Consider

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{\omega_1}{2}x^2 - \frac{\omega_2}{2}y^2$$

x equation of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \quad \text{is} \quad \frac{d}{dt}(\dot{x}) + \omega_1 x = \ddot{x} + \omega_1 x = 0$$

y equation of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = 0 \quad \text{is} \quad \frac{d}{dt}(\dot{y}) + \omega_2 y = \ddot{y} + \omega_2 y = 0$$

Consider a "rotation" to NEW x, y

$$(*) \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_N \\ y_N \end{pmatrix} \qquad \text{ie} \quad \begin{array}{l} x = \cos\theta\, x_N + \sin\theta\, y_N \\ y = \cos\theta\, y_N - \sin\theta\, x_N \end{array}$$

where $x_N \equiv x_N(t)$, $y_N \equiv y_N(t)$ and θ is constant

Since θ is constant we have

$$\dot{x} = \cos\theta\, \dot{x}_N + \sin\theta\, \dot{y}_N$$
$$\dot{y} = \cos\theta\, \dot{y}_N - \sin\theta\, \dot{x}_N$$

Then notice that

$$\begin{aligned} \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 &= \frac{1}{2}(\cos\theta\, \dot{x}_N + \sin\theta\, \dot{y}_N)^2 + \frac{1}{2}(\cos\theta\, \dot{y}_N - \sin\theta\, \dot{x}_N)^2 \\ &= \frac{1}{2}\cos^2\theta\, \dot{x}_N^2 + \frac{1}{2}\sin^2\theta\, \dot{x}_N^2 + \frac{1}{2}\sin^2\theta\, \dot{y}_N^2 + \frac{1}{2}\cos^2\theta\, \dot{y}_N^2 \\ &= \frac{1}{2}\dot{x}_N^2 + \frac{1}{2}\dot{y}_N^2 \end{aligned}$$

Similarly, if $\omega_1 = \omega_2 \equiv \omega$

$$\begin{aligned} \frac{\omega}{2}x^2 + \frac{\omega}{2}y^2 &= \frac{\omega}{2}(\cos\theta\, x_N + \sin\theta\, y_N)^2 + \frac{\omega}{2}(\cos\theta\, y_N - \sin\theta\, x_N)^2 \\ &= \frac{\omega}{2}x_N^2 + \frac{\omega}{2}y_N^2 \end{aligned}$$

So if $\omega_1 = \omega_2 \equiv \omega$ we have

$$\mathcal{L}[x, \dot{x}, y, \dot{y}] = \mathcal{L}[x_N, \dot{x}_N, y_N, \dot{y}_N] \qquad (**)$$

i.e. the Lagrangian is invariant under the rotation (*) and we say that the rotation (*) is a SYMMETRY of the Lagrangian $\mathcal{L}$.

Such symmetries imply the existence of conserved quantities for the system governed by $\mathcal{L}$.

To find these conserved quantities we use the NOETHER PROCEDURE.

Consider the rotation (*) but now with $\theta = \theta(t)$ an infinitesimal parameter. Then the rotation becomes

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} x_N \\ y_N \end{pmatrix} + O(\theta^2) \quad \text{ie} \quad \begin{aligned} x &= x_N + \theta y_N + O(\theta^2) \\ y &= y_N - \theta x_N + O(\theta^2) \end{aligned}$$

With $\omega_1 = \omega_2 \equiv \omega$ it is easy to show that

$$\frac{\omega}{2} x^2 + \frac{\omega}{2} y^2 = \frac{\omega}{2} x_N^2 + \frac{\omega}{2} y_N^2$$

as before. However, because $\theta = \theta(t)$ we now have

$$\frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 = \frac{1}{2}\left(\dot{x}_N + \theta \dot{y}_N + \dot{\theta} y_N\right)^2 + \frac{1}{2}\left(\dot{y}_N - \theta \dot{x}_N - \dot{\theta} x_N\right)^2$$

$$= \frac{1}{2} \dot{x}_N^2 + \theta \dot{x}_N \dot{y}_N + \dot{\theta} \dot{x}_N y_N + \frac{1}{2} \dot{y}_N^2 - \theta \dot{x}_N \dot{y}_N - \dot{\theta} \dot{y}_N x_N + O(\theta^2)$$

$$= \frac{1}{2} \dot{x}_N^2 + \frac{1}{2} \dot{y}_N^2 + \dot{\theta}\left(\dot{x}_N y_N - \dot{y}_N x_N\right) + O(\theta^2)$$

So we find that when $\theta = \theta(t)$ the Lagrangian is no Longer invariant under (*). In fact (**) becomes

$$\mathcal{L}[x, \dot{x}, y \dot{y}] = \mathcal{L}[x_N, \dot{x}_N, y_N, \dot{y}_N] + \dot{\theta}(\dot{x}_N y_N - \dot{y}_N x_N) + O(\theta^2)$$

We can write this at the level of the action

$$\int_{t_0}^{t_1} \mathcal{L}[x,\dot{x},y,\dot{y}]dt \approx \int_{t_0}^{t_1} \mathcal{L}[x_N, \dot{x}_N, y_N, \dot{y}_N]dt + \int_{t_0}^{t_1} \frac{d\theta}{dt}(\dot{x}_N y_N - \dot{y}_N x_N)\, dt$$

$$\approx \int_{t_0}^{t_1} \mathcal{L}[x_N, \dot{x}_N, y_N, \dot{y}_N]dt - \int_{t_0}^{t_1} \theta \frac{d}{dt}(\dot{x}_N y_N - \dot{y}_N x_N)\, dt + \Big[\theta(\dot{x}_N y_N - \dot{y}_N x_N)\Big]_{t=t_0}^{t=t_1}$$

As long as we pick $\theta(t)$ s.t. $\theta(t_0) = \theta(t_1) = 0$ the last term is equal to zero and

$$\int_{t_0}^{t_1} \mathcal{L}[x, \dot{x}, y, \dot{y}]\, dt = \int_{t_0}^{t_1} \mathcal{L}[x_N, \dot{x}_N, y_N, \dot{y}_N]\, dt - \int_{t_0}^{t_1} \theta \frac{d}{dt}(\dot{x}_N y_N - \dot{y}_N x_N)$$

Recall, we showed that for $\theta = cst$ the action was invariant under (*). For the equation above that implies that when $\theta = cst$ the last term on rhs is zero

$$0 = \int_{t_0}^{t_1} \theta \frac{d}{dt}(\dot{x}_N y_N - \dot{y}_N x_N)\, dt = \theta (\dot{x}_N y_N - \dot{y}_N x_N)\Big|_{t=t_0}^{t=t_1}$$

Since θ is arbitrary this says that $\dot{x}_N y_N - \dot{y}_N x_N$ takes the same value at $t = t_0$ and $t = t_1$. In our derivation t_0, t_1 were arbitrary. So we conclude

$$\dot{x}_N y_N - \dot{y}_N x_N = cst$$

We have found that $\dot{x}_N y_N - \dot{y}_N x_N$ is a constant of motion! Notice that the subscript N plays no role: (x_N, y_N) have the same E-L eqns as (x, y). So in fact we find

$$\frac{d}{dt}(\dot{x}y - \dot{y}x) = 0$$

But is this really true? Lets use the EL eqns to check this

$$\frac{d}{dt}(\dot{x}y - \dot{y}x) = \ddot{x}y + \dot{x}\dot{y} - \ddot{y}x - \dot{y}\dot{x}$$

$$= \ddot{x}y - \ddot{y}x$$

$$= -\omega x y + \omega y x = 0$$

This example illustrates the general Noether procedure for constructing a conserved quantity from a symmetry of the action: If an action is invariant under some time-independent symmetry the conserved quantity is obtained by evaluating the change in the action under an infinitesimal time dependent transformation. This change in the action will be non-zero, and upon integration by parts can be written in the form

$$\int_{t_0}^{t_1} \theta \frac{d}{dt}(Q)\, dt$$ with Q the conserved quantity

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 + \frac{\mu}{r} \qquad \text{(II)}$$

The EL equs are

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(r^2 \dot{\theta}\right) - 0 = \frac{d}{dt}(r^2\dot{\theta})$$

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}(\dot{r}) - r\dot{\theta}^2 + \frac{\mu}{r^2} = \ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2}$$

We recognize these as the equations of a planet moving around the sun from earlier in our course

We see in particular that $r^2\dot{\theta} \equiv h = \text{cst}$ as befor

Can we find this quantity using the Noether procedure?

Note $\theta(t) \to \theta(t) + \varepsilon$ for ε a constant is a symmetry of (II)

Since $\dot{\theta}(t) \to \frac{d\theta}{dt} + \frac{d\varepsilon}{dt} = \frac{d\theta}{dt} + 0 = \dot{\theta}$

So $\mathcal{L} = \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2\dot{\theta}^2 + \frac{\mu}{r} \to \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{\mu}{r} = \mathcal{L}$ (**)

Now, follow Noether procedure and take $\varepsilon = \varepsilon(t)$ with ε infinitesimal. Then

$$\dot{\theta}(t) \to \frac{d\theta}{dt} + \frac{d\varepsilon}{dt} = \dot{\theta} + \dot{\varepsilon}$$

So $\int_{t_0}^{t_1} \mathcal{L}\, dt \to \int_{t_0}^{t_1} \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2(\dot{\theta} + \dot{\varepsilon})^2 + \frac{\mu}{r}\, dt \approx \int_{t_0}^{t_1} \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + r^2\dot{\theta}\dot{\varepsilon} + \frac{\mu}{r} + \ldots$

$$= \int_{t_0}^{t_1} \mathcal{L}\, dt + \int_{t_0}^{t_1} r^2 \dot{\theta}\frac{d\varepsilon}{dt}\, dt$$

Let us integrate the last term by parts

$$\int_{t_0}^{t_1} \mathcal{L}\, dt \longrightarrow \int_{t_0}^{t_1} \mathcal{L} dt + \left[r^2 \dot{\theta} \varepsilon \right]_{t=t_0}^{t_1} - \int_{t_0}^{t_1} \varepsilon \frac{d}{dt}(r^2 \dot{\theta})\, dt$$

We are free to choose ε such that $\varepsilon(t_0) = \varepsilon(t_1) = 0$

So in the end

$$\int_{t_0}^{t_1} \mathcal{L}\, dt \longrightarrow \int_{t_0}^{t_1} \mathcal{L}\, dt - \int_{t_0}^{t_1} \varepsilon \frac{d}{dt}(r^2 \dot{\theta})\, dt$$

Since, for constant ε, the last term should be zero ($\because$ the action is invariant under constant ε shifts as shown in (***))

we must have

$$\frac{d}{dt}(r^2 \dot{\theta}) = 0$$

which shows that $r^2 \dot{\theta} = h$ as we desired