')$ for all i.

We will also need the following pair of lemmas.

Lemma 6.1.7 *For $\lambda \in \Gamma_r(D)$, if $\lambda_1 \in \Pi_r^1$ then*

$$\inf_{G_rT}\mathcal{F}_{M_rD}(\hat{Z}_r(\lambda)) \cong \hat{Z}_r(\lambda).$$

Proof: By [34, II 9.2 (6)], all weights μ of $\hat{Z}_r(\lambda)$ satisfy $\lambda - (p^r - 1)(1, -1) \leq \mu \leq \lambda$. So if $\lambda_1 \geq p^r - 1$, then all these weights are polynomial, and so, as $\mathcal{F}_{M_rD}$ is equivalent to $\mathcal{O}_\pi$, the result follows.

Lemma 6.1.8 *If λ, $\mu \in \Gamma_r^d(D) \cap \Pi_r^1$ and $\lambda - \mu \in p^m\mathbb{Z}\alpha$, then $\lambda \sim \mu$.*

Proof: The argument follows just as in [33, Section 5.5] as the exact sequence constructed there remains non-trivial when we apply $\mathcal{F}_{M_rD}$, by the last result.

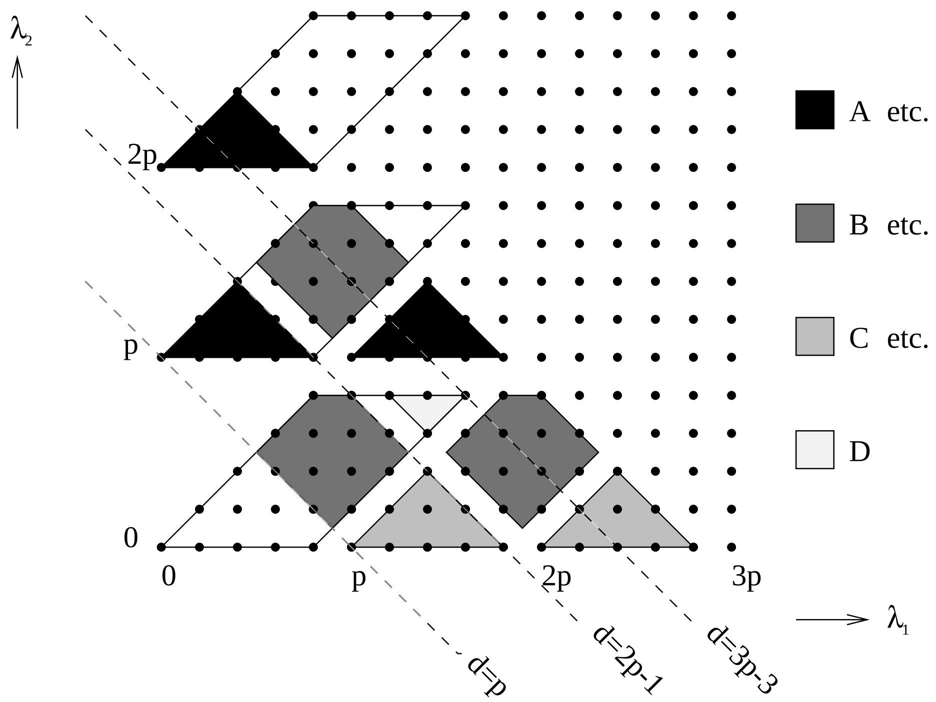

Figure 2: The case n=2, p=5, and r=1.

We now consider the case when $p^r \leq d \leq 2p^r - 1$. In this case it will be convenient to divide $\Gamma_r(D)$ into three regions; we set $A = \Pi_r^2 \cap \Gamma_r(D)$, $B = P_r(D) \cap \Gamma_r(D)$, and C to be the remainder (see Figure 2). Then

Lemma 6.1.9 *All the weights in $\mathcal{C}_r^d(\lambda) \cap B$ are linked.*

Proof: Consider $d' \in \{p^r - 1, p^r - 2\}$ such that $d - d'$ is even. Then we know that all weights in $\mathcal{C}_r^{d'}(\lambda - \frac{d-d'}{2}(1,1))$ are linked, as this reduces to the ordinary Schur algebra case. So for any two weights in this set there is a chain of simple modules with non-trivial extensions between consecutive terms. Tensoring up with $\det^{\frac{d-d'}{2}}$ then gives the result as above.

As $\Gamma_{r+1}^d(D) \subseteq \Gamma_r^d(D)$, we now consider the case where $\lambda \in \Gamma_{r+1}^d(D)$ and $\mu \in \mathcal{B}_{r+1}^d(\lambda)$. Then $\mathcal{B}_{r+1}^d(\lambda) = (W.\lambda + p^m \mathbb{Z}\Phi) \cap \Gamma_{r+1}^d(D)$, and there exists a chain $\lambda = {}_0\lambda, {}_1\lambda, \ldots, {}_t\lambda = \mu$ in $\Gamma_{r+1}^d(D)$ such that either $[\hat{I}_{r+1}({}_i\lambda) : \hat{L}_{r+1}({}_{i+1}\lambda)] \neq 0$ or $[\hat{I}_{r+1}({}_{i+1}\lambda) : \hat{L}_{r+1}({}_i\lambda)] \neq 0$ for $1 \leq i \leq t-1$. Now for all i set ${}_i\lambda = {}_i\lambda' + p^r {}_i\lambda''$, where ${}_i\lambda' \in P_r(D)$ and ${}_i\lambda'' \in P(D)$. By (6.1.2), and as $\hat{L}_{r+1}({}_i\lambda) \cong_{M_rD} \hat{L}_r({}_i\lambda') \otimes \hat{L}_1({}_i\lambda'')^{F^r}$, we have that for $2 \leq i \leq t$ there exists ${}_i\nu, {}_i\nu' \in P(D)$ such that either

$$[\hat{I}_r({}_i\lambda' + p^r {}_i\nu) : \hat{L}_r({}_{i+1}\lambda)] \neq 0,$$

or

$$[\hat{I}_r({}_{i+1}\lambda' + p^r {}_{i+1}\nu') : \hat{L}_r({}_i\lambda)] \neq 0.$$

Hence either ${}_i\lambda$ is linked to ${}_{i+1}\lambda' + p^r {}_{i+1}\nu'$ or ${}_{i+1}\lambda$ is linked to ${}_i\lambda' + p^r {}_i\nu$. With this we can now prove

Lemma 6.1.10 *For $p^r \leq d \leq 2p^r - 1$ we have that either $\mathcal{C}_r^d(\lambda)$ is a single block, or it is the union of the two blocks $\mathcal{C}_r^d(\lambda) \cap \Pi_r^2$ and $\mathcal{C}_r^d(\lambda) \backslash \Pi_r^2$.*

Proof: First note that if $\mathcal{C}_r^d(\lambda) \subseteq B$ then we are done by the previous lemma, so we may assume that this does not hold. Thus, as $\mathcal{C}_r^d(\lambda) \cap (A \cup C) \neq \emptyset$, and $C = \{a + p^r\alpha \mid a \in A\}$, we must have $\mathcal{C}_r^d(\lambda) \cap C \neq \emptyset$, and hence $\mathcal{C}_{r+1}^d(\lambda) \cap C \neq \emptyset$. Consider the sequence of linked weights introduced above, and assume — as by the last remark we may — that $\lambda \in C$. Suppose that $\mu \in B$. Now as the only weight equal to μ modulo $p^r\alpha$ is μ, and the only weights equal to those in C modulo $p^r\alpha$ lie in $A \cup C$, there exists some weight τ such that $\tau \in A \cup C$ and $\mu \sim \tau$. We will consider the following two sets of weights:

$$B_1 = \{\mu \in \mathcal{B}_{r+1}^d(\lambda) \cap B \mid \exists \tau \in A \text{ with } \mu \sim \tau\},$$

and

$$B_2 = \{\mu \in \mathcal{B}^d_{r+1}(\lambda) \cap B \mid \exists \tau \in C \text{ with } \mu \sim \tau\}.$$

By (6.1.6), all the weights in $\mathcal{B}^d_{r+1}(\lambda) \cap A$ are linked, and by tensoring up with $p^r(1,-1)$ we see that all the weights in $\mathcal{B}^d_{r+1}(\lambda) \cap C$ are linked also. So if $B_1 = B_2 = \emptyset$ we are done. Otherwise there are two possibilities: $B_1 = B_2 = B$, or $B_1 \cap B_2 = \emptyset$.

Choose a minimal weight $\tau \in \mathcal{B}^d_{r+1}(\lambda) \cap C$ (this exists by our initial assumption). By (6.1.7), $\hat{Z}_r(\tau)$ has polynomial weights, and so (as it is not simple by [34, II 11.8 Lemma]) we see by [34, II 9.1 (6)] that τ is linked to some lower weight. By minimality this weight lies in B or A. If it is in A then $B_1 = B_2 = B$, while if it is in B then $B_2 \neq \emptyset$. So by the previous lemma we either have $B_1 = B_2 = B$, or $B_1 = \emptyset$ as required.

Now we consider the case when $2p^r \leq d \leq 3p^r - 1$. Once again it will be convenient to divide our weights into regions. For a set of weights X, we will set $X' = \{x + p^r(0,1) | x \in X\}$, and $X'' = \{x + p^r(1,0) | x \in X\}$. We also denote by D the set of weights with $2p^r \leq d \leq 3p^r - 1$ that are not contained in $(A \cup B \cup C)' \cup (A \cup B \cup C)''$.

Lemma 6.1.11 *For $2p^r \leq d \leq 3p^r - 1$ we have that either $\mathcal{C}^d_r(\lambda)$ is a single block, or it is the union of the two blocks $\mathcal{C}^d_r(\lambda) \cap \Pi^2_r$ and $\mathcal{C}^d_r(\lambda) \backslash \Pi^2_r$.*

Proof: First consider $\mathcal{C}^d_r(\lambda) \cap (B' \cup B'' \cup C' \cup D)$. Let $d' \in \{2p^r - 1, 2p^r - 2\}$ be such that $d - d'$ is even. Then as all the weights in $A \cup B \cup C$ are linked by the induction hypothesis, we see by tensoring up by $\det^{\frac{d-d'}{2}}$ that all of these weights are linked also. If $\mathcal{C}^d_r(\lambda) \cap C'' \neq \emptyset$ then these weights can be linked to those in C' by (6.1.8).

We now show that $\mathcal{C}^d_r(\lambda)$ is in fact a single block for $p \leq 3p^r - 1$. For this we will need to define two further regions of the plane. Decomposing $\lambda = \lambda' + p^r \lambda''$ as usual, we set

$$\Pi^3_r = \{\lambda + (0,1) \in \Gamma_r(D) \mid \lambda'_1 \geq p^r - 1,\ \lambda'_1 + \lambda'_2 < 2p^r - 1, \text{ and } \lambda''_1 = 0\}$$

and

$$\Pi^4_r = \{\lambda + p^r(1,0) \mid \lambda \in \Pi^2_r\}.$$

We can now show

Lemma 6.1.12 *For $d \leq 3p^r - 1$ and $\lambda \in \Gamma_r^d$ we have $\mathcal{C}_r^d(\lambda) = \mathcal{B}_r^d(\lambda)$.*

Proof: First suppose that $\lambda \in \Pi_r^1$. By (6.1.7), the lowest weight in $\hat{A}_r(\lambda)$ is $\lambda - (p^r - 1)(1, -1)$. Now for $\tau = \tau' + p^r\tau''$, with $\tau' \in P_r(D)$ and $\tau'' \in P(D)$, we have $\hat{L}_r(\tau) \cong \hat{L}_r(\tau') \otimes p^r\tau''$, and by [34, II 3.15 Proposition] $\hat{L}_r(\tau') \cong L(\tau')$. Now the lowest weight in $L(\tau')$ is $w_0\tau'$ (where w_0 is the non-trivial element of the Weyl group), and hence the lowest weight in $\hat{L}_r(\tau)$ is $w_0\tau' + p^r\tau''$. Clearly $\hat{L}_r(\tau)$ and $\hat{L}_r(\mu)$ have the same lowest weight if, and only if, $\tau = \mu$, and so $\hat{A}_r(\lambda)$ has a composition factor $\hat{L}_r(\tau)$, where $w_0\tau' + p^r\tau'' = \lambda - (p^r - 1)(1, -1)$.

Now we may assume that $\mathcal{C}_r^d(\lambda) \cap \Pi_r^2 \neq \emptyset$ (else the result holds by our earlier calculations). Then, as $\Pi_r^4 = \{\mu + p^r\alpha \mid \mu \in \Pi_r^2\}$, we have that $\mathcal{C}_r^d(\lambda) \cap \Pi_r^4 \neq \emptyset$. Modulo $p^r\alpha$, we have that $\Pi_r^4 = w_0.\Pi_r^3$, and so $\mathcal{C}_r^d(\lambda) \cap \Pi_r^3 \neq \emptyset$. So we may assume that $\lambda \in \Pi_r^3$. Then, by considering Figure 1, along with the above remarks, we see that $\tau \in \Pi_r^2$, and we are done (by (6.1.6), (6.1.10) and (6.1.11)).

To complete the proof we require

Lemma 6.1.13 *If $d \geq 2p^r - 1$ then, for all $\lambda \in \Gamma_{r+1}^d$ and $w \in W$, there is some element of the form $w.\lambda + p^m z\alpha$ in $\Gamma_{r+1}^d \cap \Pi_r^1$.*

Proof: For each $w \in W$ there is one such representative in any chain of p^m consecutive weights in Γ_{r+1}^d. So it is enough to show that such a chain exists. But all (μ_1, μ_2) with $\mu_1 + \mu_2 = d$ and $\mu_1 \geq \mu_2$ lie in $\Gamma_{r+1}^d \cap \Pi_r^1$, so such a chain always exists if $d \geq 2p^m - 1$.

To conclude we suppose that $d \geq 3p^r - 1$. Then there exists integers a and d' such that $2p^r \leq d' \leq 3p^r - 1$ and $d = d' + p^r a$. By the previous lemma, there is a representative of each $w.\lambda$ class in $\mathcal{C}_r^d(\lambda) \cap \Pi_r^1 \cap \{\mu + p^r(0, a) \mid \mu \in \Gamma_r^{d'}\}$, and all the weights in $\mathcal{C}_r^d(\lambda) \cap \{\mu + p^r(0, a) \mid \mu \in \Gamma_r^{d'}\}$ are linked by tensoring up the corresponding chains from $\Gamma_r^{d'}$. All other weights in $\mathcal{C}_r^d(\lambda)$ are linked to these, as they are linked to their corresponding $w.\lambda$ class representative by (6.1.8).

6.2 The quantum case

In this section we will verify that the results of the previous section, appropriately modified, also hold in the quantum case. Most of the work is devoted to determining the blocks of the quantum Jantzen subgroups, at least in the cases of interest to us. To begin, we give an alternative description of induced modules.

Given K a subgroup of a quantum group H, and V a K-module, we define a map

$$\Theta : V \otimes k[H] \longrightarrow V \otimes k[K] \otimes k[H]$$

as follows. We will use the convention that we suppress certain summations, indicated by primes, in a similar manner to Sweedler's notation (see [43]). Thus we shall write the structure map τ of V as $v \longmapsto v' \otimes g''$, and comultiplication δ in $k[H]$ by $f \longmapsto f' \otimes f''$. Denoting the antipode in $k[K]$ by σ, and the image of $f \in k[H]$ in $k[K]$ by $\bar{f}$, we define Θ on elements of the form $v \otimes f$ by $v \otimes f \longmapsto v' \otimes g''\sigma(\bar{f}') \otimes f''$, and extend by linearity. We define the fixed points under this map to be those elements $\sum_i v_i \otimes f_i$ satisfying $\Theta(\sum_i v_i \otimes f_i) = \sum_i v_i \otimes 1 \otimes f_i$, and denote the set of these by $(V \otimes k[H])^K$. Then we have

Proposition 6.2.1 *Given H, K and V as above, we have*

$$\mathrm{ind}_K^H(V) = (V \otimes k[H])^K .$$

Proof: We first show $\mathrm{ind}_K^H(V) \subseteq (V \otimes k[H])^K$. Consider $\sum_i v_i \otimes f_i \in \mathrm{ind}_K^H(V)$. Now

$$\Theta\Big(\sum_i v_i \otimes f_i\Big) = (\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \bar{\delta})\Big(\sum_i v_i' \otimes g_i'' \otimes f_i\Big),$$

where $\bar{\delta} = (\bar{\ } \otimes \mathrm{id})\delta$ and m is the usual multiplication map. By the definition of induction (see Section 1.4) we have

$$\sum_i v_i' \otimes g_i'' \otimes f_i = \sum_i v_i \otimes \bar{f}_i' \otimes f_i'',$$

and so by applying our alternative description of Θ to this we obtain

$$\Theta\Big(\sum_i v_i \otimes f_i\Big) = \sum_i v_i \otimes \bar{f}_i'\sigma(\bar{f}_i'') \otimes f_i'''.$$

As $\delta(\bar{f}) = \bar{\delta(f)}$, we have by the axioms for a coalgebra that this equals $\sum_i v_i \otimes 1 \otimes f_i$ as required.

Next we consider the reverse inclusion. As

$$(\mathrm{id} \otimes m \otimes \bar{\delta})\Big(\sum_i v_i \otimes 1 \otimes f_i\Big) = \sum_i v_i \otimes \bar{f}_i' \otimes f_i'',$$

we have by our alternative description of Θ above that it is enough to show that

$$(\mathrm{id} \otimes m \otimes \bar{\delta})(\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \bar{\delta}) = \mathrm{id}.$$

So consider the left-hand side acting on some element $v \otimes a \otimes b$. The image of this is

$$v \otimes a\sigma(\bar{b'})\bar{b''} \otimes b''' = v \otimes a \otimes b$$

as required, and the result now follows.

With this last result we can now prove

Lemma 6.2.2 *For all $i \geq 0$, $B(n,k)$-modules M and G-modules V, we have*

$$R^i \operatorname{ind}_{G_rB}^{G}(V \otimes M^{F^r}) \cong V \otimes (R^i \operatorname{ind}_{\mathrm{B}(n,k)}^{\mathrm{GL}(n,k)} M)^{F^r}.$$

Proof: By the generalised tensor identity (1.4.1), it is enough to show that

$$R^i \operatorname{ind}_{G_rB}^{G}(M^{F^r}) \cong (R^i \operatorname{ind}_{\mathrm{B}(n,k)}^{\mathrm{GL}(n,k)} M)^{F^r}.$$

We first consider the case $i = 0$. Let us denote G_rB by H. Now G_r is a subgroup of H, and we shall denote the corresponding factor group by $\bar{H}^r$. Then by the last proposition we have

$$\begin{aligned} \operatorname{ind}_H^G(M^{F^r}) &\cong (M^{F^r} \otimes k[G])^H \\ &\cong ((M^{F^r} \otimes k[G])^{G_r})^{\bar{H}^r} \\ &\cong (M^{F^r} \otimes k[\bar{G}^r])^{\bar{H}^r} \\ &\cong \operatorname{ind}_{\bar{H}^r}^{\bar{G}^r} M^{F^r}, \end{aligned}$$

using for the penultimate step (1.4.2(ii)). Now we have $\bar{H}^r \cong \mathrm{B}(n,k)$ and $\bar{G}^r \cong \mathrm{GL}(n,k)$, both via F^r. Hence $\operatorname{ind}_H^G(M^{F^r}) \cong (\operatorname{ind}_{\mathrm{B}(n,k)}^{\mathrm{GL}(n,k)} M)^{F^r}$ as required.

The argument for the general case now proceeds much as in [34, I 6.11]. We replace appeals to [34, I 4.5(c)] by [20, Proposition 1.2], and note that induction is exact where required by the results in [20, Section 1].

With this lemma, we can now prove the following important proposition, relating filtrations of $\hat{Z}_r(\lambda)$ and $\nabla(\lambda)$.

Proposition 6.2.3 *Given $\lambda \in X(T)^+$, suppose that each composition factor of $\hat{Z}_r(\lambda)$ has the form $\hat{L}_r(\mu' + lp^{r-1}\mu'')$, with $\mu' \in P(D)$ and $\mu'' \in X(T)$, such that $\langle \mu'' + \rho, \alpha \rangle \geq 0$ for all $\alpha \in \Pi$. Then $\nabla(\lambda)$ has a filtration with factors of the form $L(\mu') \otimes \bar{\nabla}(\mu'')^{F^r}$, with $\mu' \in P(D)$ and $\mu'' \in X(T)^+$. Each such module occurs as often as $\hat{L}_r(\mu' + lp^{r-1}\mu'')$ occurs in a composition series of $\hat{Z}_r(\lambda)$.*

Proof: We first note that $H^0(\lambda) \cong \mathrm{ind}_{G_rB}^G \hat{Z}_r(\lambda)$, as in [34, 9.8 Lemma]. The result now follows, by the previous lemma and Kempf's Vanishing Theorem (1.6.2) just as in the classical case (see [34, II 9.11 Proposition]).

Recall the definition of $m(= m(\lambda))$ from Chapter 4. With this we now have

Corollary 6.2.4 *Let λ, $\mu \in X(T)$,*

i) if $\hat{L}_r(\mu)$ is a composition factor of $\hat{Z}_r(\lambda)$, then $\mu \in W.\lambda + lp^{\min(m,r)-1}\mathbb{Z}\Phi$;

ii) if $L_r(\mu)$ is a composition factor of $Z_r(\lambda)$ then $\mu \in W.\lambda + lp^{m-1}\mathbb{Z}\Phi + lp^{r-1}X(T)$.

Proof: This is a strengthened version of the classical result [34, II 9.12 Corollary], and follows from the previous proposition just as there, but replacing the appeal to the strong linkage principle with an application of the description of the blocks of G in (4.2.12).

Lemma 6.2.5 *For all λ, $\mu \in X(T)$,*

$$\mathrm{Ext}^i_{G_r}(L_r(\lambda), L_r(\mu)) = \bigoplus_{\tau \in X(T)} \mathrm{Ext}^i_{G_rT}(\hat{L}_r(\lambda + lp^{r-1}\tau), \hat{L}_r(\mu)).$$

Proof: This follows just as in [34, I 6.9(5)], once we note that (by the remarks before [10, 3.1(9)]) G_r and G_rT satisfy the hypotheses of (1.4.2), giving the required spectral sequence.

We can now give one of the desired inclusion of blocks.

Lemma 6.2.6 *For λ, $\mu \in X(T)$,*

i) if $\mathrm{Ext}^1_{G_rT}(\hat{L}_r(\lambda), \hat{L}_r(\mu)) \neq 0$, then $\mu \in W.\lambda + lp^{\min(m,r)-1}\mathbb{Z}\Phi$;

ii) if $\mathrm{Ext}^1_{G_r}(L_r(\lambda), L_r(\mu)) \neq 0$, then $\mu \in W.\lambda + lp^{m-1}\mathbb{Z}\Phi + lp^{r-1}X(T)$.

Proof: To define a contravariant duality as described before [39, (11.1.3)], we note that the coalgebra anti-automorphism used there translates via (1.9.4) to one for the Dipper–Donkin quantisation. By considering the explicit description of this, it is clear that it now restricts to an anti-automorphism of G_rT. Then arguing as in [34, II 2.12] we see that for all $i \in \mathbb{N}$ and λ, $\mu \in X(T)$ we have

$$\mathrm{Ext}^i_{G_rT}(\hat{L}_r(\lambda), \hat{L}_r(\mu)) \cong \mathrm{Ext}^i_{G_rT}(\hat{L}_r(\mu), \hat{L}_r(\lambda)).$$

With this, the lemma now follows from the previous two results just as in [34, II 9.16 Lemma].

For the reverse inclusion we will need a few technical lemmas. The first of these is a straightforward adaptation of the corresponding calculation in [34, page 329].

Lemma 6.2.7 *For all $\lambda \in X(T)$ and $w \in W$, there exists a $\tau \in X(T)$ such that $\lambda - lp^{r-1}\tau$ and $w.\lambda - lp^{r-1}w\tau$ are linked as G_rT-weights.*

Proof: By [10, 3.1(20)ii)] we have

$$\mathrm{ch}\hat{Z}_r(\lambda) = e(\lambda - (lp^{r-1}-1)\rho)\chi((lp^{r-1}-1)\rho).$$

Hence, as $\chi((lp^{r-1}-1)\rho) \in \mathbb{Z}[X(T)]^W$, we have

$$\begin{aligned} \mathrm{ch}\hat{Z}_r(w.\lambda + lp^{r-1}\rho) &= e(w(\lambda+\rho))\mathrm{ch}\hat{Z}_r(lp^{r-1}\rho) \\ &= w[e(\lambda+\rho)\mathrm{ch}\hat{Z}_r(lp^{r-1}\rho)], \end{aligned}$$

and so

$$\mathrm{ch}\hat{Z}_r(w.\lambda + lp^{r-1}\rho) = w\mathrm{ch}\hat{Z}_r(\lambda + lp^{r-1}\rho). \tag{6.6}$$

Now any $\mu \in X(T)$ can be written uniquely in the form $\mu = \mu' + lp^{r-1}\mu''$, with $\mu' \in P_r(D)$ and $\mu'' \in X(T)$, so for any finite dimensional module M we have

$$\mathrm{ch}M = \sum_{\mu' \in P_r(D)} \sum_{\mu'' \in X(T)} [M : \hat{L}_r(\mu)]e(lp^{r-1}\mu'')\mathrm{ch}L(\mu').$$

Taking $M = \hat{Z}_r(\lambda + lp^{r-1}\rho)$ and applying w, we see from (6.6) that

$$\mathrm{ch}\hat{Z}_r(w.\lambda + lp^{r-1}\rho) = \sum_{\mu' \in P_r(D)} \sum_{\mu'' \in X(T)} [\hat{Z}_r(\lambda + lp^{r-1}\rho) : \hat{L}_r(\mu)]e(lp^{r-1}w\mu'')\mathrm{ch}L(\mu').$$

Comparing coefficients for $M = \hat{Z}_r(w.\lambda + lp^{r-1}\rho)$ we see that

$$[\hat{Z}_r(\lambda + lp^{r-1}\rho) : \hat{L}_r(\mu)] = [\hat{Z}_r(w.\lambda + lp^{r-1}\rho) : \hat{L}_r(\mu' + lp^{r-1}w\mu'')]. \tag{6.7}$$

Hence, by tensoring up with suitable one-dimensional modules, we obtain that

$$[\hat{Z}_r(\lambda - lp^{r-1}\mu'') : \hat{L}_r(\mu' - lp^{r-1}\rho)] = [\hat{Z}_r(w.\lambda - lp^{r-1}w\mu'') : \hat{L}_r(\mu' - lp^{r-1}\rho)].$$

Now taking $\tau = \mu''$ for some μ for which the left hand side of (6.7) is non-zero gives the result.

Lemma 6.2.8 *For $\lambda \in X(T)$, if $\langle \lambda + \rho, \alpha \rangle \in \mathbb{Z}lp^{r-1}$ for all $\alpha \in \Pi$ then $\hat{Z}_r(\lambda)$ is simple.*

Proof: This follows just as in [34, II 11.8 Lemma], using [10, 3.1(22), 3.1(13)(i), and 3.1(20)(ii)].

For our next lemma, it is necessary to restrict to the case when $n = 2$. However, as the results in the previous section only hold in this case, this will be sufficient for our needs. Recall that we denote the unique simple root in this case by α. We will also use the $\theta(m)$ notation from Section 2.4.

Lemma 6.2.9 *For $\lambda \in X(T)$, if $\langle \lambda+\rho, \alpha \rangle = alp^{m-1} + b\theta(m-2)$ for some $1 \leq m \leq r$, $a \in \mathbb{Z}$ and $0 < b < p$ (or $0 < b < l$ if $m = 1$), then*

$$[\hat{Z}_r(\lambda) : \hat{L}_r(\lambda - b\theta(m-2)\alpha)] \neq 0.$$

Proof: We first note that, by [10, 3.1(20)(ii)], we have

$$\mathrm{ch}\hat{Z}_r(\lambda) = e(\lambda)[1 + e(-\alpha) + \cdots + e(-(lp^{r-1} - 1)\alpha)].$$

First assume that $m > 1$. Then we have

$$\begin{aligned} e(\lambda_1, \lambda_2) &= e(\lambda_2 + lp^{m-2} - 1, \lambda_2)e(alp^{m-1} + blp^{m-2} - 1, 0) \\ &= e(\lambda_2 + lp^{m-2} - 1, \lambda_2)e(ap + b - 1, 0)^{F^{m-1}}. \end{aligned}$$

Similarly,

$$[1+\cdots+e(-(lp^{r-1}-1)\alpha)] = [1+\cdots+e(-(lp^{m-2}-1)\alpha)][1+\cdots+e(-(p^{r+1-m}-1)\alpha)]^{F^{m-1}},$$

and hence we obtain that

$$\mathrm{ch}\hat{Z}_r(\lambda) = [\mathrm{ch}\hat{Z}_{m-1}(\lambda_2 + lp^{m-2} - 1, \lambda_2)][\mathrm{ch}\bar{Z}_{r+1-m}(ap + b - 1, 0)]^{F^{m-1}},$$

where $\bar{Z}_s(\mu)$ is the classical induced module for the sth Jantzen subgroup of $GL(2, k)$. Now $\hat{L}_{m-1}(\lambda_2 + lp^{m-2} - 1, \lambda_2) \cong_{G_{m-1}T} L(\lambda_2 + lp^{m-2} - 1, \lambda_2)$, which has dimension lp^{m-2} by Steinberg's Tensor Product Theorem. Hence $\hat{Z}_{m-1}(\lambda_2 + lp^{m-2} - 1, \lambda_2) \cong \hat{L}_{m-1}(\lambda_2 + lp^{m-2} - 1, \lambda_2)$. Again by Steinberg's Tensor Product Theorem, the result will now follow in this case if we can show that

$$[\bar{Z}_{r+1-m}(ap + b - 1, 0) : \bar{L}_{r+1-m}((a-1)p + p - b - 1)] \neq 0,$$

where $\bar{L}(\mu)$ is the usual simple module for $GL(2,k)$. But this follows from the calculations in [33, Section 5.5].

We now consider the case $m = 1$. Now $[\hat{Z}_r(\lambda) : \hat{L}_r(\lambda)] = 1$ by [10, 3.1(13)(i) and (20)(ii)], so we consider $\mathrm{ch}\hat{Z}_r(\lambda) - \mathrm{ch}\hat{L}_r(\lambda)$. Writing $a = a' + p^{r-1}a''$ with $0 \leq a' < p^{r-1}$ we have that

$$\hat{L}_r(\lambda) \cong_{G_rT} L(\lambda_2 + b - 1, \lambda_2) \otimes \bar{L}(a')^F \otimes lp^{r-1}a'',$$

and so, as $b < l$, the highest remaining weight in $\mathrm{ch}\hat{Z}_r(\lambda) - \mathrm{ch}\hat{L}_r(\lambda)$ is

$$(\lambda_2 + (a-1)l + l - 1, \lambda_2 + b) = \lambda - b\alpha$$

as required.

We are now able to determine the desired blocks. As in the previous section, we denote the blocks of G_rT and G_r containing λ by $\hat{\mathcal{B}}_r(\lambda)$ and $\mathcal{B}_r(\lambda)$ respectively.

Theorem 6.2.10 *For $n = 2$, $r > 0$ and $\lambda \in X(T)$, we have*

$$\hat{\mathcal{B}}_r(\lambda) = \begin{cases} W.\lambda + lp^{m-1}\mathbb{Z}\Phi & \text{if } m \leq r, \\ \{\lambda\} & \text{if } m > r, \end{cases}$$

and

$$\mathcal{B}_r(\lambda) = W.\lambda + lp^{m-1}\mathbb{Z}\Phi + lp^{r-1}X(T).$$

Proof: We first consider the G_rT case. For $m > r$ the result follows from (6.2.8). For $m \leq r$, one inclusion comes from (6.2.6). For the reverse inclusion, given two weights in $W.\lambda + lp^{m-1}\mathbb{Z}\Phi$, we use (6.2.7) and (6.2.9) to construct a chain of weights linking them in the G_rT case. Finally we deduce the G_r case from the G_rT result using (6.2.5).

The determination of the blocks of the infinitesimal q-Schur algebras (in the case $n = 2$) will now follow just as in the classical case described earlier, once we have verified a few remaining technical results. We first collect together those results whose proofs are just appropriate modifications of the G_1T results obtained in [10].

Lemma 6.2.11 *For $\lambda = \lambda' + lp^{r-1}\lambda'' \in X(T)$, with $\lambda' \in P_r(D)$ and $\lambda'' \in X(T)$, we have*

i) $\hat{Q}_r(\lambda) \cong \hat{Q}_r(\lambda') \otimes lp^{r-1}\lambda''$;

ii) all weights of $\hat{Z}_r(\lambda)$ satisfy $\lambda - 2(lp^{r-1}-1)\bar{\rho} \leq \mu \leq \lambda$.

Proof: See [10, 3.2(10)(ii) and 3.1(20)(ii)] respectively.

It now only remains to check

Lemma 6.2.12 *For $\lambda = \lambda' + lp^{r-1}\lambda'' \in X(T)$ with $\lambda' \in P_r(D)$ and $\lambda'' \in X(T)$ we have*

$$\hat{Q}_{r+1}(\lambda) \cong_{G_rT} \hat{Q}_r(\lambda') \otimes \hat{Q}_1(\lambda'')^{F^r}.$$

Proof: This follows just as in [34, II 11.15 Lemma], once we have shown that $\hat{Q}_{r+1}(\lambda)$ is injective as a G_rT module, and that the appropriate spectral sequence exists. Set $H = G_{r+1}T$, and denote by $\bar{H}$ the factor group generated by $d_q^{-lp^{r-1}}$ and the $c_{ij}^{lp^{r-1}}$, for all $1 \leq i,j \leq n$. It is routine to check that this is a sub-Hopf algebra, indeed $\bar{H} \cong GL(n,k)_1T$ under the map taking $c_{ij}^{lp^{r-1}} \longmapsto x_{ij}$ and $d_q^{-lp^{r-1}} \longmapsto d^{-1}$. The corresponding subgroup H_1 (in the notation of Section 1.3) has defining ideal generated by the elements $c_{ij}^{lp^{r-1}} - \delta_{ij}$ and $d_q^{-lp^{r-1}} - 1$, for all $1 \leq i,j \leq n$. Hence $H_1 \cong G_r$.

Arguing as in [5, (1.3.3)], we see that $k[H]$ is free (so certainly faithfully flat) as a $k[\bar{H}]$-module. So by (1.4.2(iv)) we get the spectral sequence required in the proof of the lemma. Now by (1.4.2(iii)), or the main theorem in [19], $\hat{Q}_{r+1}(\lambda)$ is an injective G_r-module. Also, by [10, 3.1(9)], $\mathrm{Ind}_{G_r}^{G_rT}$ is exact so, as σ_H and σ_{H_1} are anti-automorphisms (see [20, Remark 2.2]), we have that $\hat{Q}_{r+1}(\lambda)$ is an injective G_rT-module by [39, (2.9.1)].

Now the argument of the previous section, along with the above results, gives

Theorem 6.2.13 *For $n = 2$ and $d \geq 0$ we have for all $\lambda \in \Gamma_r^d(D)$ that*

$$\mathcal{B}_r^d(\lambda) = \hat{\mathcal{B}}_r(\lambda) \cap \Gamma_r^d(D).$$

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