DIFFERENTIAL EQUATIONS FOR FINANCE

Course Outline

A. Basic Options Theory

1. Options and markets
2. Asset price random walks
3. Black-Scholes model

B. Partial Differential Equations

1. Classification of 2nd order pdes
2. Heat diffusion equation
3. Black-Scholes equation

C. Numerical Methods

1. Finite difference approximation
2. Methods for parabolic systems
3. Accuracy and stability
4. Black-Scholes equation

Reading List

The Mathematics of Financial Derivatives

P. Wilmott, S. Howison, J. Dewynne (CUP)

Numerical Methods for Partial Differential Equations

W. F. Ames (Academic)

Numerical Solution of Partial Differential Equations: Finite Difference Methods

G. D. Smith (Clarendon)

Partial Differential Equations

P. Duchateau, D. W. Zachmann (Schaum)

A. Basic Options Theory

1. Options and markets

A company can sell SHARES in itself to investors to raise money. The company is then 'owned' by its shareholders and, if it makes a profit, may pay part of this to its shareholders as a DIVIDEND.

A share, or collection of shares, may also be referred to as an ASSET.

Trading in shares occurs on the STOCK MARKET (e.g. in London, New York, Tokyo).

As markets have become more sophisticated, more complex contracts than simple buy/sell trades have been introduced - these are known as FINANCIAL DERIVATIVES.

Call Options

One of the simplest forms of such a derivative is a EUROPEAN CALL OPTION, a contract with the conditions:

> At a prescribed future time (the EXPIRY DATE) the HOLDER of the option may purchase a prescribed asset (the UNDERLYING ASSET) for a prescribed amount known as the EXERCISE PRICE (or strike price).

The other party to the contract is the WRITER who must sell the asset if the holder chooses to buy it.

The writer sells the option to the holder for a price (the OPTION PRICE). How much should this be, i.e. what is the value of the option? Predicting this is the main aim of options theory.

Example Call option sold on 1/1/2007 : the holder may purchase 1 share of CITYTEC for 100 p on 1/1/2008.

If the share price is 120 p on 1/1/2008 the holder can purchase the asset (exercise the option) for 100 p, making a profit of 20 p (since he can then sell it on the market for 120 p)

$$120\text{ p} - 100\text{ p} = 20\text{ p profit}$$

If the share price is 80 p on 1/1/2008 then it would not be sensible to exercise the option.

If the CITYTEC share takes the values 80 p and 120 p with equal probability, the expected profit is

$$\tfrac{1}{2} \times 0\text{ p} + \tfrac{1}{2} \times 20\text{ p} = 10\text{ p}$$

so the rough value of the option price might be 10 p.

If the share did take the value 120p on 1/1/2008 then:

Profit on exercise	=	20 p
Cost of option	=	10 p
Net profit	=	10 p

This would be a good outcome for the holder since he would make a gain of 100% relative to his initial outlay of 10 p for the option.

In general, the value of the option will depend on:

- Current share price
- Exercise price
- Time to expiry
- Randomness of asset price (VOLATILITY)
- Bank interest rate (the money paid for the option could otherwise be invested in a bank)

Put Options

At a prescribed future time the holder of the option may <u>sell</u> a prescribed asset for a prescribed amount.

Other types of option

American option: this may be exercised at <u>any</u> time prior to expiry. Here it is important to try and predict not only the value of the option but <u>when</u> it is best to exercise it.

Exotic (path-dependent) options: these are options whose value depends on the <u>history</u> of the asset price as well as its current value:

e.g. Barrier options, where the option can be initiated or terminated if the asset price reaches some prescribed value.

Asian options, where the option value depends on a form of average over time.

Lookback options, where the option value depends on a maximum or minimum of the asset price.

Bank interest rate $r(t)$

We assume this is known. If $M(t)$ is the amount of money deposited at time t then assuming continuously compounded interest,

$$\frac{dM}{M} = r(t)\,dt$$

Integrate:

$$\ln M = \int_{t'=T}^{t'=t} r(t')\,dt' + C \qquad (1)$$

where C, T are constants.

Suppose $M = E$ at some future time $t = T$.

Then (1) $\Rightarrow$ $\ln E = C$ so

$$\ln M = \int_{t'=T}^{t'=t} r(t')\,dt' + \ln E$$

$$\therefore \quad M = E e^{\int_T^t r(t')\,dt'}$$

$$\therefore \quad M = E e^{-\int_t^T r(t')\,dt'} \qquad (2)$$

This gives the amount $M(t)$ needed at time t to produce a given amount E at time T. This must be taken into account in valuing options and is known as DISCOUNTING.

Example $\quad r = 0.05$ per annum (const) $\quad$ (5% p.a.)

Then $\quad M = E e^{-0.05[t']_t^T} = E e^{-0.05(T-t)}$

If $E = 100\text{p}$, $t = 0$ (years) and $T = 1$ (years) then

$$M = 100 e^{-0.05} = 95.123 \text{ p}$$

i.e. 95.123 p is needed to produce 100 p one year later.

Example $\quad r = 0.05t$ per annum $\quad (t \geq 0)$.

Then $\quad M = E e^{-0.05\left[\frac{1}{2}t'^2\right]_t^T} = E e^{-0.025(T^2 - t^2)}$

If $E = 100\text{p}$, $t = 0$ (years), $T = 1$ (years) then

$$M = 100 e^{-0.025} = 97.531 \text{ p}$$

2. Asset Price Random Walks

We assume

(i) the past history of an asset price is fully reflected in the present price, which does not hold any further information,

(ii) markets respond immediately to any new information about an asset.

This is known as an EFFICIENT MARKET HYPOTHESIS.

We do however know about the past history of asset prices, including such information about typical jumps in value, and their mean and variance.

Let $S(t)$ be the asset price (or share price) at time t and $S + dS$ the new price at time $t + dt$.

The RETURN on the asset is defined as the change in its price relative to its original value:

$$\frac{dS}{S}$$

We model this as composed of two parts;

(a) a predictable part equivalent to money invested in a bank, giving a contribution

$$\mu\, dt$$

where μ is a measure of the <u>average</u> rate of growth of the asset price (also known as the DRIFT) and often assumed to be constant, and

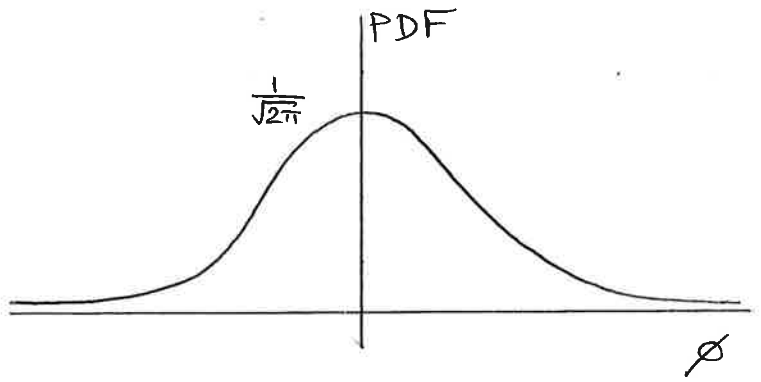
(b) a random change in response to external effects such as unexpected news. It is represented by a random sample drawn from a normal distribution with mean zero (so as not to affect μ) giving a contribution

$$\sigma\, dX$$

where σ is the VOLATILITY and measures the standard deviation of the return, and

$$dX = \phi\sqrt{dt}\,.$$

Here $-\infty < \phi < \infty$ is a random variable drawn from a standard normal distribution with mean zero and variance 1, i.e. with $\text{PDF} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}$.



Standard Normal Distribution

N.B. Area under PDF = Total Probability = 1.

<u>Proof</u>: Let $I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx$

Then $\quad I^2 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx\,dy \qquad \text{Use polars}$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}r^2} r\,dr\,d\theta$$

$$= 2\pi\left[-e^{-\frac{1}{2}r^2}\right]_0^{\infty} = 2\pi$$

Hence $\quad I = \sqrt{2\pi}$ as required.

The term $\sqrt{dt}$ ensures that the variance of dX is proportional to the time increment dt (see below).

Properties of ϕ:

Let $\mathcal{E}(F(\phi))$ denote the expectation of $F(\phi)$, i.e.

$$\mathcal{E}(F(\phi)) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\phi) e^{-\frac{1}{2}\phi^2} d\phi$$

$\therefore$ Mean of ϕ $= \mathcal{E}(\phi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi e^{-\frac{1}{2}\phi^2} d\phi = 0$

(by symmetry)

Variance of ϕ $= \mathcal{E}(\phi^2) - (\mathcal{E}(\phi))^2$

$$= \mathcal{E}(\phi^2)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi^2 e^{-\frac{1}{2}\phi^2} d\phi$$

$$= \frac{1}{\sqrt{2\pi}}\left[(\phi)(-e^{-\frac{1}{2}\phi^2})\right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1)(-e^{-\frac{1}{2}\phi^2})\,d\phi$$

$$= 0 + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\phi^2} d\phi = 1$$

We now have

$$\frac{dS}{S} = \sigma\, dX + \mu dt \qquad (3)$$

which is a STOCHASTIC DIFFERENTIAL EQUATION describing a random walk (the randomness introduced through dX).

Properties of (3):

1. The new asset price $S + dS$ depends solely on today's price S and is independent of the past price (MARKOV PROPERTY).

2. $$\mathcal{E}\left(\frac{dS}{S}\right) = \mathcal{E}(\sigma dX + \mu dt)$$
$$= \mu dt$$
since this part is not random and
$$\mathcal{E}(dX) = \mathcal{E}(\phi\sqrt{dt}) = \sqrt{dt}\,\mathcal{E}(\phi) = 0.$$

3. $$\mathrm{Var}\left(\frac{dS}{S}\right) = \mathcal{E}\left(\left(\frac{dS}{S}\right)^2\right) - \left[\mathcal{E}\left(\frac{dS}{S}\right)\right]^2$$
$$= \mathcal{E}(\sigma^2 dX^2 + \mu^2 dt^2 + 2\mu\sigma\, dX dt) - (\mu dt)^2$$
$$= \mathcal{E}(\sigma^2 dX^2) + 2\mu\sigma\, dt\, \mathcal{E}(dX)$$
$$= \sigma^2\, \mathcal{E}(dX^2) \quad + \quad 0$$
$$= \sigma^2\, \mathcal{E}(\phi^2 dt)$$
$$= \sigma^2 dt\, \mathcal{E}(\phi^2) \quad = \quad \sigma^2 dt$$

i.e. Variance of $\frac{dS}{S}$ is proportional to dt and σ^2.

Standard deviation of $\frac{dS}{S} = \left(\mathrm{Var}\left(\frac{dS}{S}\right)\right)^{1/2} = \sigma\sqrt{dt}$

Substituting in (4) and retaining terms of the size of dt or larger, and using (3),

$$df = \frac{df}{dS}(\sigma S\,dX + \mu S\,dt) + \frac{1}{2}\sigma^2 S^2 \frac{d^2f}{dS^2}dt.$$

$$\therefore \quad df = \sigma S \frac{df}{dS}dX + \left(\mu S \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2f}{dS^2}\right)dt \qquad (5)$$

This is the first version of Ito's Lemma, relating the small change df in the function of a random variable to the small change dS in the variable itself, when dS, dX and dt are related by (3).

<u>Generalization for $f = f(S,t)$</u>

We now need to use the 2D Taylor expansion

$$df = \frac{\partial f}{\partial S}dS + \underbrace{\frac{\partial f}{\partial t}dt}_{\text{extra term}} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 + \ldots$$

and proceed as before to obtain

$$df = \sigma S \frac{\partial f}{\partial S}dX + \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)dt \qquad (6)$$

This is the second, more general version of Ito's Lemma.

Ito's Lemma

This enables us to interpret (3) in the limit as $dt \to 0$, in particular the random part involving dX.

Let $f(S)$ be a smooth function of S.
If S varies by a small amount dS it follows that f varies by a small amount df where Taylor's theorem $\Rightarrow$

$$df = \frac{df}{dS} dS + \frac{1}{2} \frac{d^2 f}{dS^2} (dS)^2 + \ldots \qquad (4)$$

Now dS is given by (3) so

$$(dS)^2 = (\sigma S \, dX + \mu S dt)^2$$
$$= \sigma^2 S^2 dX^2 + 2\sigma\mu S^2 dt \, dX + \mu^2 S^2 dt^2$$

But $dX = \phi \sqrt{dt}$ with ϕ finite so these 3 terms are proportional to dt, $dt^{3/2}$ and dt^2 respectively. Thus to a first approximation as $dt \to 0$,

$$(dS)^2 = \sigma^2 S^2 dX^2 + \ldots$$

But $dX^2 = \phi^2 dt$ and $\mathcal{E}(\phi^2) = 1$ so as $dt \to 0$, $dX^2 \to dt$. Thus

$$(dS)^2 \to \sigma^2 S^2 dt \qquad \text{as} \qquad dt \to 0.$$

3. Black-Scholes Model

Arbitrage

This is the idea that an instantaneous risk-free profit might be possible by dealing in different markets. In option pricing we assume this cannot occur – if such profits were possible, prices would move instantaneously to eliminate them.

Option values

Let $V = V(S,t)$ be the value of an option.

Where necessary we shall use

$C = C(S,t)$ to denote the value of a call option,

$P = P(S,t)$ to denote the value of a put option.

We assume V, (P, or C) to be a function of the current value of the asset S and time t.

The value of the option may also depend on the following parameters:

σ the volatility of the asset

E the exercise price

T the expiry time

r the bank interest rate

Payoffs

(i) Call option

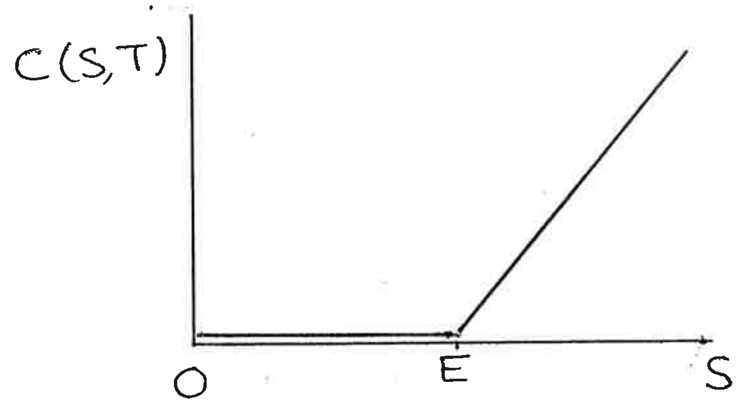
Consider what happens at expiry when $t = T$.

If $S > E$ the call will be exercised, handing over E to obtain an asset worth S. The profit is $S - E$.

If $S < E$ at expiry, the call option will not be exercised (because it would make a loss $E - S$) and the option expires worthless.

Thus the value of the call option at expiry (called the PAYOFF) is

$$C(S,T) = \max(S-E, 0)$$



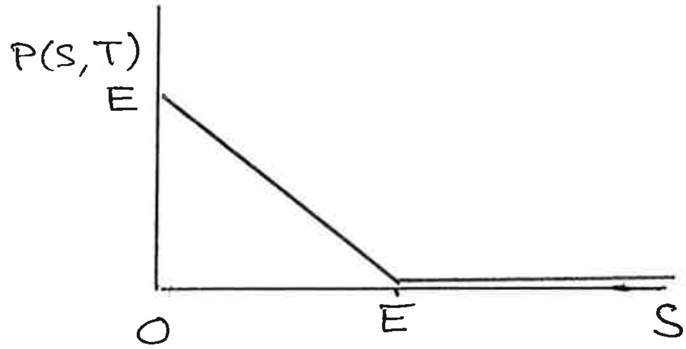
Payoff diagram

Note that our calculation of the value C (V, or P) excludes the original cost of the option – we choose to define C (V, or P) in this way for convenience.

(ii) Put option

Here the option at expiry is worthless if $S > E$ but has the value $E - S$ if $S < E$. Thus the payoff (the value of the put option at expiry) is

$$P(S,T) = \max(E-S, 0)$$



Payoff diagram

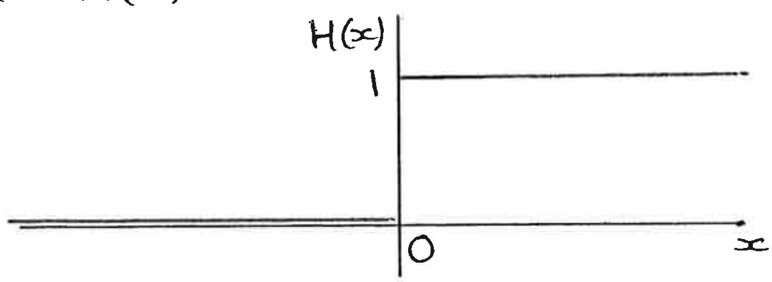
(iii) Other types of option

Example: Cash-or-Nothing Call

Here the payoff is

$$V(S,T) = B\,H(S-E)$$

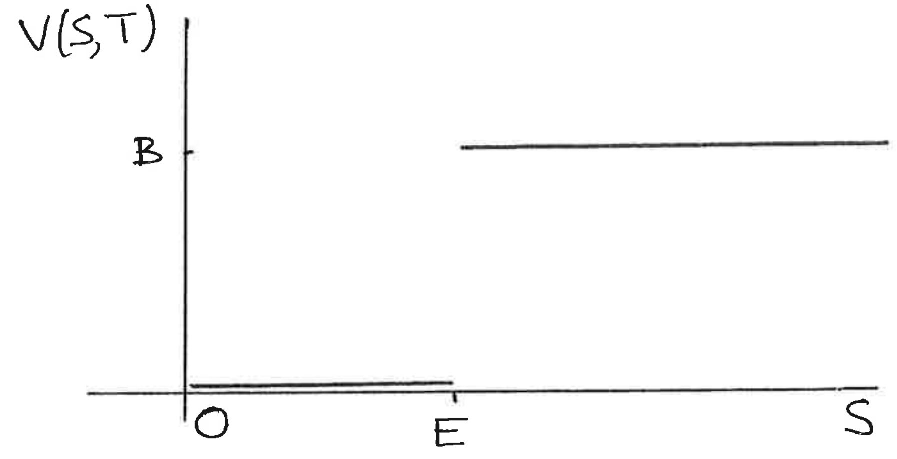
where $H(x)$ is the Heaviside function, defined as



$$H(x) = 1, \quad x > 0$$

$$H(x) = 0, \quad x < 0$$

This is a straight bet on the asset price – if $S > E$ at expiry then the value of the option is $V = B$ whereas if $S < E$ the payoff is nothing. The holder is gambling an amount B that the share price will beat the exercise price.



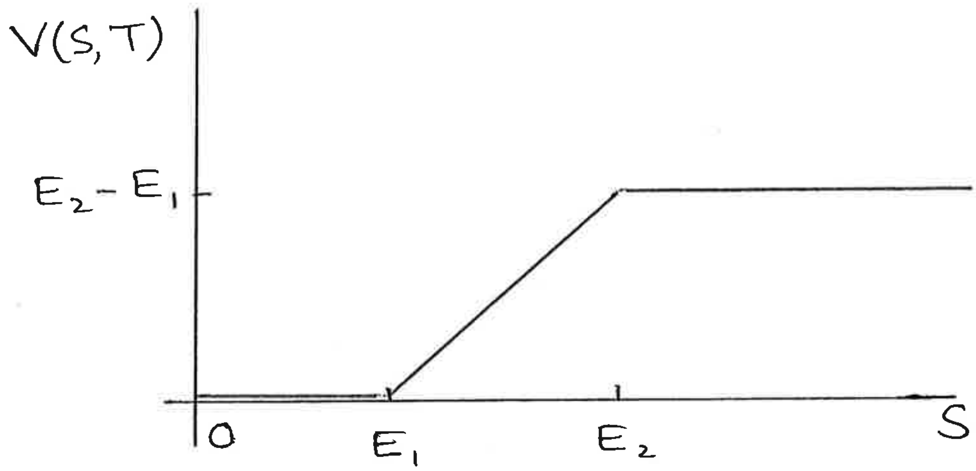
Payoff diagram

Example Bullish Vertical Spread

Here the payoff may have the form

$$V(S,T) = \max(S-E_1, 0) - \max(S-E_2, 0)$$

where $E_2 > E_1$, and results from buying one call option and writing one call option with the same expiry date but a larger exercise price.



Payoff diagram

Put-Call Parity

Suppose we have an asset S and hold a put option (to sell) and have written a call option (to sell), both with the same expiry date.

Let Π denote the overall value of this 'portfolio', at time t. Then

$$\Pi = S + P - C$$

where P and C are the value of the put and call options respectively. The payoff at expiry is

$$S + \max(E-S, 0) - \max(S-E, 0)$$

$$\begin{cases} = S + (E-S) - 0 = E & \text{if } S \leq E \\ = S + 0 - (S-E) = E & \text{if } S \geq E \end{cases}$$

Since the value at expiry is always E, the value at a general time t must be the same value as if the money were in a bank, i.e. allowing for discounting (and assuming no arbitrage):

$$E e^{-r(T-t)}$$

provided the bank rate r is constant. Thus

$$S + P - C = E e^{-r(T-t)} \qquad (7)$$

This relation between the asset price S and its option values is called PUT-CALL PARITY.

For non-constant r the rhs must be replaced by the rhs of (2).

If we know C, put-call parity gives a simple way of calculating P (or vice-versa).

Black-Scholes analysis

We now derive an equation for the option value V. This analysis, originally due to Black and Scholes, is based on the following assumptions.

1. The asset price follows the random walk (3).
2. The bank rate $r(t)$ and asset volatility $\sigma(t)$ are known functions of time t over the life of the option.
3. There are no transaction costs.
4. There are no dividends.
5. Arbitrage cannot occur.
6. Trading of an asset can take place continuously
7. Any number of the underlying asset can be bought or sold.

Let $V(S,t)$ be the value of an option (put or call).

From Ito's Lemma (6) (taking $f = V$) we have

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) dt \qquad (8)$$

Construct a portfolio consisting of one option and a number $-\Delta$ of the underlying asset, which can be chosen arbitrarily and is as yet unspecified

The value of the portfolio is

$$\Pi = V - \Delta S \qquad (9)$$

The jump in value in one time increment is

$$d\Pi = dV - \Delta dS$$

where we assume Δ is held fixed during the time step.

Substituting for dS from (3) and dV from (8) gives

$$d\Pi = \sigma S \left(\frac{\partial V}{\partial S} - \Delta\right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S\right) dt$$

We can now eliminate any randomness in $d\Pi$ by choosing Δ so that the coefficient of dX vanishes. i.e. Choose

$$\Delta = \frac{\partial V}{\partial S} \qquad (10)$$

Now

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

If an amount Π had been invested in a bank the return would see a growth $r\Pi\,dt$ in time dt. Since arbitrage cannot occur, it therefore follows that

$$r\Pi\,dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt \qquad (11)$$

Substituting (9) and (10) into (11) and dividing by dt we obtain

$$\underline{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0} \qquad (12)$$

This is the BLACK-SCHOLES PDE for the value V of an option as a function of the share price S and time t. It also involves σ and r but not the mean growth μ (i.e. the option value is independent of how rapidly or slowly the asset grows).

The quantity

$$\Delta = \frac{\partial V}{\partial S}$$

is known as DELTA and gives the rate of change of the option value with asset price at a fixed time (and is a measure of the correlation between the two).

PDEs such as (12) have many solutions. Unique solutions are generally only obtained when conditions are specified on V.

Since (12) is second-order in S and first-order in t it needs two conditions at specified values of S (usually $S=0$ and $S=\infty$) and one condition at a specified value of t (usually $t=T$). The solution is then determined in the domain

$$0 \leq S < \infty \quad , \quad t \leq T$$

i.e. for share prices from 0 to ∞ and for times less than the expiry time, T.

(a) <u>Final condition at $t=T$</u>

This is provided by the payoff. For a European call it is

$$V = C(S,T) = \max(S-E, 0) \qquad (13)$$

and for a European put it is

$$V = P(S,T) = \max(E-S, 0) \qquad (14)$$

(b) <u>Boundary conditions at $S=0$ and as $S \to \infty$</u>

(i) <u>Call option</u>

If ever $S=0$ then $dS=0$ from (3) so S never changes from zero. But if $S=0$ at expiry then the payoff is zero and the call option worthless. Thus

$$C(0,t) = 0 \qquad \text{for } t \leq T \qquad (15)$$

As S increases the option will be exercised and the magnitude of E becomes less significant.

The value of the option therefore becomes that of the asset:

$$C(S,t) \sim S \qquad \text{as} \qquad S \to \infty \quad \text{for } t \leq T \qquad (16)$$

i.e. C approaches S asymptotically as $S \to \infty$,

or $\dfrac{\partial C}{\partial S} \to 1$ as $S \to \infty$.

(ii) <u>Put option</u>

If ever S reaches zero it remains zero and the payoff is then E (from (14)). Thus $P(0,t)$ is the value that gives an amount E at time T. From (2), this is

$$P(0,t) = E e^{-\int_t^T r(t')\,dt'} \qquad \text{for } t \leq T \qquad (17)$$

or, if r is constant,

$$P(0,t) = E e^{-r(T-t)} \qquad \text{for } t \leq T \qquad (18)$$

As $S \to \infty$ the option will not be exercised, so

$$P(S,t) \to 0 \qquad \text{as} \qquad S \to \infty \qquad \text{for } t \leq T. \qquad (19)$$

We shall find solutions of (12) subject to either (13), (15), (16) or (14), (18), (19) once we have developed the necessary techniques in Section B.