Differential Equations for Finance MA3607, 2013/14

Functions defined by integrals

The Dirac delta function, $\delta(x)$, (or more accurately *distribution*) is defined as

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ ”\infty” & x = 0 \,. \end{cases} \tag{1}$$

In other words $\delta(x)$ is zero for all $x \neq 0$.

Exercises

1. Find the values of x for which the following functions are zero

$$\begin{aligned} (i) \quad & \delta(x+4)\,, \\ (ii) \quad & \delta(3x-20)\,, \\ (iii) \quad & \delta(x^2-3x)\,, \\ (iv) \quad & \delta(e^x)\,, \qquad\qquad x \in \mathbf{R}\,. \end{aligned}$$

Integrating the delta function

The value of $\delta(x)$ at $x = 0$, denoted ”∞” above, only really makes sense when we consider integrals of the delta function. This is why $\delta(x)$ is not really a function: it only makes sense when it is inside some integral. The simplest integral involving $\delta(x)$ is

$$\boxed{I = \int_{-\infty}^{\infty} \delta(x) = 1\,.} \tag{2}$$

Let us see how this expression is consistent with the definition (1). It is far beyond the scope of this course to define precisely both $\delta(x)$ and the integral above - this area of mathematics is called measure theory. However, we should think of the $\delta(x)$ as being very large at $x = 0$ - infinite, so to say. This needs to be the case for $\int_{-\infty}^{\infty} \delta(x)$ to be non-zero. Roughly speaking the integral of any function measures the area between the function and the x-axis. If a function is equal to zero for some part of the x-axis then, of course, the area will also be zero. So for example

$$\begin{aligned} \int_{-7}^{-3} \delta(x)dx = \int_{-7}^{-3} 0dx \quad &= \quad 0\,, \\ \int_{\frac{1}{10}}^{10000} \delta(x)dx = 0\,. \end{aligned} \tag{3}$$

In fact, since $\delta(x) = 0$ for $x \neq 0$ we have for any $y > 0$

$$I = \int_{-\infty}^{\infty} \delta(x) = \int_{-\infty}^{-y} \delta(x) + \int_{-y}^{y} \delta(x) + \int_{y}^{\infty} \delta(x) = \int_{-y}^{y} \delta(x)\,. \tag{4}$$

If $\delta(0)$ was a finite number, we could easily find an upper bound for the value of I: since $\delta(x) \leq \delta(0)$ we see that

$$I = \int_{-y}^{y} \delta(x) \leq 2y\delta(0)\,, \tag{5}$$

which holds for any value of $y > 0$. We can make y as small as we like so, if $\delta(0)$ were a finite number, the above argument would show that $I = 0$. But we want a function for which $I = 1$ - the only way to do that while having $\delta(x \neq 0) = 0$ is to have $\delta(0)$ "infinite".

Exercises

2. Evaluate the following integrals

$$\begin{aligned}
&(i) \quad \int_{-3}^{1} \delta(x)dx\,,\\
&(ii) \quad \int_{-5}^{2} \delta(x-3)dx\,,\\
&(iii) \quad \int_{-\infty}^{\infty} \delta(x-7)dx\,,\\
&(iv) \quad \int_{-\infty}^{\infty} \delta(2x)dx\,,\\
&(v) \quad \int_{-\infty}^{\infty} \delta(10-5x)dx\,.
\end{aligned}$$

Integrating with the delta function

We will be interested in integrals of the form

$$J = \int_{-\infty}^{\infty} f(x)\delta(x)dx\,, \tag{6}$$

where $f(x)$ is a function which is continuous at $x = 0$. To evaluate this integral we note that just as above

$$J = \int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-y}^{y} f(x)\delta(x)dx\,, \tag{7}$$

for any $y > 0$ - the value of the integral is *the same* for each such y. Since $f(x)$ is continuous at $x = 0$, as we make y smaller and smaller the value of $f(x)$ approaches $f(0)$. Using this observation, the integral J then can be evaluated in the small-y limit since

$$J = \lim_{y\to 0} \int_{-y}^{y} f(x)\delta(x)dx = \lim_{y\to 0} \int_{-y}^{y} f(0)\delta(x)dx = f(0) \lim_{y\to 0} \int_{-y}^{y} \delta(x)dx = f(0)I = f(0)\,. \tag{8}$$

So we have found the central formula of integrating functions with the delta-function

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)\,.} \tag{9}$$

Notice that the integral depends only on the value of the function $f(x)$ at the point where the delta function is infinite - $x = 0$ in this case.

Exercises

2. Show that equation (9) implies equation (2).
3. Evaluate the following integrals

$$(i) \quad \int_{-\infty}^{\infty} \cos(x)\delta(x)dx\,,$$

$$(ii) \quad \int_{-\infty}^{\infty} (6x^3 + 24x^2 - 3x + 17)\delta(x)dx\,,$$

$$(iii) \quad \int_{-\infty}^{\infty} e^x\delta(x)dx\,,$$

$$(iv) \quad \int_{-\infty}^{\infty} \sin(x)\delta(x-4)dx\,,$$

$$(v) \quad \int_{-\infty}^{\infty} (17x^2 - 6x^4 + 21)\delta(10-5x)dx\,,$$

$$(vi) \quad \int_{-2}^{4} \tan(x)\delta(x+1)dx\,.$$