Chapter 1

1. The survival model

1.1 Survival probabilities for a life age 0

Let us define T_0 to be the random variable representing the future lifetime of a life aged 0. In other words, T_0 is the eventual age at death of a new-born baby picked at random from a homogeneous population of new-born babies.

If we assume that T_0 is a continuous random variable then the distribution function of T_0, $F_0(t)$, is given by:

$$F_0(t) = \Pr(T_0 \le t) \tag{1.1.1}$$

Conversely, the survival function, $S_0(t)$, is given by:

$$S_0(t) = \Pr(T_0 > t) = 1 - F_0(t)$$

Example 1

Suppose $\qquad F_0(t) = 1 - e^{-0.008t}$

Find the probability that a new-born baby dies between age 60 and age 70

Solution

$$\Pr(60 \le T_0 < 70) = F_0(70) - F_0(60)$$

$$= e^{-0.48} - e^{-0.56}$$
$$= 0.04757$$

1.2 Survival probabilities for age x (x>0)

Consider a life aged x. We denote this throughout the book as (x).

Let us define T_x to be the random variable representing the future lifetime of a person who is currently aged x exact.

The distribution function of T_x, $F_x(t)$, is given by:

$$F_x(t) = \Pr(T_x \le t) \tag{1.2.1}$$

and the corresponding survival function, $S_x(t)$, is given by:

$$S_x(t)=\Pr(T_x>t)=1-F_x(t) \tag{1.2.2}$$

We can now find expressions for $F_x(t)$ and $S_x(t)$ in terms of $F_o(t)$ and $S_o(t)$, respectively:

$$F_x(t)=\Pr(T_0 \le x+t \mid T_0>x)$$

$$=\frac{\Pr(x<T_0 \le x+t}{\Pr(T_0>x)} \tag{1.2.3}$$

Note that (1.2.3) uses Bayes Theorem: given events X and Y,

$$\Pr(X \mid Y)=\frac{\Pr(X \text{ and } Y)}{\Pr(Y)}$$

Hence, from (1.2.3),

$$F_x(t)=\frac{F_O(x+t)-F_O(x)}{1-F_O(x)} \tag{1.2.4}$$

Similarly,

$$S_x(t)=\Pr(T_0>x+t \mid T_0>x)$$

$$=\frac{\Pr(T_o>x+t)}{\Pr(T_0>x)}$$

$$=\frac{S_o(x+t)}{S_o(x)} \tag{1.2.5}$$

Formula (1.2.5) can be re-written as

$$S_0(x+t)=S_0(x)\ .\ S_X(t) \tag{1.2.6}$$

Formula (1.2.6) has the following intuitive explanation:

The probability that a life aged 0 survives to age $x+t$ is the probability that a life aged 0 survives to age x and then survives a further t years to age $x+t$.

This explanation can be shown diagrammatically:

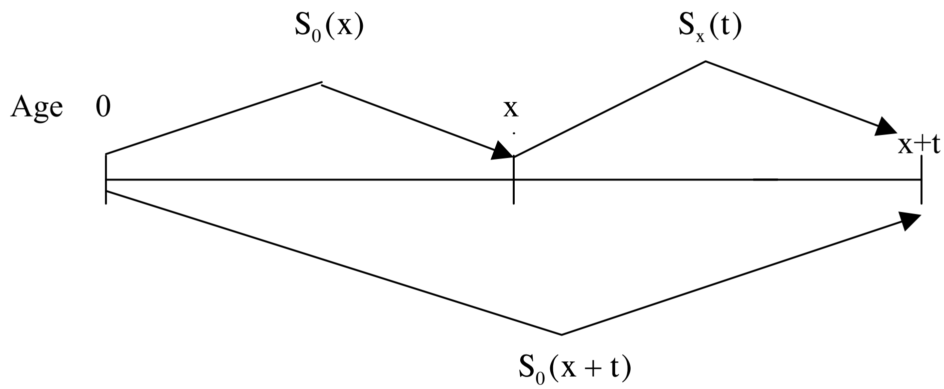

Figure 1.2

Similar reasoning can be applied to obtain the following additional useful results:

$$S_X(t+a) = S_x(t)\ .\ S_{x+t}(a) \qquad \text{for a >70} \qquad (1.2.7)$$
$$= S_x(a)\ .\ S_{x+a}(t)$$

1.3 The force of mortality

An important component within the survival model is the force of mortality at age x. This is denoted μ_X and is defined as follows:

$$\mu_X = \lim_{dx\to 0}\left(\frac{1}{dx}\ .\ \Pr\left(x<T_0 \le x+dx \middle| T_0>x\right)\right) \qquad (1.3.1)$$

For an interpretation of μ_X, it is helpful to re-arrange formula (1.3.1) as follows:

$$\mu_X.dx \simeq \Pr\left((x)\text{ dies between age x and age x+dx}\right) \qquad (1.3.2)$$

where dx is very small.

Using formula (1.2.4), we can re-write formula (1.3.1) in terms of distribution and survival functions:

$$\mu_X = \lim_{dx\to 0}\left(\frac{1}{dx}\ .\ \frac{F_0(x+dx)-F_0(x)}{1-F_0(x)}\right)$$

$$= \frac{1}{1-F_O(x)} \cdot \frac{d}{dx} F_O(x) \tag{1.3.3}$$

$$= \frac{-1}{S_O(x)} \cdot \frac{d}{dx} S_O(x) \tag{1.3.4}$$

1.4 Actuarial notation for survival probabilities

The probability that a life aged x survives at least one year is denoted by p_x in actuarial notation.

Hence, $p_x = \Pr((x) \text{ survives at least one year})$ (1.4.1)

The probability that a life aged x dies during the year is denoted by q_X.
This is also known as "the rate of mortality".

Hence $q_X = \Pr((x) \text{ dies during the next year})$ (1.4.2)

We can relate (1.4.1) and (1.4.2) to the survival and distribution functions as follows:

$$p_X = S_X(1) = \Pr(T_X > 1) \tag{1.4.3}$$

$$q_X = F_X(1) = \Pr(T_X \leq 1) \tag{1.4.4}$$

Since a life aged x either survives the following year or dies during the year, it follows that:

$$p_X + q_X = 1 \tag{1.4.5}$$

The probability that (x) survives at least t years is denoted by ${}_t p_x$

Hence, ${}_t p_x = \Pr((x) \text{ survives at least t years})$ (1.4.6)

The probability that (x) dies during the next t years is given the symbol ${}_t q_x$.

Hence, ${}_t q_x = \Pr((x) \text{ dies during the next t years})$ (1.4.7)

It follows that:

$$ {}_t p_x = S_x(t) = \Pr(T_x > t) \tag{1.4.8}$$

$$ {}_t q_x = F_x(t) = \Pr(T_x \leq t) \tag{1.4.9}$$

$$ {}_t p_x + {}_t q_x = 1 \tag{1.4.10}$$

It should be noted that if $t=1$, we write p_x and q_x rather that ${}_1p_x$ and ${}_1q_x$.

The following useful result follows from formula (1.2.7):

$$ {}_{t+a}p_x = {}_tp_x \cdot {}_ap_{x+t} = {}_ap_x \cdot {}_tp_{x+a} \tag{1.4.11} $$

In addition, it is worth noting the following:

$$ \mu_x = \lim_{h\to 0}\left(\frac{{}_hq_x}{h}\right) \tag{1.4.12} $$

Example 2

$$ \text{If } {}_np_x = \frac{x}{x+n} \qquad (n>0) $$

find μ_x

Solution

$$ {}_hq_x = 1 - {}_hp_x $$

$$ = 1 - \frac{x}{x+h} $$

$$ = \frac{h}{x+h} $$

Hence

$$ \mu_x = \lim_{h\to 0}\left(\frac{{}_hq_x}{h}\right) $$

$$ = \lim_{h\to 0}\frac{1}{x+h} $$

$$ = \frac{1}{x} $$

1.5 Probability density function of T_x

Since T_x is a continuous random variable, it is natural to consider the probability density function of T_x which is denoted $f_x(t)$.

$$f_x(t) = \frac{d}{dt} F_x(t)$$

$$= \frac{d}{dt} {}_t q_x$$

$$= \frac{d}{dt}(1 - {}_t p_x)$$

$$= -\frac{d}{dt}({}_t p_x) \qquad (1.5.1)$$

Now, from (1.3.4),

$$\mu_x = -\frac{1}{{}_x p_o} \cdot \frac{d}{dx}({}_x p_o) \qquad (1.5.2)$$

Hence, using (1.5.1)

$$f_0(x) = {}_x p_0 \cdot \mu_x \qquad (1.5.3)$$

Thus,

$$f_x(t) = \frac{d}{dt} F_x(t)$$

$$= \frac{d}{dt}\left(\frac{F_0(x+t) - F_0(x)}{1 - F_0(x)} \right) \quad \text{from (1.2.4)}$$

$$= \frac{f_0(x+t)}{1 - F_0(x)}$$

$$= \frac{{}_{x+t} p_o \cdot \mu_{x+t}}{{}_x p_o} \quad \text{from (1.5.3)}$$

$$= \frac{{}_{x}p_{0} \cdot {}_{t}p_{x} \cdot \mu_{x+t}}{{}_{x}p_{0}} \text{ from (1.4.11)}$$

$$= {}_{t}p_{x} \cdot \mu_{x+t} \tag{1.5.4}$$

It also follows from (1.5.1) and (1.5.4) that

$$\frac{d}{dt}\,{}_{t}p_{x} = -{}_{t}p_{x} \cdot \mu_{x+t} \tag{1.5.5}$$

Example 3

Assume that, for a certain life, the survival function for age x, $S_x(t)$, $t \geq 0$, is of the form:

$$S_x(t) = \left(\frac{\lambda}{\lambda + t}\right)^{\alpha} \qquad \lambda, \alpha > 0.$$

Obtain and simplify expressions involving λ, α and t for:

(i) $F_x(t)$

(ii) ${}_{n}p_{x}$ (n>0)

(iii) $f_x(t)$

(iv) μ_{x+n}.

Solution

(i) $F_x(t) = 1 - S_x(t)$

$$= 1 - \left(\frac{\lambda}{\lambda + t}\right)^{\alpha}$$

(ii) ${}_{n}p_{x} = S_x(n)$

$$= \left(\frac{\lambda}{\lambda + n}\right)^{\alpha}$$

(iii) $f_x(t) = \frac{d}{dt}F_x(t)$

$$= \frac{d}{dt}\left\{1 - \left(\frac{\lambda}{\lambda + t}\right)^{\alpha}\right\}$$

$$= \left(\frac{\alpha\lambda^{\alpha}}{(\lambda + t)^{\alpha+t}} \right)$$

(iv) $$\mu_{x+n} = \frac{f_x(n)}{S_x(n)} = \frac{\alpha}{\lambda + n}$$

1.6 Formula for ${}_tq_x$

We can find an important formula for ${}_tq_x$ using (1.5.4):

$${}_tq_x = F_x(t)$$

$$= \int_0^t f_x(s)ds$$

$$= \int_0^t {}_sp_x \;\; \mu_{x+s} \; ds \qquad (1.6.1)$$

It is helpful to explain the formula in (1.6.1) by means of the following general reasoning:

${}_sp_x$ is the probability that a life aged x survives a period of s to age x + s.

We also know from (1.3.2) that $\mu_{x+s}.ds$ can be thought of as being the probability that a life aged x+s dies in the next short interval of time. Hence, ${}_sp_x.\mu_{x+s}$ is the probability that a life aged x survives for a period of s (where s≤t) and dies in the next instant. The sum of these probabilities for all possible value of s between 0 and t (which is obtained by integrating the expression) must equal the probability that a life aged x dies during the next period of t.

This explanation can be represented diagrammatically as follows:

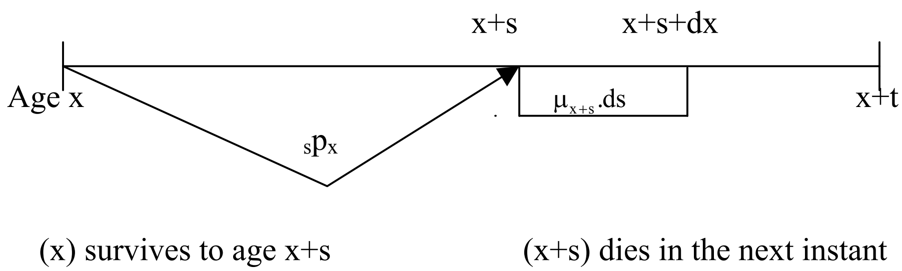

Figure 1.6

1.7 Formula for ${}_t p_x$

We can find an expression for ${}_t p_x$ by first noting:

$$\mu_s = -\frac{1}{{}_s p_0} \frac{d}{ds}\left({}_s p_0\right)$$
$$= -\frac{d}{ds} \log_e\left({}_s p_0\right)$$

Hence

$$-\int_x^{x+t} \mu_s \, ds = \int_x^{x+t} \frac{d}{ds} \log_e\left({}_s p_0\right) ds.$$

$$\therefore -\int_x^{x+t} \mu_s \, ds = \left[\log_e\left({}_s p_0\right)\right]_x^{x+t}$$
$$= \log_e\left(\frac{{}_{x+t} p_0}{{}_x p_0}\right)$$
$$= \log_e\left({}_t p_x\right) \qquad \text{from (1.4.11)}$$

Hence

$${}_t p_x = e^{-\int_x^{x+t} \mu_s \, ds} \qquad (1.7.1.)$$

It is also worth noting from (1.7.1) that:

$$\frac{d}{dx}\left({}_t p_x\right) = \frac{d}{dx}\left(e^{-\int_x^{x+t} \mu_s \, ds}\right)$$

$$= \frac{d}{dx}\left(-\int_x^{x+t} \mu_s \, ds\right).e^{-\int_x^{x+t} \mu_s \, ds}$$

$$= {}_t p_x.(\mu_x - \mu_{x+t}) \qquad (1.7.2)$$

$$\neq \frac{d}{dt}({}_t p_x) \qquad \text{(see (1.5.5))}$$

1.8 The Life Table

Having defined the survival probability, ${}_t p_x$, it is now possible to construct the life table. It will be seen that the life table is a convenient way of summarizing the information contained within the survival model.

Given the survival probabilities, ${}_t p_x$, for a particular group of lives for all values of t and x, the life table can be constructed as follows:

An initial age, α, for the life table is chosen. For example, α would be 0 if the life table for a country's population is being constructed. α is known as the "starting age".

Next, an arbitrary positive number, l_α, is chosen. l_α is known as the "radix".

Now, for $x \geq \alpha$, a function of x, l_x, is defined as follows:

$$l_x = {}_{x-\alpha} p_x . l_\alpha$$

l_x, known as the "life table", is therefore the product of the survival probability, ${}_{x-\alpha} p_\alpha$, and the radix, l_α. Hence, l_x can be found for any age x, provided all the survival probabilities within the survival model are known.

It is useful to continue this analysis a stage further:-

For $t>0$ and $x \geq \alpha$,

$${}_{t+x-\alpha} p_x = {}_{x-\alpha} p_\alpha \cdot {}_t p_x \qquad \text{from (1.4.11)}$$

Multiplying by l_α gives

$$\begin{aligned} {}_{t+x-\alpha} p_\alpha . l_\alpha &= {}_{x-\alpha} p_\alpha . l_\alpha \cdot {}_t p_x \\ \therefore \qquad l_{x+t} &= l_x \cdot {}_t p_x \end{aligned}$$

Hence

$${}_t p_x = \frac{l_{x+t}}{l_x} \tag{1.8.1}$$

Therefore, the survival probability, ${}_t p_x$, is the ratio of l_{x+t} to l_x within the life table.

For most survival models there is some age, ω, beyond which survival is assumed to be impossible. ω is known as the "limiting age".

Hence, for all $x \geq 0$ and $t \geq \omega - x$,

$$ {}_tp_x = 0 $$
$$ \therefore l_y = 0 \quad \text{for } y \geq \omega $$

Example 4

Given
$$p_{20} = 0.98$$
$${}_2p_{20} = 0.95$$
$${}_3p_{20} = 0.91$$

and choosing a radix of $l_{20} = 100{,}000$, calculate l_{21}, l_{22} and l_{23}.

Solution

From (1.8.1),
$$l_{21} = l_{20}.p_{20} = 98{,}000$$

$$l_{22} = l_{20} \cdot {}_2p_{20} = 95{,}000$$
$$l_{23} = l_{20} \cdot {}_3p_{20} = 91{,}000$$

Hence the life table is:

x	l_x
20	100,000
21	98,000
22	95,000
23	91,000

It is helpful to interpret the results from this example from a deterministic rather than stochastic viewpoint. If there are 100,000 lives in the population subject to this survival model who are aged exactly 20, 91,000 of them will survive to age 23. This is an important intuitive interpretation of formula (1.8.1).

In general, if there are Z lives in a population aged x then

$Z.{}_tp_x =$ Number of lives expected to reach age x+t (ignoring the fact that this quantity will probably not be an integer).

Having introduced the life table, it can be convenient to establish the following relationship between l_x and μ_x.

From (1.5.2),
$$\mu_x = -\frac{1}{{}_xp_0}.\frac{d}{dx}\left({}_xp_0\right)$$
$$= -\frac{l_0}{l_x}.\frac{d}{dx}\left(\frac{l_x}{l_0}\right)$$
$$= -\frac{1}{l_x}.\frac{d}{dx}\left(l_x\right) \tag{1.8.2}$$

1.9 The d_x function

d_x is defined as follows:

$$d_x = l_x - l_{x+1} \tag{1.9.1}$$

Interpreting this function from a deterministic viewpoint, d_x can be thought of as representing the number of lives dying between age x and age x+1 out of l_x lives alive at age x.

Note that

$$q_x = 1 - p_x$$
$$= 1 - \frac{l_{x+1}}{l_x}$$
$$= \frac{l_x - l_{x+1}}{l_x}$$
$$= \frac{d_x}{l_x} \tag{1.9.1}$$

This result follows from general reasoning since the expression on the right hand side can be interpreted as being the proportion of lives alive at age x who die between age x and age x+1.

Similarly

$$
\begin{aligned}
{}_k q_x &= 1 - {}_k p_x \\
&= 1 - \frac{l_{x+k}}{l_x} \\
&= \frac{l_x - l_{x+k}}{l_x} \qquad (1.9.2)
\end{aligned}
$$

Example 5

Calculate d_{20}, d_{21} and d_{22} for Example 4.

Solution

$$
\begin{aligned}
d_{20} &= l_{20} - l_{21} = 100{,}000 - 98{,}000 = 2{,}000 \\
d_{21} &= l_{21} - l_{22} = 98{,}000 - 95{,}000 = 3{,}000 \\
d_{22} &= l_{22} - l_{23} = 95{,}000 - 91{,}000 = 4{,}000
\end{aligned}
$$

1.10 The ${}_{n|m}q_x$ function

Insurance problems often involve the probability that a person survives n years but dies in the m years which follow. The symbol for this probability is ${}_{n|m}q_x$.

Hence ${}_{n|m}q_x = p_r$((x) survives to age x+n but dies before reaching age x+n+m).
$= p_r$ $(n \leq T_x < n+m)$.

$$
\begin{aligned}
&= {}_n p_x - {}_{n+m} p_x \qquad (1.10.1) \\
&= {}_n p_x - {}_n p_x \cdot {}_m p_{x+n} \quad \text{using (1.4.11)} \\
&= {}_n p_x \left(1 - {}_m p_{x+n}\right) \\
&= {}_n p_x \cdot {}_m q_{x+n} \qquad (1.10.2)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{l_{x+n}}{l_x} \cdot \frac{l_{x+n} - l_{x+n+m}}{l_{x+n}} \\
&= \frac{l_{x+n} - l_{x+n+m}}{l_x} \qquad (1.10.3)
\end{aligned}
$$

If m=1, the notation becomes ${}_{n|}q_x$ and

$$ {}_{n|}q_x = {}_np_x . q_{x+n} = \frac{d_{x+n}}{l_x} \qquad (1.10.4) $$

Example 6

A group of lives experience mortality which can be represented between ages 80 and 90 by the English Life Tables No 12-Males mortality table with a deduction from the force of mortality of 0.05 at age 80, the deduction increasing linearly with age to a deduction of 0.15 at age 90. Find the probability of a life aged 80 dying within the next 10 years.

Solution

Let μ_x be the force of mortality according to the English Life Tables No 12 – Males mortality table.

Let μ_x^* be the force of mortality experienced by the particular group of lives.

Then

$$ \mu_{80+t}^* = \mu_{80+t} - [0.05 + 0.01t] \qquad 0 \le t \le 10 $$

and

$$ \begin{aligned} {}_{10}p_{80}^* &= e^{-\int_0^{10} \mu_{80+t}^* dt} \\ &= e^{\int_0^{10} (\mu_{80+t} - (0.05+0.01\,t))dt} \\ &= e^{-\int_0^{10} \mu_{80+t} dt} \quad .e^{\int_0^{10} 0.05\,dt} \quad .e^{\int_0^{10} 0.01t\,dt} \\ &= {}_{10}p_{80} \quad .e^{0.5} \quad .e^{[0.01t^2/2]_0^{10}} \\ &= {}_{10}p_{80} \quad .e^{0.5} \quad .e^{0.5} \\ &= {}_{10}p_{80} .e \end{aligned} $$

(where ${}_{10}p_{80}$ is calculated according to the ELT No 12 mortality table).

$$ = \frac{l_{90}}{l_{80}} .e $$

$$ = \frac{3047.2}{22933} .e = 0.36119 $$

Hence, ${}_{10}q_{80}^{*} = 1 - 0.36119$.

$= 0.63881$