DIFFERENTIAL EQUATIONS FOR FINANCE
SHEET 3: NUMERICAL METHODS

1. Use an explicit method to solve
$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}$$
with $U = \sin \pi x$ at $\tau = 0$ and $U = 0$ at $x = 0$ and $x = 1$ for $\tau \geq 0$. Find the solution at intervals of 0.2 in x and 0.02 in τ up to $\tau = 0.06$. Repeat your calculations for the first time step using the Crank-Nicolson method. Solve the problem by the method of separation of variables and compare the solution obtained by each method at $x = 0.4$ and $\tau = 0.02$. What happens to U as $\tau \to \infty$?

2. Derive an explicit finite difference scheme for solution of the equation
$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} + U\frac{\partial U}{\partial x}$$
in the region $-1 \leq x \leq 1$ based on a forward difference in τ and central differences in x. Find the principal part of the local truncation error at $x = i\Delta x$, $\tau = j\Delta\tau$ where Δx and $\Delta\tau$ are the step lengths in x and τ respectively and hence confirm that your scheme is consistent.
Use two time steps of the scheme to find a numerical solution for U at intervals of $\frac{1}{2}$ in x at time $\tau = \frac{1}{4}$, given that $U = -1$ at $x = -1$ and $U = 1$ at $x = 1$ for $\tau \geq 0$ and that $U = x$ at $\tau = 0$ for $-1 \leq x \leq 1$.

3. Derive an explicit scheme for solution of the equation
$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} + 8xU^2$$
in the region $0 \leq x \leq 1$, $\tau \geq 0$ subject to $\partial U/\partial x = 0$ at $x = 0$ and $x = 1$, and $U = 0.1$ at $\tau = 0$. Find the principal part of the local truncation error at $x = i\Delta x$, $\tau = j\Delta\tau$ where Δx and $\Delta\tau$ are the step lengths in x and τ respectively and hence confirm that your scheme is consistent.
Use two time steps of the scheme to find a numerical solution for U at intervals of $\frac{1}{2}$ in x at time $\tau = \frac{1}{4}$.

4. The equation
$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial x^2} = \sin \pi x$$
is to be solved in the region $0 \leq x \leq 1$, $\tau \geq 0$ subject to $U = 0$ at $x = 0$ and $x = 1$ and $U = 0$ at $\tau = 0$. Use the Crank-Nicolson scheme with step lengths of $\frac{1}{3}$ in x and $\frac{1}{9}$ in τ to find numerical approximations for U at $x = \frac{1}{3}$ and $x = \frac{2}{3}$ when $\tau = \frac{1}{9}$. By writing $U = F(\tau)\sin \pi x$, or otherwise, find the exact solution for U. What is the error in your two numerical approximations?

5. Show that the scheme

$$\frac{\theta u_{i,j-1} + (1-\theta)u_{i,j+1} - u_{i,j}}{\Delta\tau} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2}$$

is a consistent discretization of the equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}$$

at $(i\Delta x, j\Delta\tau)$ only if $\theta = 0$. What equation does the scheme consistently represent for general θ and for what value of $\alpha = \Delta\tau/(\Delta x)^2$ is the truncation error significantly reduced for general θ?

6. By considering the error $e_{i,j} = U_{i,j} - u_{i,j}$, prove that under certain assumptions, which should be stated, the solution $u_{i,j}$ of the scheme

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta\tau} = \frac{G(i\Delta x)}{(\Delta x)^2}\{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\}$$

converges as Δx and $\Delta\tau$ tend to zero, to the solution U of the equation

$$\frac{\partial U}{\partial \tau} = G(x)\frac{\partial^2 U}{\partial x^2}$$

at any point $x = i\Delta x$, $\tau = j\Delta\tau$ in the region $0 \le x \le 1$, given that $G(x) \ge 0$ for $0 \le x \le 1$. Determine the restriction on the value of $\alpha = \Delta\tau/(\Delta x)^2$ for the case where $G(x) = x^2(1-x)$. If in this case $U = x(1-x)$ at $\tau = 0$, use one time step of the scheme with $\Delta x = 0.1$ to find u at $x = 0.5$, $\tau = 0.01$.

7. State an explicit scheme based on a forward difference in τ and central differences in x to solve

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} + 10\frac{\partial U}{\partial x} + 25U^2$$

in the region $0 \le x \le 1, \tau \ge 0$ subject to the boundary conditions $U = 0$ at $x = 0$ and $U = 1$ at $x = 1$, and the initial condition $U = x$ at $\tau = 0$. Find the principal part of the local truncation error at $x = i\Delta x, \tau = j\Delta\tau$ where Δx and $\Delta\tau$ are the step lengths in x and τ respectively, and hence confirm that your scheme is consistent.

Use two time steps of the scheme to find a numerical approximation to U at intervals of 0.2 in x at time $\tau = 0.008$, giving your answers to four significant figures.

8. State a Crank-Nicolson scheme for solving the equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} - 10U$$

in the domain $0 \le x \le 1, \tau \ge 0$ subject to the boundary conditions $U = 0$ at $x = 0$ and $x = 1$, and the initial condition $U = \sin \pi x$ at $\tau = 0$. Use one time step of the scheme to find a numerical approximation to U at intervals of 0.2 in x at time $\tau = 0.04$, giving your answers to four significant figures.

Compare your numerical values with those given by the exact solution $U = F(\tau)\sin \pi x$, where F is a function of τ to be determined.