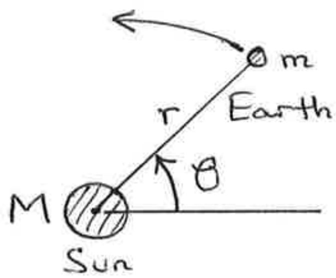
Example Planetary Orbits

Consider the Earth's motion around the Sun, ignoring the effect of other planets and assuming the Sun is fixed.



Assume motion is 2D and use plane polars taking the Sun at the origin.

On the length scales involved the Sun and Earth are point masses and have a mutual attraction governed by Newton's law of gravitation. Thus the Earth is attracted to the Sun by a force of magnitude

$$\frac{GMm}{r^2} \qquad \text{(inverse square law)}$$

where G is called the gravitational constant.

We now apply Newton's law to the Earth, resolving in the r and θ directions:

Radial: $$-\frac{GMm}{r^2} = m(\ddot{r} - r\dot{\theta}^2) \qquad (1)$$

Azimuthal: $$0 = m(2\dot{r}\dot{\theta} + r\ddot{\theta}) \qquad (2)$$

Unlike the pendulum, we cannot assume r is constant.

$$(2) \Rightarrow \qquad 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0$$

$$\frac{d}{dt}(r^2)\dot{\theta} + r^2\frac{d}{dt}(\dot{\theta}) = 0$$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

Integrate: $\qquad r^2\dot{\theta} = h \qquad (\text{const}) \qquad (3)$

h is called the angular momentum (radius × angular speed $r\dot{\theta}$) per unit mass and is constant for the Earth (and other planets).

To integrate (1), first set $\mu = GM$ so that

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \qquad (4)$$

and introduce

$$u = \frac{1}{r}$$

Then since $r = u^{-1}$,

$$\dot{r} = -u^{-2}\frac{du}{d\theta}\dot{\theta} = -r^2\dot{\theta}\frac{du}{d\theta} = -h\frac{du}{d\theta}$$

$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2}$$

so

$$\ddot{r} - r\dot{\theta}^2 = -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) \quad \text{and } (4)$$

becomes

$$\underline{\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}}$$

This has general solution (comp soln + particular soln)

$$u = \frac{\mu}{h^2} + A\cos\theta + B\sin\theta \qquad (A, B \text{ consts})$$

or alternatively

$$u = \frac{\mu}{h^2} + C\cos(\theta - \theta_0) \qquad (C, \theta_0 \text{ consts})$$

Thus

$$\frac{1}{r} = \frac{\mu}{h^2}\left(1 + \frac{Ch^2}{\mu}\cos(\theta-\theta_0)\right)$$

We can choose our polar coordinates so that $\theta_0 = 0$ and then

$$\frac{1}{r} = \frac{\mu}{h^2}\left(1 + \frac{Ch^2}{\mu}\cos\theta\right) \qquad (5)$$

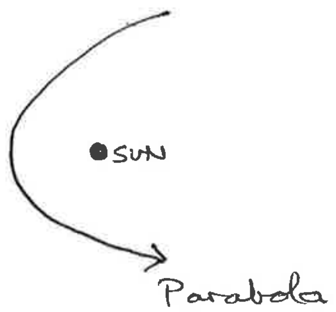
This is the equation of a conic in polars with focus at the origin $r=0$ (the standard form is $\frac{1}{r} = \frac{1}{l}(1+e\cos\theta)$ where l is the semi-latus rectum and e is the eccentricity).

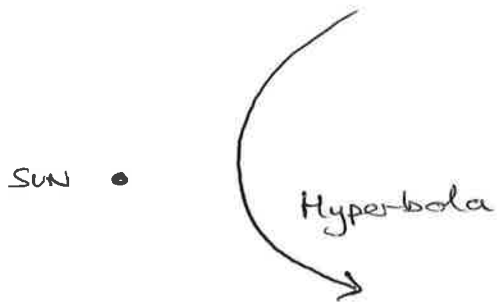
For $e = \frac{Ch^2}{\mu} < 1$ we have an ellipse.

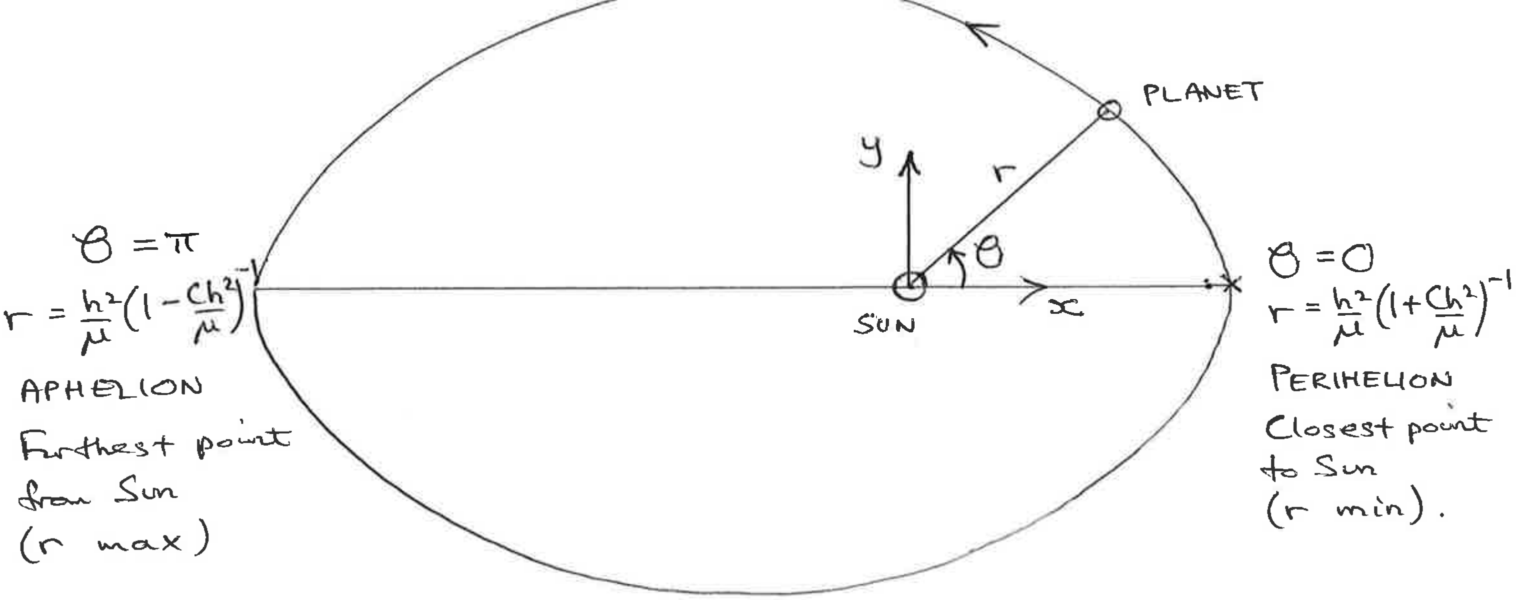
For $e = \frac{Ch^2}{\mu} = 1$ we have a parabola.

For $e = \frac{Ch^2}{\mu} > 1$ we have a hyperbola.

Thus the type of motion depends on the value of C which in turn depends on the initial conditions (which determines the total energy). For all the planets in the solar system $C < \frac{\mu}{h^2}$ so they travel in ellipses with the Sun at one focus. Bodies with $\frac{Ch^2}{\mu} \geq 1$ do not remain in the solar system. Comets such as Haley's comet have large elliptical paths.







To show that (5) is an ellipse when $\frac{Ch^2}{\mu} < 1$:

$$(5) \Rightarrow \qquad 1 = \frac{\mu r}{h^2} + Cr\cos\theta$$

Change to Cartesians $x = r\cos\theta,\ y = r\sin\theta \Rightarrow$

$$1 = \frac{\mu}{h^2}(x^2+y^2)^{\frac{1}{2}} + Cx$$

$$\Rightarrow (1 - Cx)^2 = \frac{\mu^2}{h^4}(x^2+y^2)$$

$$\therefore \quad x^2\left(1 - \frac{C^2h^4}{\mu^2}\right) + \frac{2Ch^4}{\mu^2}x + y^2 = \frac{h^4}{\mu^2}$$

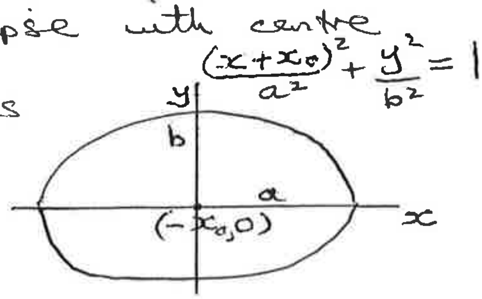
Complete square

$$\left(1 - \frac{C^2h^4}{\mu^2}\right)\left(x + \frac{Ch^4}{\mu^2\left(1 - \frac{C^2h^4}{\mu^2}\right)}\right)^2 + y^2 = \frac{h^4}{\mu^2} + \frac{C^2h^8}{\mu^4\left(1 - \frac{C^2h^4}{\mu^2}\right)}$$

$$\underline{\left(1 - \frac{C^2h^4}{\mu^2}\right)\left(x + \frac{Ch^4}{\mu^2 - C^2h^4}\right)^2 + y^2 = \frac{h^4}{\mu^2 - C^2h^4}}$$

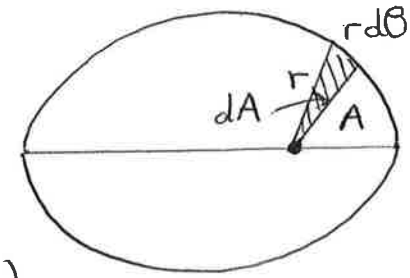
Thus if $\frac{Ch^2}{\mu} < 1$ we have an ellipse with centre at $\left(-\frac{Ch^4}{\mu^2 - C^2h^4}, 0\right)$ and semi-axes

$$a = \frac{h^2\mu}{\mu^2 - C^2h^4}, \quad b = \frac{h^2}{(\mu^2 - C^2h^4)^{1/2}}$$



Period of orbit

Since $\quad \dot{\theta} = \dfrac{h}{r^2} \quad$ we have



$$r^2 d\theta = h\,dt$$

But if A is the area swept out in time t,

$$dA = \tfrac{1}{2} r^2 d\theta \qquad \text{(from diagram)}$$

$$\therefore \quad 2\,dA = h\,dt$$

Integrate $\therefore \quad \dfrac{2A}{h} = t \qquad$ (assuming constant of integration fixed by $A = 0$ at $t = 0$)

Thus the area swept out by the planet is proportional to the time taken.

Period $\quad P = \dfrac{2}{h} \times$ Area of ellipse

$$= \frac{2}{h} \pi a b \qquad \text{where } a, b \text{ are the semi-axes.}$$

$$= \frac{2\pi}{h} \frac{h^2 \mu h^2}{(\mu^2 - C^2 h^4)^{3/2}}$$

$$\therefore \quad P = \frac{2\pi h^3 \mu}{(\mu^2 - C^2 h^4)^{3/2}}$$

Kepler's Laws

Newton's mathematical theory was consistent with observations made by the astronomer Kepler, who observed that

(i) Each planet describes an ellipse with the Sun at one focus,

(ii) The radius vector from the Sun to a planet sweeps out equal areas in equal times,

(iii) The squares of the periodic times of the planets are proportional to the cubes of the major semi-axes of their orbits.

We have already seen (i) and (ii). (iii) also follows because

$$a = \frac{h^2\mu}{\mu^2 - C^2h^4}$$

so the period P can be written as

$$P = \frac{2\pi a^{3/2}}{\mu^{1/2}}$$

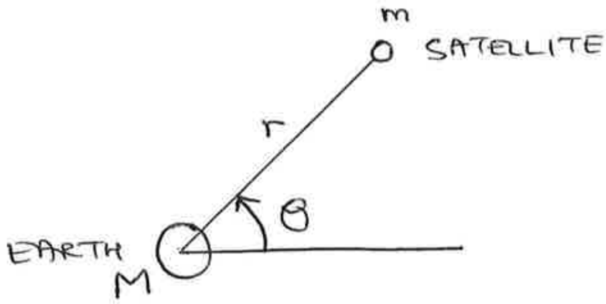
Since $\mu = GM$ is an absolute constant ($G = 6.67 \times 10^{-11}\ m^3\,kg^{-1}\,s^{-2}$, $M = 1.993 \times 10^{30}$ kg) for all planets in the solar system it follows that

$$P^2 = \frac{4\pi^2}{\mu} a^3$$

i.e. P^2 is proportional to a^3.

<u>Example</u> Stationary orbits

A satellite is to be launched into a stationary circular orbit above the Earth. What speed should be given to the satellite? What height should the orbit be? How many satellites would be needed to ensure contact with all points on the equator?



Previous theory applies but now M is mass of Earth and m is mass of satellite.

We need $C = 0$ for a circular orbit, and then

$$\frac{1}{r} = \frac{\mu}{h^2}$$

$$\therefore \quad r = \frac{h^2}{\mu} = a \qquad (\text{const})$$

Also $r^2\dot{\theta} = h \Rightarrow \dot{\theta} = \frac{h}{r^2} = \frac{h}{\left(\frac{h^2}{\mu}\right)^2} = \frac{\mu^2}{h^3}$

so period $P = \int_0^P dt = \int_0^{2\pi} \frac{h^3}{\mu^2} d\theta = \frac{2\pi h^3}{\mu^2} = \frac{2\pi a^{3/2}}{\mu^{1/2}}$

For stationary orbit we need

$$P = 24 \text{ hours} = 8.64 \times 10^4 \text{ secs}$$

Also $\mu = GM = \underbrace{(6.67 \times 10^{-11})}_{\substack{G \\ (m^3 kg^{-1} s^{-2})}} \underbrace{(5.976 \times 10^{24})}_{\text{Mass of Earth (kg)}}$

$$\therefore \quad 8.64 \times 10^4 = 2\pi a^{3/2} (6.67 \times 5.976 \times 10^{13})^{-1/2}$$

$$\Rightarrow \quad \underline{a = 4.22 \times 10^4 \text{ km}}$$

Equatorial radius of Earth is $R = 0.64 \times 10^4$ km.

$\therefore$ Height of orbit above surface of Earth is 3.58×10^4 km.

Speed of satellite is $r\dot{\theta} = a\dot{\theta} = a\frac{h}{a^2} = \frac{(a\mu)^{\frac{1}{2}}}{a} = \left(\frac{\mu}{a}\right)^{\frac{1}{2}}$

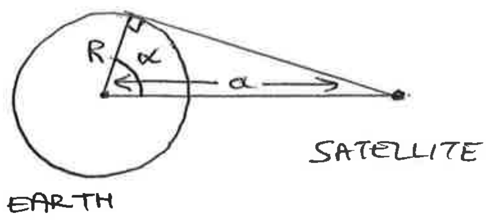
$$= \left(\frac{6.67 \times 5.976 \times 10^{13}}{4.22 \times 10^7}\right)^{1/2}$$

$$= 3.07 \times 10^3 \text{ m/sec}$$

$$= 11064 \text{ km/hour}$$

Angle α is given by

$$\cos\alpha = \frac{R}{a}$$



$\Rightarrow$

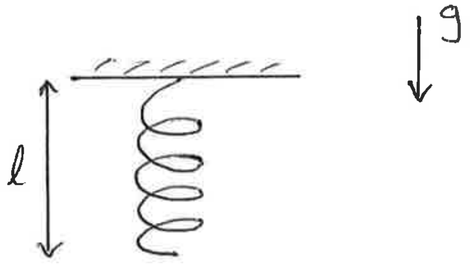
$$\alpha = \cos^{-1}\left(\frac{0.64\times10^4}{4.22\times10^4}\right)$$

$$= 81.28^\circ$$

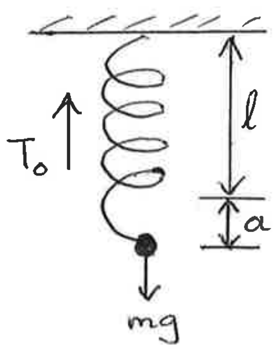
$\therefore$ One satellite covers $\approx 162^\circ$ so 3 satellites needed to cover entire surface of equator.

Example Springs and suspension systems

The tension in an elastic wire or spring is proportional to its extension (or compression for a spring) provided the extension is not too large (i.e. within the 'elastic limit'). This is known as Hooke's Law.



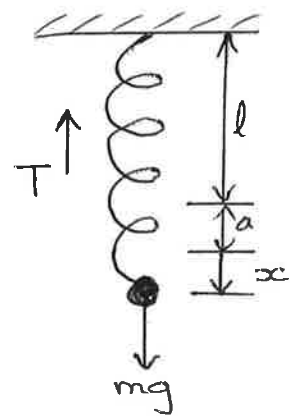
Light spring, natural length l



Mass m attached; in equilibrium

$$mg = T_0 = \lambda_0 a$$

where λ_0 is the 'spring constant'.



Deflected down a further distance x

$$mg - T = m\ddot{x}$$

by Newton's law where

$$T = \lambda_0(a+x)$$

Thus motion of the mass is governed by

$$mg - \lambda_0(a+x) = m\ddot{x}$$

But $mg = \lambda_0 a$ so

$$\ddot{x} + \left(\frac{\lambda_0}{m}\right)x = 0$$

The spring 'modulus' λ is defined by $\lambda = \lambda_0 l$ and then

$$\ddot{x} + \omega^2 x = 0 \qquad (1)$$

where $\omega = \left(\frac{\lambda}{ml}\right)^{1/2}$.

Thus we have SHM of frequency ω and period $\frac{2\pi}{\omega}$.

Energy: one integration gives from (1)

$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2 = \text{Const}$$

$$\Rightarrow \quad \underbrace{\tfrac{1}{2}m\dot{x}^2}_{KE} + \underbrace{\tfrac{1}{2}m\omega^2 x^2}_{\text{PE due to extension of spring}} = \text{Const}$$

The system is conservative, with $KE + PE = \text{Const}$.

Initial conditions will specify x and $\dot{x}$ at $t=0$, allowing the solution of (1) to be found completely in the form $x = A\cos\omega t + B\sin\omega t$ with A, B fixed by the initial conditions.