B. PARTIAL DIFFERENTIAL EQUATIONS

1. Classification of Second-Order PDEs

Consider a general linear PDE in 2 variables for $U(x,y)$:

$$a\frac{\partial^2 U}{\partial x^2} + b\frac{\partial^2 U}{\partial x \partial y} + c\frac{\partial^2 U}{\partial y^2} + d\frac{\partial U}{\partial x} + e\frac{\partial U}{\partial y} + fU = g \qquad (20)$$

where $a = a(x,y)$ etc.

This is second order because the highest derivatives are second.

It is linear because there are no products of terms in U or its partial derivatives.

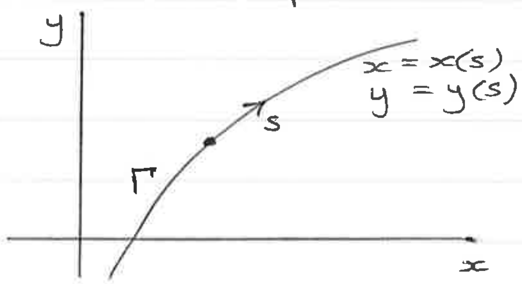
Characteristics:

These are special curves in the x, y plane which are identified with the general nature of the equation.

Let Γ be a curve

$$x = x(s), \qquad y = y(s)$$

where s is a parameter along the curve.



Suppose that U, $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$ are specified along Γ so that

$$U = F(s), \quad \frac{\partial U}{\partial x} = G(s), \quad \frac{\partial U}{\partial y} = H(s) \quad \text{say.}$$

Then we have

$$\frac{d}{ds}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial^2 U}{\partial x^2}x'(s) + \frac{\partial^2 U}{\partial x \partial y}y'(s) = G'(s) \qquad (21)$$

$$\frac{d}{ds}\left(\frac{\partial U}{\partial y}\right) = \frac{\partial^2 U}{\partial x \partial y}x'(s) + \frac{\partial^2 U}{\partial y^2}y'(s) = H'(s) \qquad (22)$$

Now $\frac{\partial^2 U}{\partial x^2}$, $\frac{\partial^2 U}{\partial x \partial y}$ and $\frac{\partial^2 U}{\partial y^2}$ can be found along Γ provided (20)–(22) can be solved consistently to find them.

These can be written

$$\begin{bmatrix} a & b & c \\ x' & y' & 0 \\ 0 & x' & y' \end{bmatrix} \begin{bmatrix} \frac{\partial^2 U}{\partial x^2} \\ \frac{\partial^2 U}{\partial x \partial y} \\ \frac{\partial^2 U}{\partial y^2} \end{bmatrix} = \begin{bmatrix} g - dG - eH - fF \\ G' \\ H' \end{bmatrix}.$$

We see that the second derivatives cannot be determined if the determinant of the lhs matrix vanishes i.e.

$$a y'^2 - b x' y' + c x'^2 = 0$$

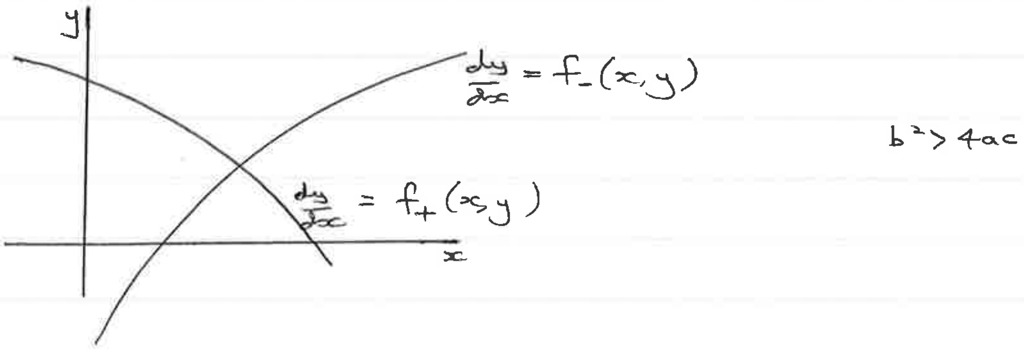
or

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

The roots of this quadratic for dy/dx are

$$\frac{dy}{dx} = \frac{b \pm (b^2 - 4ac)^{1/2}}{2a} \equiv f_\pm(x,y) \quad \text{say}$$

These are real and distinct if $b^2 > 4ac$ and real and equal if $b^2 = 4ac$. If $b^2 < 4ac$ no real roots exist



For $b^2 > 4ac$ the roots define two families of curves in the x, y plane called the CHARACTERISTICS of the pde (19). In this case the pde (19) is said to be of HYPERBOLIC TYPE and the characteristics provide a useful coordinate system which can be used to help solve the equation (see later).

For $b^2 = 4ac$ there is one family of characteristics

$$\frac{dy}{dx} = \frac{b}{2a}$$

and the equation is said to be of PARABOLIC TYPE.

For $b^2 < 4ac$ there are no characteristics and the equation is said to be of ELLIPTIC TYPE.

Since $\frac{\partial^2 U}{\partial x^2}, \frac{\partial^2 U}{\partial x \partial y}, \frac{\partial^2 U}{\partial y^2}$ are not determined along the characteristics, these p.d.s of U can be discontinuous <u>across</u> the characteristic; i.e. these are curves <u>along</u> which discontinuities in a solution can propagate. Since elliptic equations have no such curves, their solutions are generally smooth everywhere.

Note also that a pde can change type in different regions of the x, y plane if the coefficients a, b and c vary with x and y so that the sign of $b^2 - 4ac$ changes.

<u>Example</u>

The 2D wave equation $\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial \tau^2} = 0$ for $U(x, \tau)$ is hyperbolic, with characteristics

$$x \pm \tau = \text{Constant}$$

(Here y is replaced by τ).

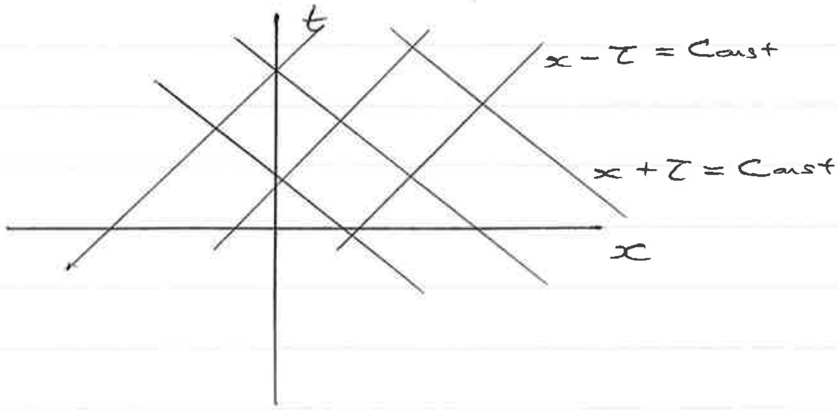
Here $a = 1$, $b = 0$, $c = -1$ so

$$b^2 - 4ac = 4 > 0 \Rightarrow \text{hyperbolic}$$

Characteristics are

$$\frac{dx}{d\tau} = \frac{b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a} = \pm \frac{\sqrt{4}}{2} = \pm 1$$

$$\frac{dx}{d\tau} = \pm 1 \Rightarrow x = \pm \tau + \text{Const}$$



Example

The 2D Laplace eqtn $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$ is elliptic.

Here $a = 1$, $b = 0$, $c = 1$ so

$$b^2 - 4ac = -4 < 0$$

No characteristics.

Example

The heat diffusion equation $\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}$ is parabolic with characteristics $\tau = \text{Constant}$.

Here $$\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial \tau} = 0$$

$$\Rightarrow a = 1, \quad b = c = 0$$

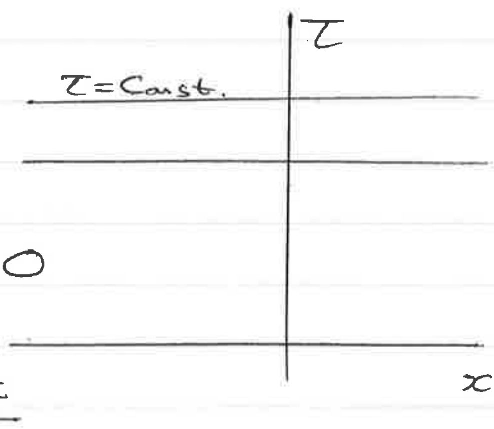
so

$$b^2 - 4ac = 0$$

and characteristics are

$$\frac{d\tau}{dx} = \frac{b}{2a} = 0$$

$$\Rightarrow \tau = \text{Constant}$$



2. Heat Diffusion Equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} \qquad (23)$$

This governs the diffusion (spreading) of heat in a continuous medium, where $U(x, \tau)$ is the temperature at position x and time τ, for example in a thin uniform rod of n-d length 1.

Since information propagates along the characteristic τ = Constant, if a change is made to U at a particular point, for example on the boundary of the solution region (e.g. at $x = 0$) its effect is felt instantaneously everywhere else; information propagates with infinite speed along the characteristic τ = Constant.

Physically, diffusion is a smoothing process; heat flows from hot to cold so evens out temperature differences as time goes on.

(i) Finite domain

We first consider (23) with initial condition

$$U(x, 0) = U_0(x) \qquad (24)$$

and boundary conditions

$$U = 0 \quad \text{at} \quad x = 0, 1 \qquad (25)$$

The solution can be found by separation of variables. First try

$$U = X(x)\,T(\tau)$$

to obtain

$$XT' = X''T$$

so

$$\frac{X''}{X} = \frac{T'}{T} = \text{Const}, -\lambda^2 \quad (\text{say}) \qquad (*)$$

since x and τ are independent variables.

Then

$$X'' + \lambda^2 X = 0 \quad \Rightarrow \quad X = A\cos\lambda x + B\sin\lambda x$$

$$T' + \lambda^2 T = 0 \quad \Rightarrow \quad T = Ce^{-\lambda^2\tau}$$

$$X(0) = 0 \quad \Rightarrow \quad A = 0$$

$$X(1) = 0 \quad \Rightarrow \quad B\sin\lambda = 0 \Rightarrow \quad B = 0 \quad (\text{no use})$$

$$\text{or } \sin\lambda = 0 \Rightarrow \lambda = n\pi$$

[Note +ve or zero const in (*) gives no non-zero solns satisfying the b.c.s] $\quad$ (n integer)

Hence general soln can be constructed as a linear combination:

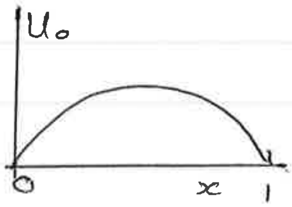
$$U = \sum_{n=1}^{\infty} BC\,e^{-n^2\pi^2\tau}\sin n\pi x$$

$$= \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2\tau}\sin n\pi x, \quad \text{say.}$$

Finally

$$U_0(x) = \sum_1^{\infty} B_n \sin n\pi x \qquad (\text{F. sine series})$$

$$\Rightarrow B_n = \frac{2}{1}\int_0^1 U_0(x)\sin n\pi x\,dx$$



e.g. 1. If $\quad U_0 = x(1-x)$ then

$$B_n = 2\int_0^1 x(1-x)\sin n\pi x\,dx$$

$$= 2\left[-(x-x^2)\frac{\cos n\pi x}{n\pi}\right]_0^1 + 2\int_0^1 (1-2x)\frac{\cos n\pi x}{n\pi}\,dx$$

$$= 2\left[(1-2x)\frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 + 4\int_0^1 \frac{\sin n\pi x}{n^2\pi^2}\,dx$$

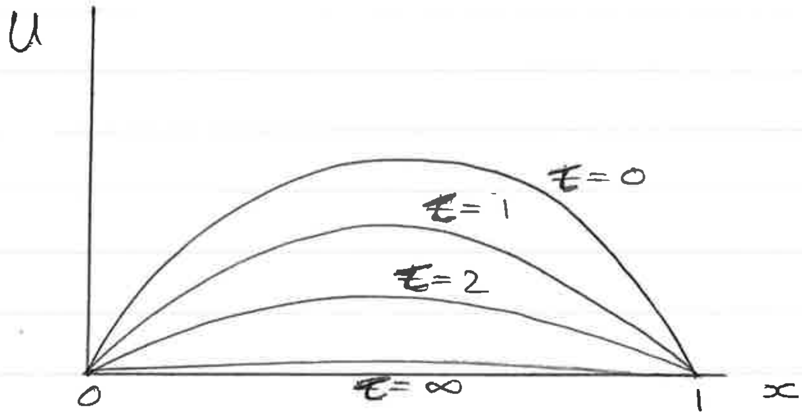
$$= -\frac{4}{n^3\pi^3}\left[\cos n\pi x\right]_0^1$$

$$= \frac{4}{n^3\pi^3}\left[1 - \cos n\pi\right] = \frac{4}{n^3\pi^3}\left(1 - (-1)^n\right)$$

$$\begin{cases} = 0 & (n = 2m) \quad (\text{even } n) \\ = \dfrac{8}{(2m+1)^3\pi^3} & (n = 2m+1) \quad (\text{odd } n) \end{cases}$$

$\therefore$ Only odd n contribute to U, so

$$U = \frac{8}{\pi^3} \sum_{m=0}^{\infty} \frac{e^{-(2m+1)^2\pi^2 t}}{(2m+1)^3} \sin (2m+1)\pi x$$



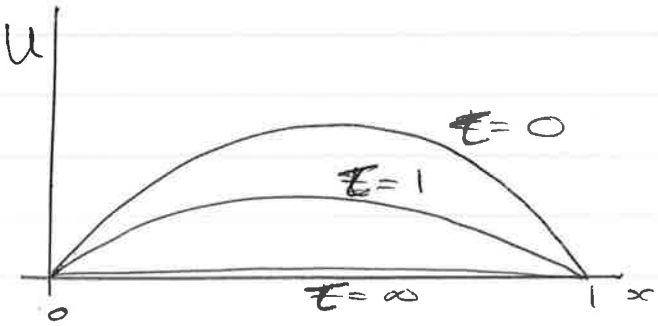
The temperature in the rod smoothes out to zero everywhere as $t \to \infty$.

e.g. 2. If $\quad U_0 = \sin \pi x$

Then by inspection, since
$$\sin \pi x = \sum_1^{\infty} B_n \sin n\pi x$$

we have $B_1 = 1, \quad B_n = 0 \quad (n > 1)$

so
$$U = e^{-\pi^2 t} \sin \pi x$$



(ii) Infinite domain

Now suppose we have (23) on $-\infty < x < \infty$ with

$$U(x, 0) = U_0(x) \tag{26}$$

and b.c.s

$$U \to 0 \qquad \text{as} \qquad x \to \pm\infty \qquad (27)$$

We build up to a solution of this problem gradually.

(a) <u>Similarity solution I</u>

Here we introduce the idea of a SIMILARITY SOLUTION. Since the equation (23) involves the combination x^2/τ we look for a solution $U(x,\tau)$ that depends on x and τ only through the combination $\xi = x/\sqrt{\tau}$, so that

$$U = F(\xi).$$

Differentiate:
$$\frac{\partial U}{\partial \tau} = -\frac{1}{2\tau}\xi F'(\xi)$$

$$\left(\text{since } \frac{\partial U}{\partial \tau} = \frac{dF}{d\xi}\frac{\partial \xi}{\partial \tau} = F'.\left(-\tfrac{1}{2}x\tau^{-3/2}\right) = -\frac{1}{2\tau}\xi F'\right)$$

$$\frac{\partial U}{\partial x} = \tau^{-\frac{1}{2}}F', \qquad \frac{\partial^2 U}{\partial x^2} = \tau^{-1}F''$$

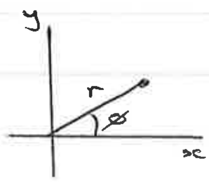
$\therefore$ (23) becomes
$$F'' + \tfrac{1}{2}\xi F' = 0$$

$$\therefore \quad F' = Ae^{-\frac{1}{4}\xi^2}$$

$$\therefore \quad F = A\int_0^{\xi} e^{-\frac{1}{4}\xi'^2}\,d\xi' + B$$

<u>Note:</u>
$$\int_0^\infty e^{-\frac{1}{4}\xi'^2}\,d\xi' = \sqrt{\pi}$$

<u>Proof:</u> Let $I = \int_0^\infty e^{-\frac{x^2}{4}}dx$



$$\therefore \quad I^2 = \int_0^\infty e^{-\frac{x^2}{4}}dx \int_0^\infty e^{-\frac{y^2}{4}}dy$$
$$= \int_0^\infty\int_0^\infty e^{-\frac{1}{4}(x^2+y^2)}\,dx\,dy$$
$$= \int_0^{\pi/2}\int_0^\infty e^{-\frac{1}{4}r^2}\,r\,dr\,d\phi$$
$$= \left[-2e^{-\frac{1}{4}r^2}\right]_0^\infty \cdot \left[\phi\right]_0^{\pi/2} = \pi, \quad \text{as reqd.}$$

This similarity soln can be used to find a solution for which

$$U \to \pm 1 \quad \text{as} \quad x \to \pm\infty .$$

This requires

$$F \to \pm 1 \quad \text{as} \quad \xi \to \infty$$

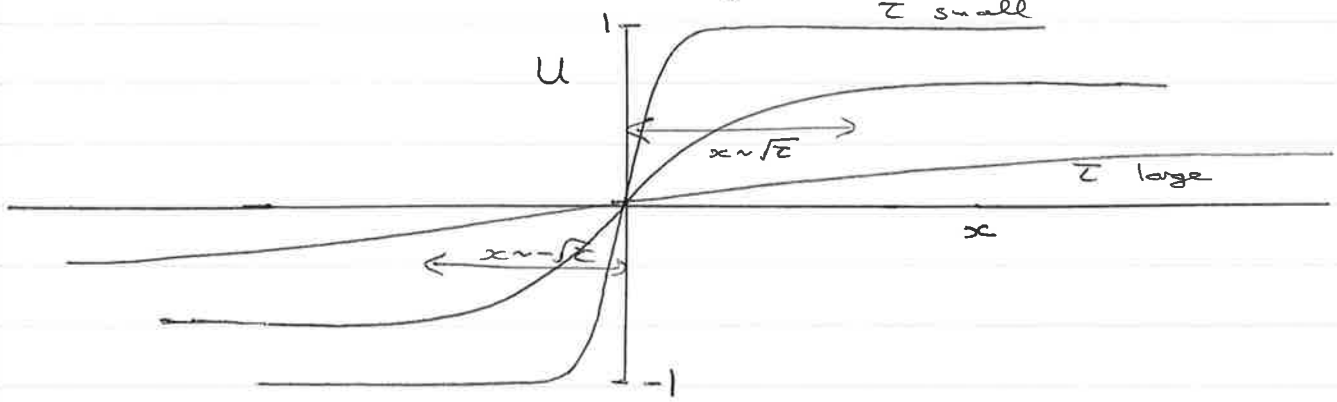
so

$$A\sqrt{\pi} + B = 1 \qquad \text{from } \xi = \infty \text{ condition}$$
$$-A\sqrt{\pi} + B = -1 \qquad \text{" } \xi = -\infty \text{ "}$$

$$\Rightarrow \quad B = 0, \quad A = \frac{1}{\sqrt{\pi}}$$

$$\therefore \quad U = F = \frac{1}{\sqrt{\pi}} \int_0^{\xi} e^{-\frac{1}{4}\xi'^2} \, d\xi'$$

$$\therefore \quad U = \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{\tau}} e^{-\frac{1}{4}\xi'^2} \, d\xi'$$



Note 'spread' of soln as τ increases, with $x \sim \pm\sqrt{\tau}$ defining region of variation of U.

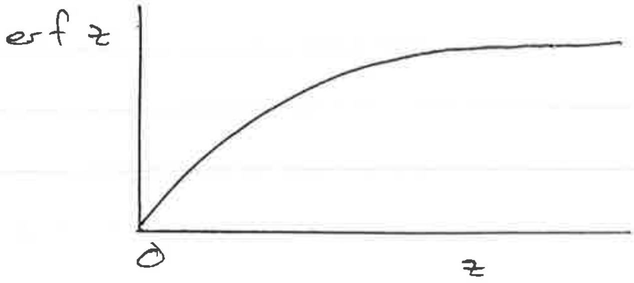
Note: Error function erf

This is defined by $\quad \operatorname{erf} z \equiv \frac{2}{\sqrt{\pi}} \int_0^{z} e^{-z'^2} \, dz'$

(since $\int_0^{\infty} e^{-\frac{1}{4}\xi'^2} d\xi' = \sqrt{\pi}$ Put $\xi' = 2z' \Rightarrow \int_0^{\infty} e^{-z'^2} dz' = \frac{1}{2}\sqrt{\pi}$

Thus $\quad \operatorname{erf} z \to 1 \quad$ as $\quad z \to \infty$

and $\quad \operatorname{erf} 0 = 0$



The soln for U can thus be expressed as

$\xi' = 2z'$

$$U = \frac{1}{\sqrt{\pi}}\int_0^{\xi = x/\sqrt{\tau}} e^{-\frac{1}{4}\xi'^2}\, d\xi' = \frac{1}{\sqrt{\pi}}\int_0^{z' = \frac{x}{2\sqrt{\tau}}} e^{-z'^2}\, 2\, dz'$$

$$\therefore \quad \underline{U = \operatorname{erf}\left(\frac{x}{2\sqrt{\tau}}\right)}$$

Note that the shape of the solution at different times is similar (it is just stretched more in x as τ increases), hence the name SIMILARITY SOLUTION.

Note also the soln $U = F = A\int_0^{\xi} e^{-\frac{1}{4}\xi'^2}\, d\xi' + B$ cannot be used to obtain a non-zero solution for which $F \to 0$ as $\xi \to \pm\infty$ (ie $U \to 0$ as $x \to \pm\infty$); then A and B would both be zero (since

$$A\sqrt{\pi} + B = 0$$
$$-A\sqrt{\pi} + B = 0 \quad)$$

so $U \equiv 0$.

(b) Similarity solution 2

Now try
$$U = \tau^{-\frac{1}{2}} G(\xi), \qquad \xi = x/\sqrt{\tau}$$

In this case
$$\frac{\partial U}{\partial \tau} = \tau^{-3/2}\left(-\tfrac{1}{2}G - \tfrac{1}{2}\xi G'\right), \qquad \frac{\partial^2 U}{\partial x^2} = \tau^{-3/2} G''$$

so (23) becomes

$$G'' + \tfrac{1}{2}G + \tfrac{1}{2}\xi G' = 0$$

$$\Rightarrow \quad G'' + \tfrac{1}{2}(\xi G)' = 0$$

$$\Rightarrow \quad G' + \tfrac{1}{2}\xi G = A$$

But if we require $U \to 0$ as $x \to \pm\infty$ then $G \to 0$ as $\xi \to \pm\infty$ (hence $G' \to 0$ as $\xi \to \pm\infty$ too) so

$$A = 0$$

$$\therefore \quad G' + \tfrac{1}{2}\xi G = 0$$

$$\therefore \quad G = Be^{-\frac{1}{4}\xi^2} \qquad (B \text{ arbitrary})$$

Thus we have a family of solutions which decay as $x \to \pm\infty$.

Note that for the above solution

$$\int_{-\infty}^{\infty} U\,dx = \int_{\xi=-\infty}^{\xi=\infty} \tau^{-\frac{1}{2}} G \,.\, \tau^{\frac{1}{2}} d\xi = \int_{-\infty}^{\infty} G\,d\xi$$

$$= B\int_{-\infty}^{\infty} e^{-\frac{1}{4}\xi^2}\,d\xi = 2B\sqrt{\pi}$$

If we choose B such that

$$\int_{-\infty}^{\infty} U\,dx = 1$$

(i.e. $B = \frac{1}{2\sqrt{\pi}}$) to 'normalize' the solution then

$$U = \frac{1}{2\sqrt{\pi\tau}}\, e^{-\frac{x^2}{4\tau}} \qquad (28)$$

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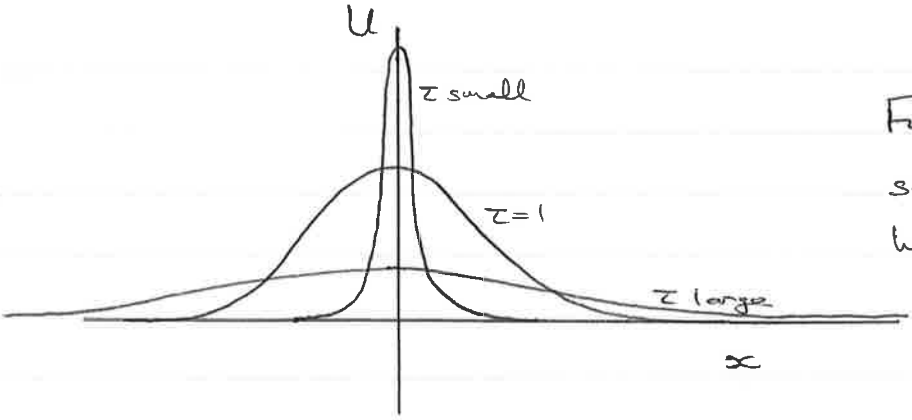
which is known as the FUNDAMENTAL SOLUTION of the heat equation. We can use this to find a solution satisfying the original problem with (26), (27) as the initial & boundary conditions.



Fundamental solution of the heat equation

Again note 'spread' as τ increases.

Area under curve is unity in each case because of normalization.

Solution approaches infinite peak at $x = 0$ as $\tau \to 0$.

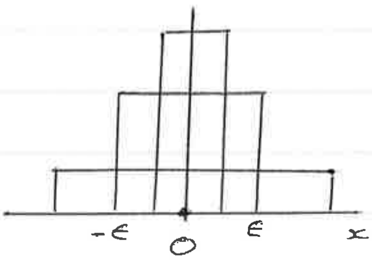
Solution $\to 0$ everywhere as $\tau \to \infty$.

(c) <u>Solution for $U = U_0(x)$ at $\tau = 0$ and $U \to 0$ as $x \to \pm\infty$.</u>

We first define the Dirac delta function $\delta(x)$:

Consider a function $f(x)$ defined as

$$f = \frac{1}{2\epsilon}, \quad |x| \le \epsilon$$
$$= 0, \quad |x| > \epsilon$$



This is such that as $\epsilon \to 0$, $f(0) \to \infty$ but $f(x) = 0$ for $x \neq 0$ (∞ peak at origin). Also

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \qquad \text{for all } \epsilon.$$

We can define

$$\delta(x) \equiv \lim_{\epsilon \to 0} f(x)$$

(The precise form of $f(x)$ is not important, for example we could take

$$f(x) = \frac{1}{2\sqrt{\pi\epsilon}} e^{-\frac{x^2}{4\epsilon}} \quad)$$

From the above defn., if $\phi(x)$ is a smooth general function

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} f(x)\phi(x)\,dx = \phi(0)$$

$$= \phi(0)\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(x)\,dx$$

More generally, for any finite x_0,

$$\int_{-\infty}^{\infty} \delta(x-x_0)\phi(x)\,dx = \phi(x_0) \qquad (29)$$

i.e. multiplying by $\delta(x-x_0)$ 'picks out' the value of ϕ at $x=x_0$.

Now take our general initial function $U_0(x)$. From (29) we can express this i.t.o δ functions as:

$$U_0(x) = \int_{-\infty}^{\infty} U_0(x')\,\delta(x'-x)\,dx' \qquad (30)$$

The fundamental solution of the heat eqtn

$$U = \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{x^2}{4\tau}}$$

has initial value

$$U(x,0) = \delta(x)$$

(since it satisfies $\int_{-\infty}^{\infty} U\,dx = 1$ for all $\tau > 0$ and is zero as $\tau \to 0$ for $x \neq 0$ and ∞ as $\tau \to 0$ for $x = 0$).

$\therefore$ The solution

$$U = \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{(x'-x)^2}{4\tau}}$$

has initial value

$$U(x,0) = \delta(x'-x)$$

for some fixed value x'.

More generally, the solution

$$U = U_0(x') \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{(x'-x)^2}{4\tau}}$$

has initial value

$$U(x,0) = U_0(x')\,\delta(x'-x)$$

Now, in view of (30) and since the heat eqtn is linear, we can 'sum' solutions of this type (ie integrate over x') to obtain the solution

$$U(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} U_0(x')\, e^{-\frac{(x-x')^2}{4\tau}}\, dx' \qquad (31)$$

satisfying the i.c. $U(x,0) = U_0(x)$.
This is the reqd. solution satisfying (26), (27).

(iii) <u>Semi-infinite domain</u>

Suppose $\quad U(x,0) = 0$
and $\quad U(0,\tau) = 1, \quad U \to 0$ as $x \to \infty$.

This is equivalent to a long rod initially at zero temperature with the end $x=0$ suddenly raised to $U=1$ for $\tau > 0$.

This can be solved using the similarity solution

$$U = F(\xi), \qquad \xi = x/\sqrt{\tau}$$

giving (as in (ii) a) a general solution

$$F = A\int_0^{\xi} e^{-\frac{1}{4}\xi'^2}\, d\xi' + B$$

Then b.c. at $\tau = 0$ <u>and</u> $x \to \infty$ $\Rightarrow$

$$A\sqrt{\pi} + B = 0$$

and so $\quad B = -A\sqrt{\pi}$.

$\therefore$ B.c at $x = 0 \Rightarrow F = 1$ at $\xi = 0$

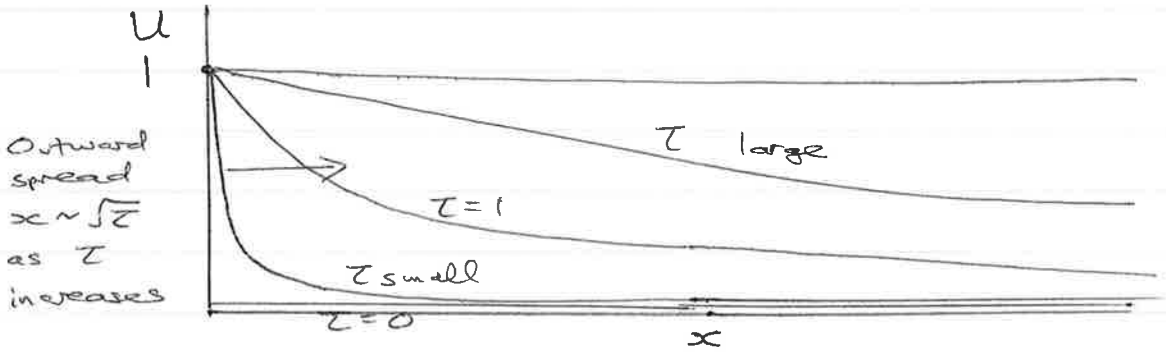
$\Rightarrow \quad B = 1$

$\therefore \quad A = -\frac{1}{\sqrt{\pi}}$

$$\therefore \quad F = 1 - \frac{1}{\sqrt{\pi}} \int_0^{\xi} e^{-\frac{1}{4}\xi'^2} \, d\xi'$$

$$F = 1 - \mathrm{erf}\left(\frac{\xi}{2}\right)$$

$$\therefore \quad \underline{U = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{\tau}}\right)}$$



Forward and backward heat equations

The equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}$$

is known as a FORWARD EQUATION because solutions can be found for increasing (forward) τ from an initial condition at $\tau = 0$.

Suppose we had a parabolic equation for $U(x,t)$

$$-\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad (32)$$

(equivalent to replacing τ by $-t$ or $k - t$).
This is known as a BACKWARD EQUATION.
However, the backward problem in $t > 0$ is ILL-POSED because ~~the~~ for most initial

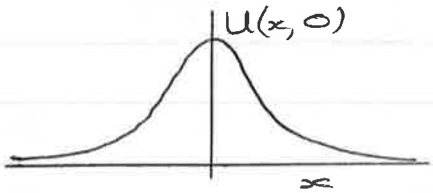
conditions at $t=0$ and boundary conditions the solution either does not exist or, if it does, will blow up $(U \to \infty)$ at a finite time $t > 0$.

e.g. If we take the fundamental solution of the heat equation (28) we could have a solution of (32) for $t>0$ given by

$$U = \frac{1}{2\sqrt{\pi(k-t)}} e^{-\frac{x^2}{4(k-t)}}$$

consistent with $U \to 0$ as $x \to \pm\infty$ (provided $t < k$) and an initial condition

$$U = \frac{1}{2\sqrt{\pi k}} e^{-\frac{x^2}{4k}} \quad \text{at } t=0$$



However, we see that $U \to \infty$ as $t \to k-$, and the solution fails (blows up) and cannot be continued.

Whereas the forward equation results in the diffusion of initial data, the backward equation causes initial data to 'focus' or 'blow up' – physically it is trying to determine from what earlier temperature distribution a given final distribution has evolved and this is an unstable process.

3. Black-Scholes equation

(a) European call option

Solution for the value $C(S,t)$ of a European call option: the problem formulated previously is

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \qquad (33)$$

with

$$C(0,t) = 0, \qquad C(S,t) \sim S \quad \text{as } S \to \infty \qquad (34)$$

and

$$C(S,T) = \max(S-E, 0) \qquad (35)$$

This is a linear parabolic equation but with non-constant coefficients (involving S and possibly t if σ and/or r depend on t). It is also a backward equation and therefore should be solved backwards in time from the final data (35) at $t = T$.

Thus it is convenient to recast the system as a forward problem by defining the non-dimensional time τ by

$$t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$$

and setting

$$r = \frac{1}{2}\sigma^2 k$$

We assume here that σ and r are <u>constants</u>

Thus (33) becomes

$$\frac{\partial C}{\partial \tau} = S^2\frac{\partial^2 C}{\partial S^2} + kS\frac{\partial C}{\partial S} - kC$$

We can remove the non-constant coefficients in S by transforming $S \to x$ with

$$S = E e^{x}$$

(where x is n-d) and we also introduce a n-d form of C by writing

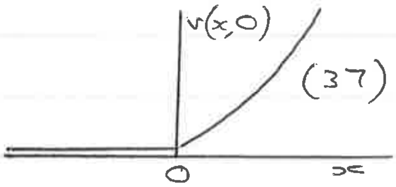
$$C = E v(x, \tau).$$

Now we obtain

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv \qquad (36)$$

with initial condition

$$v(x, 0) = \max(e^{x} - 1, 0) \qquad (37)$$



and boundary conditions

$$v \to 0 \quad \text{as} \quad x \to -\infty \qquad (38)$$

$$v \sim e^{x} \quad \text{as} \quad x \to \infty \qquad (39)$$

Note that this form of the problem (36)–(39) shows that there is really only one n-d parameter k left in the problem.

(36) now looks similar to a heat equation and in fact a further transformation allows it to be turned into one. We try

$$v = e^{\alpha x + \beta \tau} u(x, \tau)$$

with α, β constants to be determined.

Substituting this into (36) gives

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku$$

We can eliminate the u terms by choosing

$$\beta = \alpha^2 + (k-1)\alpha - k$$

and the further choice

$$0 = 2\alpha + (k-1)$$

eliminates $\frac{\partial u}{\partial x}$ as well. These give

$$\alpha = -\tfrac{1}{2}(k-1), \quad \beta = -\tfrac{1}{4}(k+1)^2$$

and now

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} \qquad \text{for} \quad -\infty < x < \infty, \quad \tau > 0$$

with

$$U(x,0) = U_0(x) \equiv e^{\frac{1}{2}(k-1)x} \max(e^x - 1, 0) \qquad (40)$$

and

$$\left.\begin{aligned} U &\sim e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}, \quad x \to \infty \\ U &\sim 0 \, . \, e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau}, \quad x \to -\infty \end{aligned}\right\} \qquad (41)$$

Derivation of b.c.s (41):

(i) $S = 0 \quad (x \to -\infty)$

In general we have $\frac{\partial C}{\partial t} - rC = 0$ on $S = 0$ (from (33)) so $C = Ae^{rt}$. $\therefore$ Condition $C(0,t) = 0$ in (34) becomes $A = 0$, or

$$C = 0 \, . \, e^{rt} \quad \text{on} \quad S = 0.$$

$$\begin{aligned} \therefore \quad v &= 0 \, . \, e^{r(T - \frac{\tau}{\frac{1}{2}\sigma^2})} \quad \text{as} \quad x \to -\infty, \\ &= 0 \, . \, e^{rT} e^{-k\tau} \quad \text{as} \quad x \to -\infty. \\ &= 0 \, . \, e^{-k\tau} \end{aligned}$$

Now since $v = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} U$

this becomes

$$U = 0 \, . \, e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k+1)^2\tau - k\tau} \quad \text{as} \quad x \to -\infty$$

$$\therefore U = 0 \, . \, e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau} \quad \text{as} \quad x \to -\infty$$

(ii) $S \to \infty \quad (x \to \infty)$

Here $C \sim S \sim Ee^x$ as $x \to \infty$

so $v \sim e^x$ as $x \to \infty$

$$\therefore U \sim e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k+1)^2\tau + x} \quad \text{as} \quad x \to \infty$$

$$\therefore U \sim e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau} \quad \text{as} \quad x \to \infty$$

Note that usually $k > 1$ in which case the condition on U as $x \to -\infty$ can be interpreted as

$$U \to 0 \quad \text{as} \quad x \to -\infty,$$

(e.g. for numerical purposes — see later).

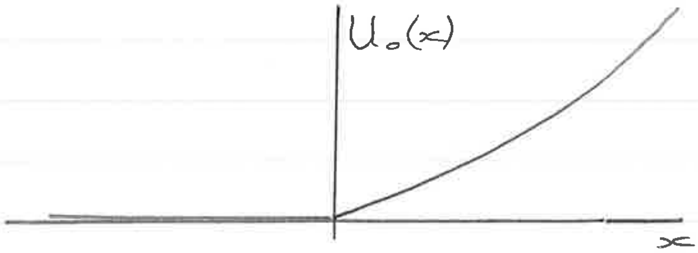
Solution for $U(x,\tau)$:

From our general soln of the heat eqtn we can infer that

$$U = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} U_0(x') e^{-\frac{(x-x')^2}{4\tau}} dx' \qquad (42)$$

where U_0 is given from (40) as

$$\left.\begin{aligned} U_0 &= e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, \quad x \geq 0 \\ &= 0, \quad x \leq 0 \end{aligned}\right\}$$



<u>Note on boundary conditions:</u>

Note that ~~as~~ $x \to \infty$, $U \not\to 0$, as in our earlier solution of the heat eqtn. However, the earlier analysis still applies provided U does not grow too rapidly as $x \to \infty$ (ie $U \ll e^{ax^2}$ as $x \to \infty$ where a is any +ve constant*). In deriving solution (31) we did not use the b.cs explicitly; what is required is that the integral $\int_{-\infty}^{\infty}$ in (42) above converges, and hence the above condition (*) on U as $x \to \infty$.

The b.cs on U do not enter the soln (42) explicitly because provided the behavior of U at $x = \pm\infty$ is not too singular, $x = \pm\infty$ is too far away to influence the solution for U generated by the initial data at $\tau = 0$; (42) in fact satisfies the conditions (41) automatically.

The solution (42) is now completed by evaluating the integral using the given form of U_0.

Use change of variable $s = \dfrac{x' - x}{\sqrt{2\tau}}$

$$\therefore\ U = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_0(s\sqrt{2\tau} + x)\, e^{-\frac{1}{2}s^2}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x + s\sqrt{2\tau})}\, e^{-\frac{1}{2}s^2}\, ds$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k-1)(x + s\sqrt{2\tau})}\, e^{-\frac{1}{2}s^2}\, ds$$

$$= I_1 - I_2 \quad \text{, say.}$$

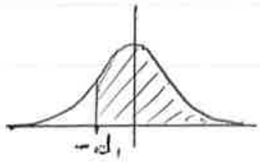
Now $$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x + s\sqrt{2\tau}) - \frac{1}{2}s^2}\, ds$$

$$= \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{4}(k+1)^2\tau}\, e^{-\frac{1}{2}(s - \frac{1}{2}(k+1)\sqrt{2\tau})^2}\, ds$$

Set $\rho = s - \frac{1}{2}(k+1)\sqrt{2\tau}$

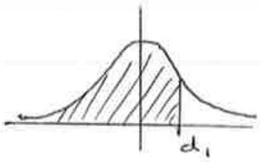
$$\therefore\ I_1 = \frac{e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau} - \frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\rho^2}\, d\rho$$

$$= e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}\, N(d_1)$$



where $$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}$$

and $$N(d_1) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{1}{2}\rho^2}\, d\rho$$



(which is the cumulative distribution function of the normal distribution). (Note $N(\infty) = 1$).

Similarly
$$I_2 = e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau}\, N(d_2)$$
where $$d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}$$

In summary, we finally have, since

$$v = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} U(x,\tau) \quad \text{with}$$

$$x = \ln(S/E), \quad \tau = \tfrac{1}{2}\sigma^2(T-t), \quad C = E\,v(x,\tau),$$

the result

$$\left.\begin{aligned} C(S,t) &= S N(d_1) - E e^{-r(T-t)} N(d_2) \\ \text{where} \quad d_1 &= \frac{\ln(S/E) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\ln(S/E) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}\right\} \quad (43)$$

This gives the value of the call option as a function of the asset price S and time $t<T$.

<u>Note</u> Since $N(\infty)=1$, $I_1 \sim e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}$ as $x\to\infty$ whereas $I_2 \sim e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau}$. Thus $U = I_1 - I_2$ <u>does</u> satisfy the b.c. (41) as $x\to\infty$. Since $N(-\infty)=0$ it also satisfies the b.c. (41) as $x\to-\infty$.

(b) <u>European put option</u>

Here the equation is the same but the final condition becomes

$$P(S,T) = \max(E-S, 0)$$

giving

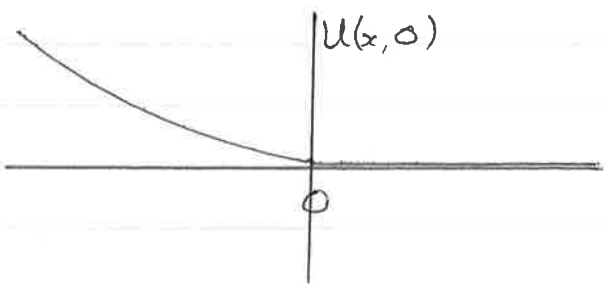
$$P = E\,v(x,\tau)$$

with

$$v = e^{\alpha x + \beta\tau} U(x,\tau)$$

and now α, β are same as before and

$$\left.\begin{aligned} U(x,0) &= e^{\frac{1}{2}(k-1)x} \max(1-e^{x}, 0) \\ &= e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, \quad x \le 0 \\ &= 0, \quad x>0 \end{aligned}\right\}$$



Alternatively we can use the put-call parity formula to give

$$P = C - S + Ee^{-r(T-t)}$$

so that in summary (and having used $N(-d) + N(d) = 1$)

$$\underline{P(S,t) = Ee^{-r(T-t)} N(-d_2) - S\,N(-d_1)} \qquad (44)$$

where N, d_1, d_2 are as defined previously.

<u>Rate of change of option value, Δ</u>

For the call option,

$$\Delta = \frac{\partial C}{\partial S} = N(d_1) + S\frac{\partial N(d_1)}{\partial S} - Ee^{-r(T-t)}\frac{\partial N(d_2)}{\partial S}$$

$$= N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S}$$

$$= N(d_1) + \underbrace{(SN'(d_1) - Ee^{-r(T-t)}N'(d_2))}_{=0 \text{ after algebra}}/S\sigma\sqrt{T-t}$$

$$= N(d_1)$$

For the put option

$$\Delta = \frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - 1 \qquad \text{(from put-call parity formula)}$$

$$= N(d_1) - 1$$

(c) <u>Other options</u>

These can be calculated, for the appropriate final conditions, in a similar fashion.

Extension to time-dependent interest rate and volatility

Suppose $r = r(t)$, $\sigma = \sigma(t)$ are given functions of time. We can avoid carrying out all the analysis again as follows.

We now have

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} + r(t)S\frac{\partial V}{\partial S} - r(t)V = 0 \qquad (45)$$

Substitute

$$\bar{S} = Se^{\alpha(t)}$$
$$\bar{V} = Ve^{\beta(t)}$$
$$\bar{t} = \gamma(t)$$

and (45) becomes

$$\mathring{\gamma}(t)\frac{\partial \bar{V}}{\partial \bar{t}} + \tfrac{1}{2}(\sigma(t))^2\bar{S}^2\frac{\partial^2 \bar{V}}{\partial \bar{S}^2} + (r(t) + \mathring{\alpha}(t))\bar{S}\frac{\partial \bar{V}}{\partial \bar{S}} - (r(t) + \mathring{\beta}(t))\bar{V} = 0$$

where $\circ \equiv d/dt$.

Eliminate coefficients of $\bar{V}$, $\frac{\partial \bar{V}}{\partial \bar{S}}$ by choosing

$$\alpha(t) = \int_t^T r(\tau)\,d\tau$$

$$\beta(t) = \int_t^T r(\tau)\,d\tau$$

and remove the remaining time-dependence by setting

$$\gamma(t) = \int_t^T \sigma^2(\tau)\,d\tau$$

Now we have

$$\frac{\partial \bar{V}}{\partial \bar{t}} = \tfrac{1}{2}\bar{S}^2\frac{\partial^2 \bar{V}}{\partial \bar{S}^2} \qquad (46)$$

If $\bar{V}$ is any solution of (46) then the solution for V is

$$V = e^{-\beta(t)}\, \bar{V}(\bar{S}, \bar{t})$$

$$\underline{V = e^{-\beta(t)}\, \bar{V}\left(Se^{\alpha(t)},\ \gamma(t)\right)} \tag{47}$$

But suppose V_{BS} is any solution of the B–S equation for <u>constant</u> r and σ (e.g. (43) or (44)). By the above, the solution can be written as

$$V_{BS} = e^{-r(T-t)}\, \bar{V}_{BS}\left(Se^{-r(T-t)},\ \sigma^2(T-t)\right) \tag{48}$$

for some function $\bar{V}_{BS}$ (since for r, σ constant $\alpha = \beta = r.(T-t)$, $\gamma = \sigma^2.(T-t)$).

By comparing (47) and (48), to go from a BS solution with constant r, σ to one with non-constant r, σ we just replace

$$r \quad \text{by} \quad \frac{1}{T-t}\int_t^T r(\tau)\,d\tau$$

and

$$\sigma^2 \quad \text{by} \quad \frac{1}{T-t}\int_t^T \sigma^2(\tau)\,d\tau$$

wherever they occur in the solution.

e.g. (43) becomes

$$C(S,t) = SN(d_1) - Ee^{-\int_t^T r(\tau)\,d\tau} N(d_2)$$

where

$$d_1 = \frac{\ln(S/E) + \left(\int_t^T r(\tau)\,d\tau + \frac{1}{2}\int_t^T \sigma^2(\tau)\,d\tau\right)}{\left(\int_t^T \sigma^2(\tau)\,d\tau\right)^{1/2}}$$

$$d_2 = \frac{\ln(S/E) + \left(\int_t^T r(\tau)\,d\tau - \frac{1}{2}\int_t^T \sigma^2(\tau)\,d\tau\right)}{\left(\int_t^T \sigma^2(\tau)\,d\tau\right)^{1/2}}$$