MA3607

CITY UNIVERSITY

London

BSc Honours Degree in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
Mathematics and Finance

Part 3

Differential Equations for Finance

2013

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the Four questions.
All necessary working must be shown.

1. (i) If the bank interest rate is $r(t)$, write down a differential equation for the amount of money $M(t)$ deposited at the bank. **[2]**

Assuming that the interest rate $r(t)$ is

$$r(t) = \begin{cases} \frac{4}{100} + \frac{1}{50}t^2 & \text{for} \quad -4 < t \le 0\,, \\ \frac{1}{1+\cosh t} & \text{for} \quad -5 \le t \le -4\,, \end{cases}$$

solve the differential equation for $M(t)$ to determine how much money you should have deposited at the bank 5 years ago to have 1000 GBP in the bank today. Give your answer to the nearest penny. **[8]**

(ii) An option value satisfies the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,,$$

where the volatility $\sigma^2 = \mu + \nu t + \lambda t^2$ and the bank rate $r = \alpha t + \beta t^3$; α, β, μ, ν, λ are constants. Find the values of the constants a, b, c and d such that

$$V = AS - BS^2 e^{-at-bt^2-ct^3-dt^4}$$

is a solution of the Black-Scholes equation where A, B are positive constants. **[8]**

For a fixed time t, find the share price S that maximizes the value of the option. **[2]**

2. For constant bank rate and volatility, the Black-Scholes equation for the value of an option can be re-written in a dimensionless form as

$$\frac{\partial V}{\partial \tau} - S^2 \frac{\partial^2 V}{\partial S^2} - kS\frac{\partial V}{\partial S} + kV = 0\,,$$

where k is a constant parameter.

(i) Use the transformation

$$V = Ae^{\alpha x + \beta \tau} U(x, \tau)\,, \tag{1}$$

where $S = Ee^x$ and E is the constant exercise price, to determine the parameters α and β such that U satisfies the heat equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}\,.$$

[5]

(ii) Consider an option with value V which is defined on the domain $0 \leq x \leq 1$, $\tau \geq 0$ with $S = Ee^x$ as before. Supposing that the option price, expressed in terms of $U(x\,,\,\tau)$ defined in (1) above, is such that U satisfies the initial condition

$$U(x, \tau = 0) = \begin{cases} 2 & 0 \leq x \leq \frac{1}{5} \\ -3 & \frac{1}{5} \leq x \leq 1 \end{cases}$$

and the boundary conditions $U = 0$ at $x = 0$ and $x = 1$. Use separation of variables for U as a function of x and τ to solve the heat equation for U. **[12]**

(iii) Using the transformation (1), write the value of the option V for the above solution as a function of S and τ. **[3]**

3. The error function is defined as

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx\,.$$

(i) Show that $\operatorname{erf}(z) \to 1$ as $z \to \infty$ **[3]**

(ii) The function $U(x\,,\,\tau)$ satisfies the equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}\,.$$

in the domain $-\infty < x < \infty$, $\tau \geq 0$ and the initial condition

$$U(x, \tau = 0) = \begin{cases} 5 & \text{for} \quad x > 4\,, \\ -2 & \text{for} \quad 4 < x\,, \end{cases}$$

By assuming a similarity solution of the form $U = F(\xi)$ where $\xi = \frac{x-4}{\sqrt{\tau}}$ find the solution for U in terms of the error function **[10]**

(iii) Recall Gauss' solution of the heat equation

$$U(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} U(y, 0) e^{-\frac{(x-y)^2}{4\tau}} dy\,.$$

Confirm your answer in (ii) above, by finding $U(x\,,\,\tau)$ using Gauss' solution above. **[7]**

4. (i) State an explicit scheme based on a forward difference in τ and a central difference in x to solve

$$\frac{\partial U}{\partial \tau} = 4x\frac{\partial^2 U}{\partial x^2} + U^2\frac{\partial U}{\partial x}\,.$$

in the region $0 \leq x \leq 2$, $\tau \geq 0$ subject to $U(x=0,\tau)=0$, $U(x=2,\tau)=4$ and the initial condition $U(x,\tau=0)=2x$. **[3]**

(ii) Find the principal part of the local truncation error at $x = i\Delta x$, $\tau = j\Delta\tau$ where Δx and $\Delta\tau$ are the step lengths in x and τ, respectively. Use this to show that your scheme is consistent. **[8]**

(iii) Use three time steps of the scheme to find a numerical solution for U at intervals of 0.5 in x at time $\tau = 0.06$. **[9]**

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