Recall the differential equations of planetary motion

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \qquad (1) \qquad \text{where } \mu = GM$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \qquad (2)$$

Consider the expression $\dot{r}(1) + r\dot{\theta}(2)$:

$$\dot{r}\ddot{r} - r\dot{r}\dot{\theta}^2 + 2r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta} = -\frac{\mu}{r^2}\dot{r}$$

This is equivalent to

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2\right) + \frac{d}{dt}\left(\frac{1}{2}r^2\right)\dot{\theta}^2 + r^2\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2\right) = \mu\frac{d}{dt}\left(\frac{1}{r}\right)$$

Combining all the terms together we have

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 - \mu\frac{1}{r}\right) + \frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}^2\right)$$

$$= \frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \frac{\mu}{r}\right) = 0$$

So $E = \underbrace{\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2}_{\text{Kinetic Energy}} \underbrace{- \frac{\mu}{r}}_{\text{Potential Energy}}$ is a constant

Recall that the velocity $\frac{d\vec{r}}{dt} \equiv \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

so $\|\vec{v}\|^2 = (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})^2 = \dot{r}^2\hat{r}\cdot\hat{r} + 2r\dot{r}\dot{\theta}\hat{r}\cdot\hat{\theta} + r^2\dot{\theta}^2\hat{\theta}^2$

$$= \dot{r}^2 + r^2\dot{\theta}^2$$

Hence $K.E = \frac{1}{2}\|\vec{v}\|^2$ as in many other systems