Differential Equations for Finance MA3607, 2013/14

Functions defined by integrals

Functions can be defined as integrals. For example, imagine that you did not know the function $\log x$. Then you could define it as

$$\log x = \int_1^x \frac{1}{z} dz \,, \tag{1}$$

where the lower bound for the integral ($z = 1$) is chosen to ensure that the the r.h.s. is precisely $\log x$ rather than $\log x + C$. Given this definition and the fundamental theorem of calculus
(http://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus#First_part) it is easy to show that

$$\frac{d}{dx} \log x = \frac{1}{x} \,. \tag{2}$$

Exercises

1. Compute $f'(x)$ if $f(x)$ is given by

$$\begin{aligned}
(i) \quad & f(x) = \int_{27}^x \frac{1}{z^2} dz \,, \\
(ii) \quad & f(x) = \int_0^x \cos y dy \,, \\
(iii) \quad & f(x) = \int_0^x \frac{1}{1+w^2} dw \,, \\
(iv) \quad & f(x) = \int_{-\infty}^x e^v dv \,, \\
(v) \quad & f(x) = \int_0^x e^{z^3} dz \,,
\end{aligned}$$

2. For $f(x)$ given in $(i) - (iv)$ above, the integrals can be solved explicitly. Find the functions $f(x)$ by doing these integrals explicitly. Verify that the derivative of the expressions agrees with $f'(x)$ you got in **1.**

Functions defined by integrals

In the exercises **1.**$(i) - (iv)$ above we could perform the integrals explicitly, so $f(x)$ can be written down in terms of well known functions - it is unnecessary to write the functions as integrals. In **1.**(v) however, we do not know how to perform the integral. As a result, **1.**(v) defines a new function. We may study its properties: for example the above exercise has already shown us what $f'(x)$ is. Using the integral representation of this $f(x)$ we may be able to express other integrals in terms of the function $f(x)$
For example consider the following integral

$$I = \int_3^7 e^{z^3} dz \,. \tag{3}$$

It looks like we should be able to express it in terms of $f(x)$ defined in exercise $\mathbf{1}.(v)$ above. to do this we re-write I as

$$I = \int_3^0 e^{z^3} dz + \int_0^7 e^{z^3} dz = -\int_0^3 e^{z^3} dz + \int_0^7 e^{z^3} dz\,. \tag{4}$$

Using the definition of $f(x)$ in $\mathbf{1}.(v)$ above we conclude that

$$I = f(7) - f(3)\,. \tag{5}$$

Exercises

3. Express the following integrals using the definition of $f(x)$ from $\mathbf{1}.(v)$ above

$$\begin{aligned}
&(i) \quad J = \int_{-4}^{23} e^{z^3} dz\,,\\
&(ii) \quad g(y) = \int_y^5 e^{z^3} dz\,,\\
&(iii) \quad h(v) = \int_1^{v+1} e^{(z+1)^3} dz\,,\\
&(iv) \quad k(x) = \int_3^x z e^{z^6} dz\,.
\end{aligned}$$

Outlook

In general given any integral that cannot be expressed in terms of elementary functions one can define new functions through it. Not all such definitions will turn out to be useful, and as the above exercises show several different unknown integrals might be expressible in terms of a single new function. Through trial and error mathematicians have identified several new functions that prove to be immensely useful. Two of the best known examples are the error function and the Gamma function, which are defined as

$$\begin{aligned}
\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz\,,\\
\Gamma(z) &= \int_0^\infty t^{z-1} e^{-t}\,.
\end{aligned}$$

The error function, defined will appear in many places in this course; it is closely related to the normal distribution, and it often appears in solutions to the heat equation. The function $\Gamma(z)$ will not feature in this course, but is one of the most beautiful functions in mathematics. Notice that the argument z of $\Gamma(z)$ does not appear in the integration limits. Hence we cannot immediately find $\frac{d}{dz}\Gamma(z)$.

Exercises

4. Show that $\Gamma(1) = 1$.
5. Show that $\Gamma(x) = (x-1)\Gamma(x-1)$.
6. Show that $\Gamma(n+1) = n!$ for any integer $n > 0$.
7. Determine $\frac{d}{dx} erf(x)$.
8. Express $\int_3^{2y+4} e^{-z^2+4z} dz$ in terms of the error function.