DIFFERENTIAL EQUATIONS FOR FINANCE
SHEET 1: BLACK-SCHOLES EQUATION

Assume σ and r to be constants unless stated otherwise.

1. Show by substitution that two exact solutions of the Black-Scholes equation are

$$\text{(i) } V(S,t) = aS \qquad \text{(ii) } V(S,t) = ae^{rt}$$

where a is an arbitrary constant. What do these solutions represent and what is the expression for Δ in each case?

2. Find all solutions of the Black-Scholes equation that can be written in the form $V(S,t) = aS^n$ where a and n are constants.

3. Find m such that $V(S,t) = aS^n e^{mt}$ is a solution of the Black-Scholes equation, where a and n are arbitrary constants.

4. State the boundary conditions at $S = 0$ and as $S \to \infty$, and the final condition at $t = T$ for a European call. Show that the solution

$$C(S,t) = SN(d_+) - Ee^{-r(T-t)}N(d_-),$$

where

$$d_\pm = \frac{\ln(S/E) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\phi^2}\, d\phi,$$

satisfies these conditions.

5. State the boundary conditions at $S = 0$ and as $S \to \infty$, and the final condition at $t = T$ for a European put. Show that the solution

$$P(S,t) = Ee^{-r(T-t)}N(-d_-) - SN(-d_+),$$

where $d_\pm$ and N are as defined in question 4, satisfies these conditions.

6. Show that the solutions for C and P given in questions 4 and 5 satisfy put-call parity.

7. Using the solutions given in questions 4 and 5, find formulae for Δ for (i) a European call (ii) a European put.

8. Starting from the Black-Scholes equation, show that for a European call option

$$C(S,t) \sim S - Ee^{-r(T-t)}$$

as $S \to \infty$.

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9. Show that if $r = \alpha + \beta t$ where α and β are positive constants,

(i) the amount of money needed at time t to produce an amount E at time $T > t$ is

$$M = Ee^{\{\alpha+\frac{1}{2}\beta(t+T)\}(t-T)}.$$

(ii) Show that

$$V = aS - bS^2 e^{-\alpha t - \frac{1}{2}\beta t^2 - \sigma^2 t}$$

where a and b are positive constants, is a solution of the Black-Scholes equation where σ is the constant volatility. Find Δ, the delta of the option value V, as a function of S and t and show that at a given time t the maximum value of the option is $V = (a^2/4b)e^{\alpha t+\frac{1}{2}\beta t^2+\sigma^2 t}$ for an asset price $S = (a/2b)e^{\alpha t+\frac{1}{2}\beta t^2+\sigma^2 t}$.

10. Find the functions $F(t)$ and $G(t)$ such that $V = S^2F(t) - S^3G(t)$ is a solution of the Black-Scholes equation and $F(0) = G(0) = 1$. Find Δ, the delta of the option value V, as a function of S and t. For a fixed time t, find the share price S that maximizes the value of the option.

11. The Black-Scholes equation for an option value V is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where S is the share price, t is time, r is the bank interest rate and σ is the volatility.

(i) If r and σ are constant, find the functions $X(t), Y(t)$ and $Z(t)$ such that

$$V = X(t) + S^2Y(t) + S^4Z(t)$$

is a solution of the Black-Scholes equation and $X(0) = Y(0) = 1, Z(0) = -1$. Define Δ, the delta of the option value V, and find it as a function of S and t. For a fixed time t, find the share price S that maximizes the value of the option.

(ii) If r is constant and $\sigma = \sqrt{2\alpha t}$ where α is a positive constant, find the constants a and b such that for $t \geq 0$

$$V = cS^2 e^{at+bt^2}$$

is a solution of the Black-Scholes equation, where c is a positive constant. Hence show that for a fixed share price S the value of the option has fallen to one half of its value at $t = 0$ when

$$t = \frac{1}{2\alpha}\left\{-r + (r^2 + 4\alpha \ln 2)^{1/2}\right\}.$$