DIFFERENTIAL EQUATIONS FOR FINANCE
SHEET 2: PARTIAL DIFFERENTIAL EQUATIONS

1. Classify the type and, where appropriate, determine the characteristic curves of the following equations:

(i) $\frac{\partial^2 U}{\partial x^2} + 4\frac{\partial^2 U}{\partial y^2} - \frac{\partial U}{\partial x} = 0,$

(ii) $2\frac{\partial^2 U}{\partial x^2} - 4\frac{\partial^2 U}{\partial x \partial y} - 6\frac{\partial^2 U}{\partial y^2} + \frac{\partial U}{\partial x} = 0,$

(iii) $4\frac{\partial^2 U}{\partial x^2} + 12\frac{\partial^2 U}{\partial x \partial y} + 9\frac{\partial^2 U}{\partial y^2} - 2\frac{\partial U}{\partial x} + U = 0.$

2. Solve the equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}, \qquad 0 \leq x \leq 1, \qquad \tau > 0,$$

subject to $U = 0$ at $x = 0$ and $x = 1$, and (i) $U = \sin 3\pi x$ at $\tau = 0$, (ii) $U = 1 - 2x$ at $\tau = 0$.

3. By assuming a similarity solution of the form $U = F(\xi)$ where $\xi = x/\sqrt{\tau}$, solve the equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}$$

subject to $U = H(x)$ at $\tau = 0$, giving your answer in terms of the error function. Check your answer using the fundamental solution of the heat equation.

4. Find similarity solutions to

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} + F(x), \qquad x > 0, \quad \tau > 0,$$

with $U(x,0) = 0, \quad x > 0$ and $U(0,\tau) = 0, \quad \tau > 0$ in the two cases (i) $F(x) = x$ (ii) $F(x) = 1$.

5. Given that $U = \tau^n F(\xi)$, where $\xi = x/\sqrt{\tau}$, is a solution of the heat equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}, \qquad x > 0, \quad \tau > 0,$$

find the equation satisfied by F. By differentiating this equation, find a solution for F'' in the case $n = \frac{1}{2}$ and hence obtain a solution for U satisfying the conditions $\partial U/\partial x = 0$ at $x = 0$ and $\partial U/\partial x \to 1$ as $x \to \infty$.

6. Use separation of variables to find all solutions of the Black-Scholes equation which can be written in the form

$$V(S,t) = T(t)R(S),$$

assuming σ and r to be constant. Using the substitution $S = e^x$, or otherwise, find the general solution for R as a function of S in the case when $T = e^{rt}$.

7. Use the fundamental solution of the heat equation to solve

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}, \qquad -\infty < x < \infty, \quad \tau > 0,$$

subject to $U = e^{-x^2}$ at $\tau = 0$. By differentiation of your solution for U, and by considering the limit $\tau \to 0$ (respectively) check that your solution satisfies both the equation and the initial condition. What happens to U as $\tau \to \infty$?

8. By assuming a similarity solution of the form $U(x,\tau) = F(\xi)$ where $\xi = x/\sqrt{\tau}$, solve the equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}$$

in the domain $-\infty < x < \infty, \tau > 0$ subject to the initial condition $U(x,0) = 8$ $(x > 0)$, $U(x,0) = -1$ $(x < 0)$, giving your answer in terms of the error function.

Confirm your answer by obtaining the same solution from the formula

$$U(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} U(x',0) e^{-\frac{(x-x')^2}{4\tau}} dx'.$$

Find $\frac{\partial U}{\partial x}$ as a function of x and τ and show that its value at $x = 0$ is $9/2\sqrt{\pi\tau}$.

9. Starting from the Black-Scholes equation in the form

$$\frac{\partial V}{\partial \tau} = S^2 \frac{\partial^2 V}{\partial S^2} + kS \frac{\partial V}{\partial S} - kV,$$

where k is a constant parameter, use the transformation

$$V = E e^{\alpha x + \beta \tau} U(x,\tau),$$

where $S = Ee^x$ and E is the constant exercise price, to determine the parameters α and β such that U satisfies the heat equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}.$$

A certain type of option defined on the domain $0 \leqslant x \leqslant 1$, $\tau \geqslant 0$ satisfies the payoff condition $U = 1$ at $\tau = 0$ and boundary conditions $U = 0$ at $x = 0$ and $x = 1$. Use separation of variables for U to show that the option value V is given by

$$V = \frac{4E}{\pi} e^{\frac{1}{2}(1-k)x - \frac{1}{4}(1+k)^2\tau} \sum_{m=0}^{\infty} \frac{e^{-(2m+1)^2\pi^2\tau} \sin(2m+1)\pi x}{2m+1}.$$