2. Mathematical Modelling of Motion

Motions observed on Earth (apart from those on molecular or inter-galactic scales) are governed by laws first formulated by Newton. In particular

$$\text{Force } \underline{F} = \text{Rate of change of momentum}$$
$$= \frac{d}{dt}(m\underline{v})$$

for a body of mass m and velocity $\underline{v}$.

If the mass is constant, then

$$\underline{F} = m\frac{d\underline{v}}{dt} \qquad (\text{mass} \times \text{acceleration})$$

Example Cricket

A cricketer throws a ball at an angle of elevation α to the wicketkeeper who is standing 50 metres away.

How long is the ball in flight?

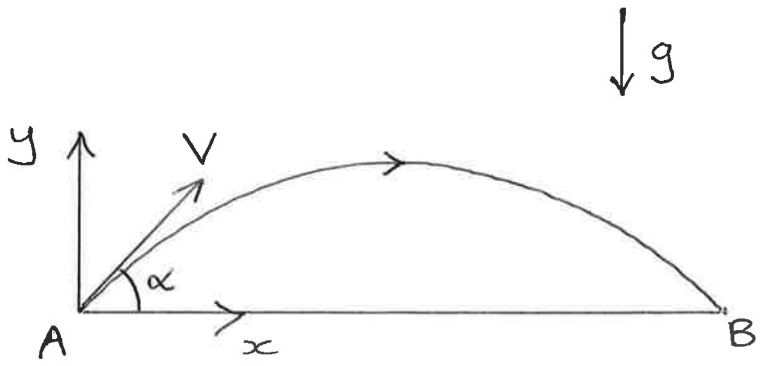
At what speed does the cricketer release the ball?

What is its maximum height and minimum speed?

Assume air resistance can be neglected.

Assume wicketkeeper catches ball at same height that it leaves the thrower's hand.

Let V be the speed of release.



Path of ball

Acceleration due to gravity is g ($\approx 9.8\,m/sec^2$ at Earth's surface)

The only force acting on the ball is its own weight, mg, due to gravity, i.e. $\underline{F} = (0, -mg)$ in 2D.

Position of ball is $\underline{r} = (x, y)$ (x, y in metres)

Velocity of ball is $\underline{v} = \frac{d\underline{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

Acceleration of ball is $\frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)$

By Newton's law

$$(0, -mg) = m\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)$$

In components

$$\frac{d^2x}{dt^2} = 0 \qquad (1)$$

$$\frac{d^2y}{dt^2} = -g \qquad (2)$$

Initial conditions

$$x = 0, \quad y = 0, \quad \frac{dx}{dt} = V\cos\alpha, \quad \frac{dy}{dt} = V\sin\alpha \quad \text{at } t = 0 \qquad (3)$$

We solve (1) and (2) subject to (3) to find x and y (the position of the ball) at any time t.

Integrate (1): $\quad \dfrac{dx}{dt} = c_1 \quad$ (const)

Apply (3) $\Rightarrow \quad c_1 = V\cos\alpha$

$$\therefore \quad \frac{dx}{dt} = V\cos\alpha$$

Integrate again: $\quad x = tV\cos\alpha + c_2$

Apply (3) $\Rightarrow \quad c_2 = 0$

$$\therefore \quad \underline{x = tV\cos\alpha} \qquad (4)$$

Integrate (2): $\quad \dfrac{dy}{dt} = -gt + c_3$

Apply (3) $\Rightarrow \quad c_3 = V\sin\alpha$

$$\therefore \quad \frac{dy}{dt} = -gt + V\sin\alpha$$

Integrate again: $\quad y = -\frac{1}{2}gt^2 + tV\sin\alpha + c_4$

Apply (3) $\Rightarrow \quad c_4 = 0$

$$\therefore \quad \underline{y = -\tfrac{1}{2}gt^2 + tV\sin\alpha} \qquad (5)$$

(4) and (5) give the position of the ball at any time t.

We know that at B, $x = 50$

$\therefore$ (4) $\Rightarrow \quad 50 = tV\cos\alpha$

$$\therefore \quad \underline{t = \frac{50}{V\cos\alpha} \text{ is the time of flight}}$$

Since $y = 0$ at B, (5) now gives

$$0 = -\frac{1}{2}g\frac{2500}{V^2\cos^2\alpha} + \frac{50}{V\cos\alpha}V\sin\alpha$$

$$\therefore \quad V^2 = \frac{25g}{\sin\alpha\cos\alpha} = \frac{50g}{\sin 2\alpha}$$

$$\therefore \quad \underline{V = \left(\frac{50g}{\sin 2\alpha}\right)^{1/2} \text{ is the speed of release}}$$

(6)

$\therefore$ Time of flight becomes

$$t = \frac{50(\sin 2\alpha)^{\frac{1}{2}}}{(50g)^{1/2}\cos\alpha} = 10\left(\frac{\tan\alpha}{g}\right)^{1/2} \qquad (7)$$

Max. height occurs when $\frac{dy}{dt} = 0$ so $t = \frac{V\sin\alpha}{g}$

and then

$$y = -\tfrac{1}{2}g\left(\frac{V\sin\alpha}{g}\right)^2 + \left(\frac{V\sin\alpha}{g}\right)V\sin\alpha = \frac{V^2\sin^2\alpha}{2g} = \frac{25}{2}\tan\alpha$$

Speed of ball at a general time is

$$v = |\underline{v}| = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{1/2}$$

$$\therefore \quad v^2 = V^2\cos^2\alpha + (V\sin\alpha - gt)^2$$

$$= V^2 - (2V\sin\alpha)gt + g^2t^2$$

$$\therefore \quad \frac{d(v^2)}{dt} = (-2V\sin\alpha)g + 2g^2t$$

$$= 0 \quad \text{when} \quad t = \frac{V\sin\alpha}{g} \qquad \text{i.e at max. height}$$

$\therefore$ Min. speed is when $\frac{dy}{dt} = 0$ and

$$v = \frac{dx}{dt} = V\cos\alpha$$

Note that to effect the fastest run out, the cricketer should throw the ball with the minimum elevation α (since $(\tan\alpha)^{1/2}$ in (7) is an increasing function of α for $0 < \alpha < \frac{\pi}{2}$).

However, the lower α is, the higher the speed of release (6) needs to be, with $V \to \infty$ as $\alpha \to 0$.

The lowest speed of release needed is when $\sin 2\alpha$ is max, i.e. $\alpha = \pi/4$.

Potential Energy and Kinetic Energy

For motion under gravity, as above, we have

$$\frac{d^2x}{dt^2} = 0 \quad , \quad \frac{d^2y}{dt^2} = -g$$

or, if we introduce the notation $\cdot$ to represent $\frac{d}{dt}$,

$$\ddot{x} = 0 \quad , \quad \ddot{y} = -g$$

One integration of each can be done using

$$\ddot{x}\,\dot{x} = 0 \Rightarrow \tfrac{1}{2}\dot{x}^2 = \text{Const}$$

$$\ddot{y}\,\dot{y} = -g\dot{y} \Rightarrow \tfrac{1}{2}\dot{y}^2 = -gy + \text{Const}$$

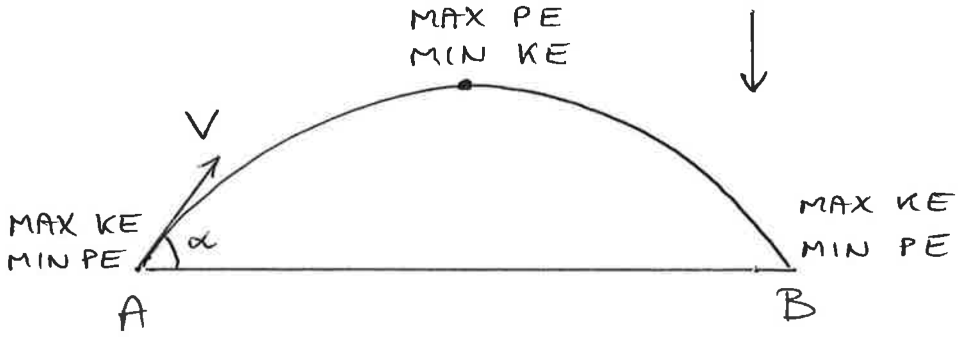
Add $\Rightarrow$

$$\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) = -gy + \text{Const}$$

$$\therefore \quad \underbrace{\tfrac{1}{2}mv^2}_{\substack{\text{Kinetic Energy} \\ \text{(Energy due to motion)}}} + \underbrace{mgy}_{\substack{\text{Potential Energy} \\ \text{(Energy achieved by doing} \\ \text{work against force field)}}} = \text{Const}$$

The energy of the motion is made up of the KE associated with the speed v of the mass m and the PE achieved by doing work against the force field (which $m \times g \times$ height, for gravity).

The above result expresses the fact that the total energy (KE + PE) is conserved during the motion (and explains why the min KE occurs at the top of the trajectory in our example, where the PE is greatest). This is true for any motion under gravity, which is therefore said to be a <u>conservative</u> force field.

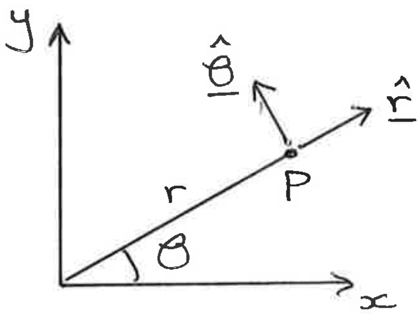


Previous example

A familiar non-conservative force is friction, where KE can be turned into heat, for example, rather than PE.

Velocity and acceleration in plane polars (r, θ)

$\hat{\underline{r}}$, $\hat{\underline{\theta}}$ are unit vectors and change direction as P changes.

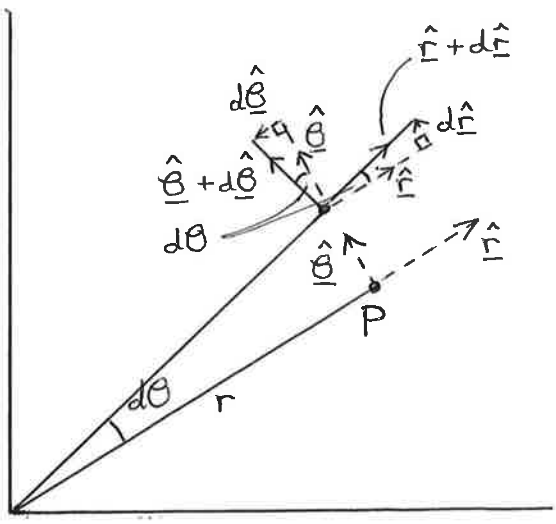


Position $\underline{r} = r\hat{\underline{r}}$

Velocity $\underline{v} = \dfrac{d\underline{r}}{dt}$

We derive expressions for velocity $\underline{v}$ and acceleration $\underline{a}$ in polar components, so that we can analyse problems involving rotation, such as a pendulum.

Consider a rotation of the radius vector through a small angle $d\theta$:



We see that

$$d\hat{\underline{r}} = \hat{\underline{\theta}}\, d\theta$$

$$d\hat{\underline{\theta}} = -\hat{\underline{r}}\, d\theta$$

Thus

$$\frac{d\hat{\underline{r}}}{d\theta} = \hat{\underline{\theta}} \quad , \quad \frac{d\hat{\underline{\theta}}}{d\theta} = -\hat{\underline{r}} \qquad (1)$$

Velocity

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}(r\hat{\underline{r}}) = \frac{dr}{dt}\hat{\underline{r}} + r\frac{d\hat{\underline{r}}}{dt}$$

$$= \dot{r}\hat{\underline{r}} + r\frac{d\hat{\underline{r}}}{d\theta}\frac{d\theta}{dt}$$

$$\therefore \quad \underline{\underline{v} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}} \qquad (2)$$

Acceleration

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}})$$

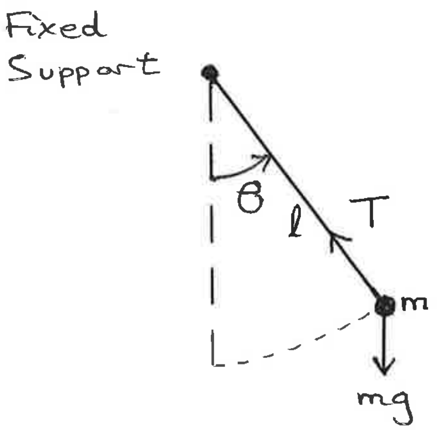
$$= \ddot{r}\hat{\underline{r}} + \dot{r}\frac{d\hat{\underline{r}}}{dt} + \dot{r}\dot{\theta}\hat{\underline{\theta}} + r\ddot{\theta}\hat{\underline{\theta}} + r\dot{\theta}\frac{d\hat{\underline{\theta}}}{dt}$$

$$= \ddot{r}\hat{\underline{r}} + \dot{r}\frac{d\hat{\underline{r}}}{d\theta}\frac{d\theta}{dt} + \dot{r}\dot{\theta}\hat{\underline{\theta}} + r\ddot{\theta}\hat{\underline{\theta}} + r\dot{\theta}\frac{d\hat{\underline{\theta}}}{d\theta}\frac{d\theta}{dt}$$

$$\therefore \quad \underline{\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}}} \qquad (3)$$

Result (3) allows us to write down Newton's law for problems expressed in terms of polar coordinates.

Example Simple Pendulum



Wire of length l

Tension T

Mass m

We assume the wire remains taut, so $r = l$, and that it is light (no mass).

Resolve Newton's law for the mass m.

Along the wire: $\quad mg\cos\theta - T = -ml\dot{\theta}^2$

Perpendicular to wire: $\quad -mg\sin\theta = ml\ddot{\theta}$

Here we have expressed mass × acceleration on r.h.s. using (3) with r constant and equal to l.

The first result just gives the tension T.

The solution for θ as a function of time t must be found from the second result

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

where $\omega = (g/l)^{1/2}$.

$$\ddot{\theta}\dot{\theta} + \omega^2(\sin\theta)\dot{\theta} = 0$$

Integrate: $$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = C \quad , \quad \text{const} \qquad (1)$$

This is the energy equation

$$\underbrace{\frac{1}{2}ml^2\dot{\theta}^2}_{\substack{\text{KE} \\ (\frac{1}{2}mv^2)}} - \underbrace{mgl\cos\theta}_{\substack{\text{PE} \\ (mg \times \text{height, taking reference level at } \theta = \frac{\pi}{2})}} = \text{Const}$$

expressing the fact that the system is conservative.

Initial conditions: suppose the pendulum is released from rest at angle $-\alpha$, so that

$$\theta = -\alpha \quad , \quad \dot{\theta} = 0 \quad \text{at} \quad t = 0.$$

Then in (1), $C = -\omega^2 \cos(-\alpha) = -\omega^2 \cos\alpha$ so

$$\dot{\theta}^2 - 2\omega^2 \cos\theta = -2\omega^2 \cos\alpha \qquad (2)$$

Separate variables:

$$\frac{d\theta}{(\cos\theta - \cos\alpha)^{1/2}} = \sqrt{2}\,\omega \, dt$$

Integrate

$$\int_{-\alpha}^{\theta} \frac{d\theta}{(\cos\theta - \cos\alpha)^{1/2}} = \sqrt{2}\,\omega\, t + D \qquad (D \text{ const})$$

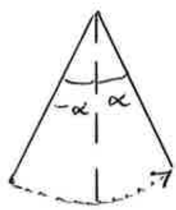
But $\theta = -\alpha$ at $t = 0$ so $D = 0$.

$$\therefore \quad t = \frac{1}{\sqrt{2}\,\omega} \int_{-\alpha}^{\theta} \frac{d\theta}{(\cos\theta - \cos\alpha)^{1/2}}$$

Unfortunately the integral cannot be evaluated exactly in terms of elementary functions, but in principle this determines θ as a function of t.

Since from (2) we will have $\dot{\theta} = 0$ again when $\theta = \alpha$, one half swing is completed in time

$$\frac{1}{\sqrt{2}\,\omega} \int_{-\alpha}^{\alpha} \frac{d\theta}{(\cos\theta - \cos\alpha)^{1/2}}$$



and the <u>period</u> of the pendulum (ie the time to complete one full swing) is

$$P = \sqrt{\frac{2}{\omega^2}} \int_{-\alpha}^{\alpha} \frac{d\theta}{(\cos\theta - \cos\alpha)^{1/2}} \qquad (3)$$

Small swings : simple harmonic motion

If α is small ($\alpha \ll 1$) the amplitude of the swing remains small, so both θ and α are small. Then $\sin\theta \approx \theta$ so the equation $\ddot{\theta} + \omega^2 \sin\theta = 0$ becomes

$$\ddot{\theta} + \omega^2\theta = 0$$

to a first approximation. This has general solution

$$\theta = A\cos(\omega t) + B\sin(\omega t)$$

which is referred to as <u>simple harmonic motion</u>.

If we apply our initial conditions $\theta = -\alpha$, $\dot{\theta} = 0$ at $t = 0$ we can find the constants A and B:

$$B = 0, \quad A = -\alpha$$

Thus

$$\theta = -\alpha \cos \omega t$$

This SHM has <u>frequency</u> ω, <u>amplitude</u> α and <u>period</u> $P = \frac{2\pi}{\omega}$.

The period is the time for one complete swing, the amplitude measures the size of the swing and the frequency measures the number of swings per unit time.

We can also confirm that $P \to \frac{2\pi}{\omega}$ as $\alpha \to 0$ using (3).

We return to the case where the swings are not small later.