Calculus of Variation

Finding the extremum of a function $f(x)$ can be done by solving

$$\frac{df}{dx} = 0$$

But sometimes we may be interested in finding a function $\phi(x)$ which extremises an integral $\qquad \phi' \equiv \frac{d\phi}{dx}$

$$S(\phi, \phi') = \int_{x_0}^{x_1} dx\, \mathcal{L}(x, \phi(x), \phi'(x))$$

where $\mathcal{L}$ is a function of x, ϕ and ϕ'

<u>Eg</u>; What is the shortest distance between the points $(x,y)=(0,0)$ and $(1,1)$?

Of course we know the answer! the straight line $y_{opt}(x) = x$. But let's see this formally

Any curve $y(x)$ s.t. $y(0)=0$, $y(1)=1$ is a candidate, its length is given by

$$S(y') = \int_0^1 dx\, \mathcal{L}(y')$$

where $\mathcal{L}(y') = \sqrt{1 + \frac{dy}{dx}}$

To find a maximum/minimum of $S(\phi, \phi')$ consider how S changes when we add a "small" function to ϕ

$$\tilde{\phi}(x) \equiv \phi(x) + \varepsilon g(x) \equiv \phi(x) + \delta\phi(x)$$

where ε is a small constant and $g(0) = 0$, $g(1) = 0$. so that $\tilde{\phi}(0) = 0$, $\tilde{\phi}(1) = 1$

$\delta\phi(x)$ is a "small" function added to ϕ called the variation

the change in S is then

$$\delta S = S(\tilde{\phi}, \tilde{\phi}') - S(\phi, \phi')$$

$$= \int_{x_0}^{x_1} dx\, \mathcal{L}(x, \phi + \varepsilon g, \phi' + \varepsilon g') - \mathcal{L}(x, \phi, \phi')$$

$$\approx \int_{x_0}^{x_1} dx\, \mathcal{L}(x, \phi, \phi') + \varepsilon g(x) \frac{\partial \mathcal{L}}{\partial \phi}(x, \phi, \phi')$$

$$+ \varepsilon g'(x) \frac{\partial \mathcal{L}}{\partial \phi'}(x, \phi, \phi') - \mathcal{L}(x, \phi, \phi')$$

$$+ O(\varepsilon^2)$$

Integrate by parts (IBP):

$$\approx \varepsilon \int_{x_0}^{x_1} dx\, g(x) \frac{\partial \mathcal{L}}{\partial \phi} - g(x) \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \phi'}$$

$$= \varepsilon \int_{x_0}^{x_1} dx\, g(x) \left[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \phi'} \right]$$

If $S(\phi,\phi')$ is to be a maximum then δS has to vanish.

To see this suppose that instead $\delta S \neq 0$

If $\delta S > 0$ then $\tilde{S}(\tilde{\phi},\tilde{\phi}') = S(\phi,\phi') + \delta S > S(\phi,\phi')$

so $S(\phi,\phi')$ not maximum $\lightning$

If $\delta S < 0$ then

$$S(\phi - \varepsilon g, \phi' - \varepsilon g') = S(\phi,\phi') - \delta S > S(\phi,\phi')$$

so again $S(\phi,\phi')$ not maximum $\lightning$

A similar argument shows that if $S(\phi,\phi')$ is to be a minimum δS has to vanish.

Notice that by the above argument when $S(\phi,\phi')$ is a min/max $\delta S = 0$ for all $g(x)$. Below we prove that this happens only when

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) = 0}$$

This is the Euler-Lagrange equation

Lemma If $f(x)$ is a continuous function on $[x_0, x_1]$ and $\int_{x_0}^{x_1} f(x)\,g(x) = 0$ for all continuously differentiable functions g with $g(x_0) = g(x_1) = 0$ then $f(x) \equiv 0 \quad \forall x \in [x_0, x_1]$

Proof Suppose $f(\tilde{x}) \neq 0$ for some $\tilde{x} \in [x_0, x_1]$

then $\exists\, \varepsilon$ s.t. $f(\tilde{\tilde{x}}) \neq 0 \ \forall\, \tilde{\tilde{x}} \in (\tilde{x} - \varepsilon, \tilde{x} + \varepsilon)$

Now pick
$$g(x) = \begin{cases} f(x)\,(x - \tilde{x} + \varepsilon)^2 (x - \tilde{x} - \varepsilon)^2 & x \in (\tilde{x} - \varepsilon, \tilde{x} + \varepsilon) \\ 0 & x \notin (\tilde{x} - \varepsilon, \tilde{x} + \varepsilon) \end{cases}$$

then
$$\int_{x_0}^{x_1} f(x)\,g(x) = \int_{\tilde{x}-\varepsilon}^{\tilde{x}+\varepsilon} (f(x))^2 (x - \tilde{x} + \varepsilon)^2 (x - \tilde{x} - \varepsilon)^2$$
$$> 0$$
since integrand > 0 on $(\tilde{x} - \varepsilon, \tilde{x} + \varepsilon)$

So $f(x) \equiv 0 \quad \forall x \in [x_0, x_1]$

Example Extremise

$$S = \int_0^1 dx \sqrt{1+(y')^2}$$

with $y(0)=0 \quad y(1)=1$

$$\delta S = \int_0^1 dx\, \delta\left(\sqrt{1+(y')^2}\right)$$

$$= \int_0^1 dx \frac{1}{2\sqrt{1+y'^2}}\, \delta(y'^2) = \int_0^1 dx \frac{y'\delta y'}{\sqrt{1+(y')^2}}$$

$$\overset{IBP}{=} \int_0^1 dx\, \delta y \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right)$$

Euler Lagrange equations are

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

CHECK USING E-L. with $\mathcal{L} = \sqrt{1+(y')^2}$ directly

$$y' = A\sqrt{1+(y')^2}$$ for some constant A

$$y'^2 = A^2(1+y'^2)$$

ie $(y')^2 = \frac{A^2}{1-A^2} = a^2$ $\quad a$ +ve constant

$y' = a$ so $y = ax+b$

$y(0) = b = 0 \quad y(1) = a+b = 1$

So optimal function is

$$y(x) = x$$

This is the straight line we expected

Example

Extremise $S[x(t), \dot{x}(t)] = \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2\right)$

Notice that we have not explicitly specified the integration limits, nor have we said what the values of x at those points is.

This is common practice — one is often just interested in what the optimal x(t) looks like for general values of t and not at the "end points"

Varying S we have

$$\delta S = \int dt \left(\dot{x}\,\delta\dot{x} - \omega^2 x\,\delta x\right)$$

$$= \int dt \left(\dot{x}\frac{d}{dt}(\delta x) - \omega^2 x\,\delta x\right)$$

integrate by parts

$$= \int dt \left(-\frac{d}{dt}(\dot{x})\delta x - \omega^2 x\,\delta x\right)$$

$$= -\int dt\,\delta x\left(\ddot{x} + \omega^2 x\right)$$

So for extremum

$$\ddot{x} + \omega^2 x = 0 \qquad (4)$$

Same result follows from using E-L equations with

$$L = \frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2$$

Equation (A) is just the simple harmonic motion eqn! So this motion extremises the integral

$$\int dt(L) = \int dt\,(K.E. - P.E)$$

$K.E. = \frac{1}{2}\dot{x}^2 \qquad P.E. = \frac{\omega^2}{2}x^2$

Example

Extremise

$$S[\theta(t), \dot{\theta}(t)] = \int dt\left(\frac{1}{2}\dot{\theta}^2 + \omega^2\cos\theta\right)$$

varying the action

$$\begin{aligned}
\delta S &= \int dt\,(\delta\dot{\theta}\,\dot{\theta} - \omega^2\,\delta\theta\sin\theta) \\
&= \int dt\left(\frac{d}{dt}(\delta\theta)\,\dot{\theta} - \omega^2\delta\theta\sin\theta\right) \quad \text{I.B.P.} \\
&= -\int dt\left(\delta\theta\,\frac{d}{dt}(\dot{\theta}) + \omega^2\delta\theta\sin\theta\right) \\
&= \int dt\,\delta\theta\,(\ddot{\theta} + \omega^2\sin\theta)
\end{aligned}$$

So for extremum

$$\ddot{\theta} + \omega^2\sin\theta = 0$$

The same result can be obtained by using the EL equations with

$$L = \frac{1}{2}\dot{\theta}^2 + \omega^2\cos\theta$$

To see this explicitly note

$$\frac{\partial}{\partial \theta}\left(\frac{1}{2}\dot{\theta}^2 + \omega^2 \cos\theta\right) = \frac{\partial}{\partial \theta}\left(\omega^2 \cos\theta\right)$$
$$= -\omega^2 \sin\theta$$

$$\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}\dot{\theta}^2 + \omega^2 \cos\theta\right) = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}\dot{\theta}^2\right)$$
$$= \dot{\theta}$$

So E.L. eqns are

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -\omega^2 \sin\theta - \frac{d}{dt}\left(\dot{\theta}\right)$$
$$= -\omega^2 \sin\theta - \ddot{\theta} = 0$$

This equation is the Pendulum equation!

So the pendulum's motion extremises

$$\int dt\,(L) = \int dt\,(\text{K.E.} - \text{P.E.})$$

where K.E. $= \frac{1}{2}\dot{\theta}^2$

P.E. $= \omega^2 \cos\theta$