Q1 i) $\frac{dM}{dt} = Mr$ so $M(t) = Ee^{-\int_t^T r(s)ds}$

where E is amount of money at time T

$E = 1000$, $T = 0$, want $M(t=-5)$

$$M(t=-5) = 1000\, e^{-\int_{-4}^{0} \frac{4}{100} + \frac{1}{50}s^2 ds}\, e^{-\int_{-5}^{-4} \frac{1}{\cosh s} ds}$$

$$= 1000\, e^{-\frac{1}{50}\left(2s + \frac{s^3}{3}\right)_{-4}^{0}}\, e^{-\left(\tanh\left(\frac{s}{2}\right)\right)_{-5}^{-4}}$$

$$\approx £543.76$$

ii) $$\frac{\partial V}{\partial t} = BS^2(a + 2bt + 3ct^2 + 4dt^3)e^{-at-bt^2-ct^3-dt^4}$$

$$\frac{\partial V}{\partial S} = A - 2SBe^{-at-bt^2-ct^3-dt^4}$$

$$\frac{\partial^2 V}{\partial S^2} = -2Be^{-at-bt^2-ct^3-dt^4}$$

Plug in to B-S eqn

$$0 = Be^{-at-bt^2-ct^3-dt^4}S^2\big(-a+\mu+t(\nu+\alpha-2b) + t^2(\lambda-3c) + t^3(\beta-4d)\big)$$

So $a = \mu$, $b = \frac{\alpha+\nu}{2}$, $c = \frac{\lambda}{3}$, $d = \beta/4$

$\frac{\partial V}{\partial S} = 0$ for $S = \frac{A}{2B} e^{\mu t + \frac{\alpha+\nu}{2}t^2 + \frac{\lambda}{3}t^3 + \frac{\beta}{4}t^4}$

$$\frac{\partial^2 V}{\partial S^2} = -2Be^{\mu t + \frac{\alpha+\nu}{2}t^2 + \frac{\lambda}{3}t^3 + \frac{\beta}{4}t^4} < 0$$

So max

Q2 i) $\dfrac{dS}{dx} = Ee^{x} = S$ so $\dfrac{dx}{dS} = \dfrac{1}{S}$

$$\frac{\partial V}{\partial \tau} = Ae^{\alpha x + \beta \tau}\left(\beta u + \frac{\partial u}{\partial \tau}\right)$$

$$\frac{\partial V}{\partial S} = \frac{dx}{dS}\frac{\partial V}{\partial x} = \frac{1}{S}Ae^{\alpha x + \beta \tau}\left(\alpha u + \frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{S^2}Ae^{\alpha x + \beta \tau}\left(-\alpha u - u_x + \alpha^2 u + 2\alpha u_x + u_{xx}\right)$$

into B-S eqn

$$0 = Ae^{\alpha x + \beta \tau}\Big(u_\tau - u_{xx} + (\beta + \alpha - \alpha^2 + k - \alpha k)u + (1 - 2\alpha - k)u_x\Big)$$

so $\alpha = \dfrac{1-k}{2}$ $\quad \beta = -\frac{1}{4}(k+1)^2$

ii) $u(x,\tau) = X(x)T(\tau)$ so $XT' = X''T \iff \dfrac{T'}{T} = -\lambda^2 = \dfrac{X''}{X}$

$X = A\cos\lambda x + B\sin\lambda x \quad T = Ce^{-\lambda^2 t}$ (constant C irrelevant for u)

$u = 0$ at $x = 0 \Rightarrow A = 0$; $u = 0$ at $x = 1 \Rightarrow \lambda = n\pi$, $n = 1, 2, 3$

so $u = \displaystyle\sum_{n=1}^{\infty} A_n e^{-n^2\pi^2\tau}\sin(n\pi x)$

$$A_n = 2\int_0^1 u(x,\tau=0)\sin(n\pi x)\,dx$$
$$= 2\int_0^{1/5} 2\sin(n\pi x)\,dx + 2\int_{1/5}^1 (-3)\sin(n\pi x)\,dx$$
$$= \frac{4}{n\pi}\left(1 - \cos\frac{n\pi}{5}\right) + \frac{6}{n\pi}\left((-1)^n - \cos\frac{n\pi}{5}\right) \quad (*)$$

iii) $V = A\left(\dfrac{S}{E}\right)^{\frac{1-k}{2}} e^{-\frac{(k+1)^2}{4}\tau}\displaystyle\sum_{n=1}^{\infty} A_n e^{-n^2\pi^2\tau}\sin\left(n\pi\log\frac{S}{E}\right)$

with A_n given in (*)

Q3 i) $I = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du$

$$I^2 = \frac{4}{\pi} \int_0^\infty \int_0^\infty e^{-x^2-y^2} dx\,dy = \frac{4}{\pi} \int_0^{\pi/2} \int_0^\infty r e^{-r^2} dr\,d\theta$$

$$= \frac{4}{\pi} \frac{\pi}{2} \left[-\tfrac{1}{2} e^{-r^2} \right]_0^\infty = 1 \quad \text{since } I > 0,\ I = 1$$

ii) $u = F(\xi) \qquad \xi = \frac{x-4}{\sqrt{\tau}} \qquad \xi_x = \frac{1}{\sqrt{\tau}} \quad \xi_\tau = -\frac{\xi}{2\tau}$

$$u_\tau = F'\xi_\tau = -\frac{\xi}{2\tau} F' \qquad u_x = F'\xi_x = \frac{1}{\sqrt{\tau}} F'$$

$$u_{xx} = \frac{1}{\sqrt{\tau}} F'' \xi_x = \frac{1}{\tau} F''$$

Heat eq $\Rightarrow \quad F'' + \frac{\xi}{2} F' = 0 \quad$ so $F' = \alpha e^{-\frac{1}{4}\xi^2}$ ↳ cst

$$\text{so } F = A \operatorname{erf}(\xi/2) + B$$

$u = 5$ for $x > 4$, $\tau = 0 \Rightarrow F \to 1$ as $\xi \to \infty$
$u = -2$ for $x < 4$, $\tau = 0 \Rightarrow F \to 0$ as $\xi \to -\infty$

So $\begin{cases} A + B = 5 \\ -A + B = -2 \end{cases}$ and $A = 7/2 \quad B = 3/2$

$$\text{So } u(x,\tau) = \frac{3}{2} + \frac{7}{2} \operatorname{erf}\left(\frac{x-4}{2\sqrt{\tau}}\right) \quad (*)$$

iii) $u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_4^\infty 5 e^{-(x-y)^2/4\tau} dy + \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^4 (-2) e^{-\frac{(x-y)^2}{4\tau}} dy$

$$= \frac{1}{2\sqrt{\pi\tau}} \int_{\frac{x-4}{2\sqrt{\tau}}}^{-\infty} 5 e^{-s^2} (-2\sqrt{\tau}) ds + \frac{1}{2\sqrt{\pi\tau}} \int_\infty^{\frac{x-4}{2\sqrt{\tau}}} (-2) e^{-s^2} (-2\sqrt{\tau}) ds$$

$$= \frac{5}{\sqrt{\pi}} \int_{-\infty}^0 e^{-s^2} ds + \frac{5}{\sqrt{\pi}} \int_0^{\frac{x-4}{2\sqrt{\tau}}} e^{-s^2} ds + \frac{2}{\sqrt{\pi}} \int_\infty^0 e^{-s^2} ds + \frac{2}{\sqrt{\pi}} \int_0^{\frac{x-4}{2\sqrt{\tau}}} e^{-s^2} ds$$

$$= \frac{5}{2} + \frac{5}{2} \operatorname{erf}\left(\frac{x-4}{2\sqrt{\tau}}\right) - 1 + \operatorname{erf}\left(\frac{x-4}{2\sqrt{\tau}}\right)$$

$= (*)$ as req d

Q4 $u_\tau = 4x u_{xx} + u^2 u_x$

$$\frac{u_{i,j+1} - u_{ij}}{\Delta\tau} = \frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{\Delta} 4i\Delta x + u_{ij}^2 \frac{u_{i+1j} - u_{i-1j}}{2\Delta x}$$

Let $F(u_{ij}) = \frac{u_{ij+1} - u_{ij}}{\Delta\tau} - 4i\frac{(u_{i+1j} - 2u_{ij} + u_{i-1j})}{\Delta x} - u_{ij}^2 \frac{u_{i+1j} - u_{i-1}}{2\Delta x}$

Local truncation error $T = F(u_{ij})$, Taylor expand

$$u_{ij+1} = u_{ij} + \Delta\tau (u_\tau)_{ij} + \frac{(\Delta\tau)^2}{2}(u_{\tau\tau})_{ij} + \cdots$$

$$u_{i\pm 1j} = u_{ij} \pm \Delta x (u_x)_{ij} + \frac{(\Delta x)^2}{2}(u_{xx})_{ij} \pm \frac{(\Delta x)^3}{6}(u_{xxx})_{i,j} + \frac{(\Delta x)^4}{24}(u_{xxxx})_{ij} + \cdots$$

Substitute to get

$$T = \underbrace{\{u_\tau - 4i\Delta x u_{xx} - u^2 u_x\}_{ij}}_{=0 \text{ from pde}} + \frac{\Delta\tau}{2}(u_{\tau\tau})_{ij} - \frac{(\Delta x)^3 i}{3}(u_{xxxx})_{ij}$$
$$- \frac{1}{6}(u^2 u_{xxx})_{i,j}(\Delta x)^2 + \cdots$$

$$T = \left[\frac{\Delta\tau}{2}(u_{\tau\tau})_{ij} - \frac{1}{6}(u^2 u_{xxx})_{ij}\Delta x^2 - \frac{x}{3}(\Delta x)^2 (u_{xxxx})_{i,j} \right]$$

$T \to 0$ as $\Delta\tau, \Delta x \to 0$ so scheme consistent

$$\Delta\tau = 0.02, \quad \Delta x = 0.5 \quad \frac{\Delta\tau}{\Delta x} = 4/100$$

$$u_{ij+1} = u_{ij} + \frac{16i}{100}(u_{i+1j} - 2u_{ij} + u_{i-1j}) + \frac{2}{100}u_{ij}^2(u_{i+1j} - u_{i-1j})$$

	$x=0$	0.5	1.0	1.5	2.0
$\tau=0$	0	1	2	3	4
$\tau=0.02$	0	1.04	2.16	3.36	4
$\tau=0.04$	0	1.10	2.40	3.51	4
$\tau=0.06$	0	1.19	2.62	3.61	4

to 2 dp