Differential Equations for Finance MA3607: Coursework

To be handed in by 12:00 25th November 2013

Hand in ONLY to SEMS Undergraduate Office C108

To answer these questions you will need two numbers α and β. These will be related to the first letter of your first name and the second letter of your last name, respectively, through the following algorithm

If the letter is an A,B,C or D the numeric value is 1
If the letter is an E,F,G or H the numeric value is 2
If the letter is an I,J,K or L the numeric value is 3
If the letter is an M,N,O or P the numeric value is 4
If the letter is an Q,R,S or T the numeric value is 5
If the letter is an U,V,W,X,Y or Z the numeric value is 6

So for example if your name is John Smith to determine α you should use the letter J an to determine β you should use the letter m. The algorithm then says that for John Smith $\alpha = 3$ and $\beta = 4$.

Once you determine your values of α and β you should use them in the questions below.

To get full marks you will need to answer two 10 mark questions and one 20 mark question.

You may attempt as many questions as you want. If you attempt more than the minimum number of questions I will select the combination of questions that gives you the highest possible marks.

Problem 1   10 marks

The Black-Scholes equation for an option value V is

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,,$$

where S is the share price, t is time, $r = r(t)$ is the bank rate and $\sigma = \sigma(t)$ is the volatility

If σ is constant and $r = \alpha + \beta t$, find the value of the constant c such that

$$V = aS - bS^2 e^{-ct - \beta t^2/2},$$

is a solution of the Black-Scholes equation, where a and b are arbitrary positive constants. Define Δ, the delta of the option value V, and find it as a function of S and t. For a fixed time t, find the share price S that maximizes the value of the option. What is this maximum value?

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Problem 2 10 marks

Using seperation of variables, solve the heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

for a function $U(x, t)$ in the range $x \in [0, 1]$ and $t \in [0, \infty)$ together with the boundary conditions

$$U(x = 0, t) = 0, \qquad U(x = 1, t) = 0,$$

and the initial condition

$$U(x, t = 0) = \begin{cases} \beta x & 0 \le x \le \frac{1}{\alpha} \\ (1 - x)\frac{\beta}{\alpha - 1} & \frac{1}{\alpha} \le x \le 1 \end{cases}.$$

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Problem 3 10 marks

(i) If the error function is defined as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-y^2} \, \mathrm{d}y,$$

re-express the following integral in terms of the error function

$$\int_{-alpha}^{\alpha\beta} \mathrm{e}^{-x^2 + \alpha x - \beta} \, \mathrm{d}x.$$

Show all your working.

(ii) Show that the partial differential equation

$$\alpha^2 \frac{\partial U}{\partial \tau} = y^{2-2\alpha} \frac{\partial^2 U}{\partial y^2} + (1 - \alpha) y^{1-2\alpha} \frac{\partial U}{\partial y}$$

can be transformed to the heat equation using the substitution $x = y^\alpha$.

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Problem 4 20 marks

Find the solution $U = U(t, x)$ to the partial differential equation

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} + \beta t^{-1} U \,,$$

using the similarity ansatz

$$U(t, x) = t^m F(x t^n) \,,$$

for suitable constants m and n and a function F.
Hint: Picking suitable values for m and n should result in the following ordinary differential equation for F

$$2\alpha F''(\zeta) + \zeta F'(\zeta) = 0 \,,$$

which can then be solved.

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Problem 5 20 marks

Using a forward difference in t and a central difference in x derive an explicit scheme for the equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \alpha \frac{\partial U}{\partial x} + (10 - \beta) U \,,$$

in the region $0 \le x \le 1$ and $t \ge 0$, subject to

$$\frac{\partial U}{\partial t} = 0 \,, \qquad \text{at } x = 1 \,,$$

and

$$U = 0 \,, \qquad \text{at } x = 0 \,,$$

as well as the initial condition

$$U(x, t = 0) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 4(1 - x)^2 & 1/2 \le x \le 1 \end{cases}$$

Find the principal part of the local truncation error at $x = i\Delta x$ and $t = j\Delta t$ where Δx and Δt are step lengths in the x and t directions, respectively. Hence confirm that your scheme is consistent.
Picking intervals of 0.2 in x and 0.01 in t, use your explicit scheme for U to find $U(x, t = 0.1)$ to 4 d.p.

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