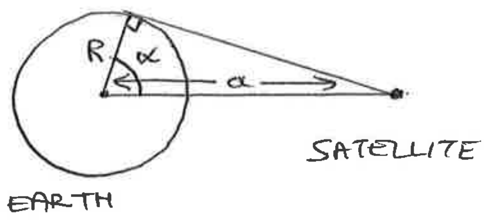
Angle α is given by



$$\cos\alpha = \frac{R}{a}$$

$$\Rightarrow$$

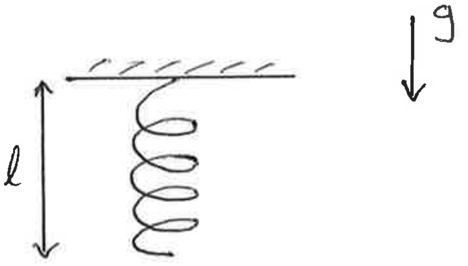
$$\alpha = \cos^{-1}\left(\frac{0.64\times10^4}{4.22\times10^4}\right)$$

$$= 81.28^\circ$$

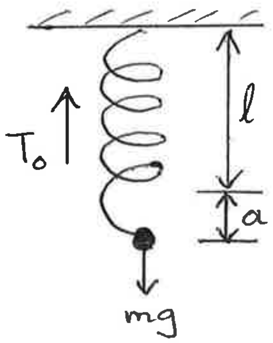
$\therefore$ One satellite covers $\approx 162^\circ$ so 3 satellites needed to cover entire surface of equator.

Example Springs and suspension systems

The tension in an elastic wire or spring is proportional to its extension (or compression for a spring) provided the extension is not ~~too~~ large (i.e. within the 'elastic limit'). This is known as Hooke's Law.



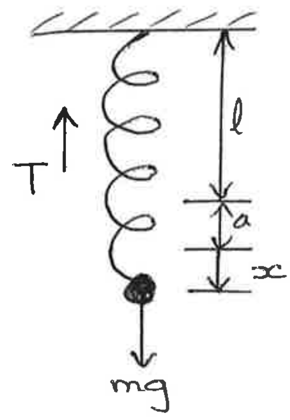
Light spring, natural length l



Mass m attached; in equilibrium

$$mg = T_0 = \lambda_0 a$$

where λ_0 is the 'spring constant'.



Deflected down a further distance x

$$mg - T = m\ddot{x}$$

by Newton's law where

$$T = \lambda_0(a+x)$$

Thus motion of the mass is governed by

$$mg - \lambda_0(a+x) = m\ddot{x}$$

But $mg = \lambda_0 a$ so

$$\ddot{x} + \left(\frac{\lambda_0}{m}\right)x = 0$$

The spring 'modulus' λ is defined by $\lambda = \lambda_0 l$ and then

$$\ddot{x} + \omega^2 x = 0 \qquad (1)$$

where $\omega = \left(\frac{\lambda}{ml}\right)^{1/2}$.

Thus we have SHM of frequency ω and period $\frac{2\pi}{\omega}$.

Energy: one integration gives from (1)

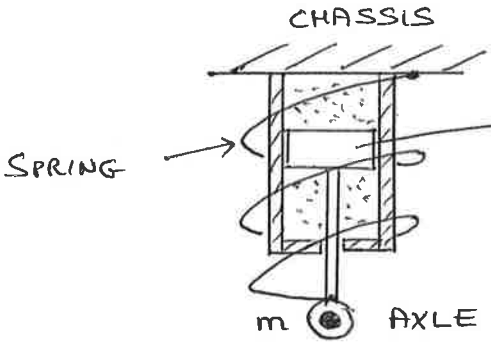
$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2 = \text{Const}$$

$$\Rightarrow \quad \underbrace{\tfrac{1}{2}m\dot{x}^2}_{\text{KE}} + \underbrace{\tfrac{1}{2}m\omega^2 x^2}_{\text{PE due to extension of spring}} = \text{Const}$$

The system is conservative, with $KE + PE = \text{Const}$.

Initial conditions will specify x and $\dot{x}$ at $t=0$, allowing the solution of (1) to be found completely in the form $x = A\cos\omega t + B\sin\omega t$ with A, B fixed by the initial conditions.

Example   Frictional damping in suspension systems



Car suspension system

Typically suspension system is damped by the movement of a piston through hydraulic fluid, to avoid too much 'bounce'.

Frictional force due to piston is roughly proportional to the speed of the piston, and opposite to the direction of motion. Thus previous example becomes

$$mg - \lambda_0(a+x) - k_0\dot{x} = m\ddot{x}$$

where $k_0\dot{x}$ is the damping force. Thus

$$\ddot{x} + \underbrace{2k\dot{x}}_{\text{DAMPING}} + \underbrace{\omega^2 x}_{\text{SPRING}} = 0 \qquad (2)$$

where $\omega^2 = \lambda/ml$ and $k = k_0/2m$.

General solution of (2) is found by trying $x = Ae^{mt}$ to get

$$m^2 + 2km + \omega^2 = 0$$

so

$$m = -k \pm (k^2-\omega^2)^{1/2}$$

Thus there are 3 cases to consider, as follows.

(i) Heavy damping $k > \omega$

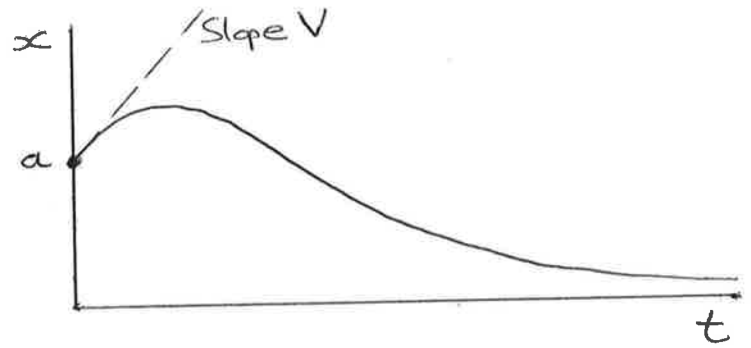
2 real roots $\quad m_1 = -k + (k^2 - \omega^2)^{1/2} = -p$, say.

$\qquad\qquad\quad m_2 = -k - (k^2 - \omega^2)^{\frac{1}{2}} = -q$, say

and
$$x = Ae^{-pt} + Be^{-qt}$$

Thus the system is exponentially damped due to large friction, and $x \to 0$ as $t \to \infty$.

If $x = a$, $\dot{x} = V$ at $t = 0$ then A, B can be found and
$$x = \frac{1}{q-p}\left((V+qa)e^{-pt} - (V+pa)e^{-qt} \right)$$



Typical solution

(ii) Critical damping $k = \omega$

Repeated roots $\quad m_1 = m_2 = -k \quad$ so
$$x = e^{-kt}(A + Bt)$$

Again the solution is exponentially damped, but not as rapidly as when $k > \omega$ (due to t term).

(iii) Light damping $k < \omega$

Here

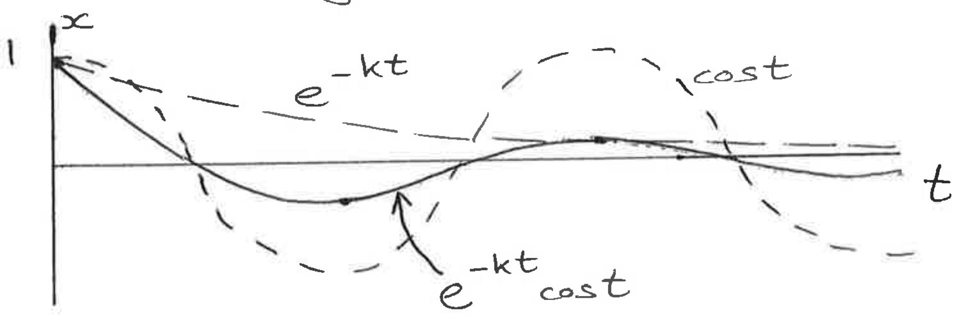
$$m_{1,2} = -k \pm i(\omega^2 - k^2)^{1/2}$$

so the real solution is

$$x = e^{-kt}\left(\bar{A}\cos\left[(\omega^2-k^2)^{\frac{1}{2}}t\right] + \bar{B}\sin\left[(\omega^2-k^2)^{\frac{1}{2}}t\right]\right)$$

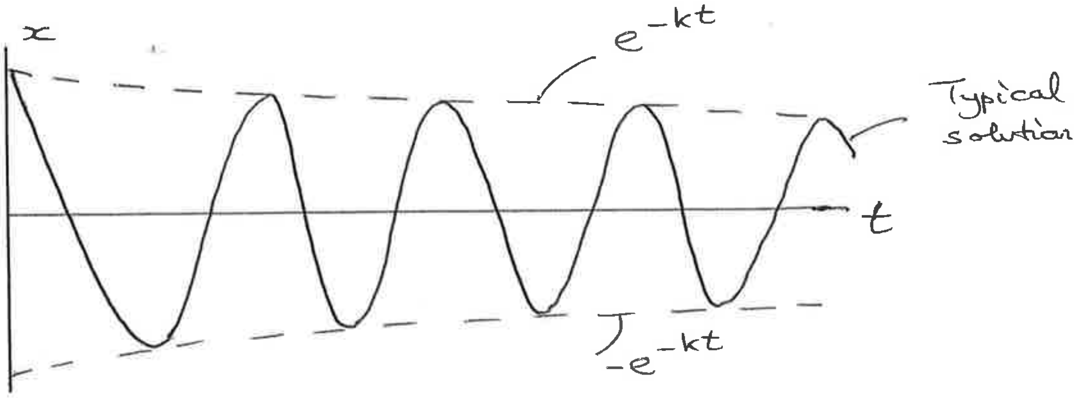
The motion now has oscillations but is still damped exponentially to reach $x = 0$ as $t \to \infty$. The smaller k, the slower the damping.

Typical solution e.g $x = e^{-kt}\cos t$



The oscillation is said to be modulated in amplitude by e^{-kt}

Very light damping ($k \ll \omega$):



Slow decay as t increases.

Energy:

For no damping ($k=0$) $\quad \ddot{x} + \omega^2 x = 0$

gives $\quad E \equiv \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2 = \text{Const}$

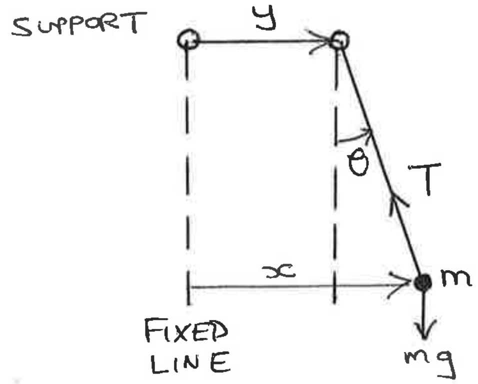
With damping ($k>0$) $\quad \ddot{x} + 2k\dot{x} + \omega^2 x = 0$

we can no longer show that $E = \text{Const}$. In fact we have seen that $\begin{cases} \dot{x} \to 0 \\ x \to 0 \end{cases}$ as $t \to \infty$, so $E \to 0$ as $t \to \infty$. The energy is dissipated by friction (into heat, typically). With friction included the system is not conservative.

Forced oscillations

Cars, planes etc are subject to forced oscillations due to their engines. As a simple model we consider a system with natural frequency ω forced by another frequency Ω.

Example $\quad$ Simple pendulum with moving support



Support moves with displacement $y(t)$. Tension T.

Assume small swings $\theta \ll 1$.

Apply Newton's law to mass m:

Vertically: $\quad T\cos\theta \simeq mg \quad$ (neglect vertical acceleration)

Horizontally: $\quad -T\sin\theta = m\ddot{x}$

Since $\theta \ll 1$, $\cos\theta \approx 1 \Rightarrow T \approx mg$

Also $\sin\theta = \dfrac{x-y}{l}$ so to a first approximation

$$m\ddot{x} = -\frac{mg}{l}(x-y)$$

$$\therefore \quad \ddot{x} + \omega^2 x = \omega^2 y \qquad \text{where } \omega = \left(\frac{g}{l}\right)^{1/2}$$

is the natural frequency of the pendulum.

If we also include friction (e.g. air resistance) then

$$\ddot{x} + 2k\dot{x} + \omega^2 x = \omega^2 y$$

Consider forcing with frequency Ω, so that

$$y = \frac{al}{g}\cos\Omega t$$

and then

$$\underline{\ddot{x} + 2k\dot{x} + \omega^2 x = a\cos\Omega t}$$

General solution = Comp. Soln + Particular Soln

We have already found the comp. solution (where rhs is set to zero) so we now consider the particular solution forced by the term on the rhs.

Try $\quad x = \bar{A}\cos\Omega t + \bar{B}\sin\Omega t$

Substitute in to get

$$\begin{cases} (\omega^2-\Omega^2)\bar{A} + 2k\Omega\bar{B} = a \\ (\omega^2-\Omega^2)\bar{B} - 2k\Omega\bar{A} = 0 \end{cases}$$

Solving simultaneously gives

$$\bar{A} = \frac{a(\omega^2-\Omega^2)}{(\omega^2-\Omega^2)^2+4k^2\Omega^2}, \qquad \bar{B} = \frac{2k\Omega a}{(\omega^2-\Omega^2)^2+4k^2\Omega^2}$$

so the general solution for x is

$$x = Ae^{-pt} + Be^{-qt} + \frac{a(\omega^2-\Omega^2)\cos\Omega t + 2k\Omega a\sin\Omega t}{(\omega^2-\Omega^2)^2 + 4k^2\Omega^2}$$

with A and B to be fixed by initial conditions at $t=0$.

As $t \to \infty$, the solution will tend to the particular solution and the effect of the initial conditions will be lost. The part of the solution depending on the initial conditions is called the transient part of the solution.

We consider the non-transient part only.

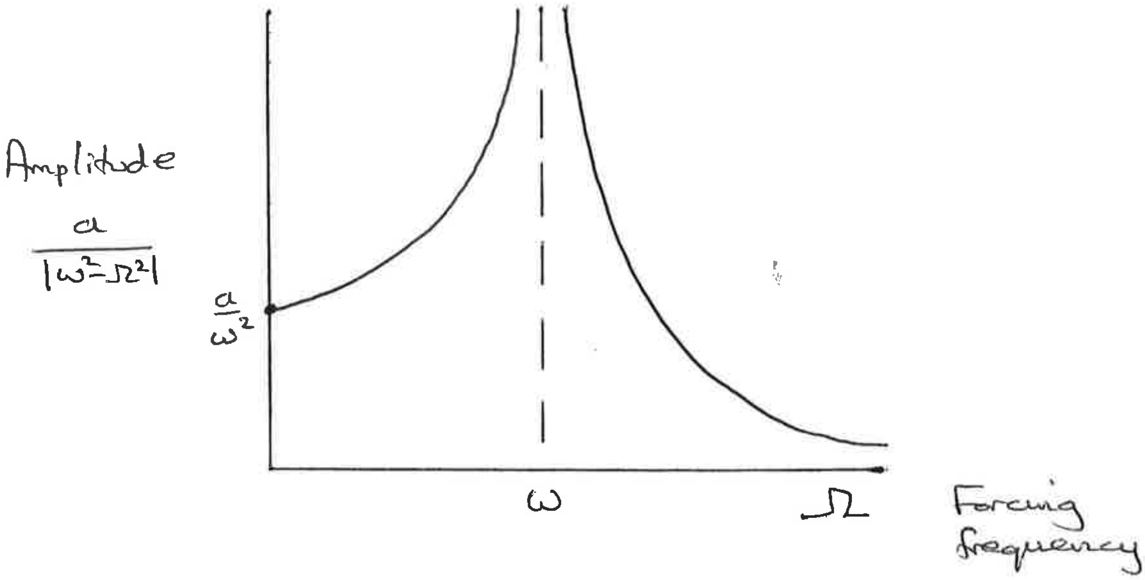
(i) No damping $k=0$

$$x = \frac{a}{\omega^2-\Omega^2}\cos\Omega t$$

If $\Omega < \omega$ (forced freq < natural freq) displacement of mass (x) is in phase with displacement of support (y).

If $\Omega > \omega$ (forced freq > natural freq) displacement of mass (x) is π out of phase with displacement of support (y)

If $\Omega = \omega$, amplitude $\frac{a}{|\omega^2-\Omega^2|} \to \infty$ and resonance is said to occur. In practice this does not happen because system fails, nonlinear effects become important (i.e. $\theta \ll 1$ no longer valid) or friction is present ($k \neq 0$).



(ii) Damping $k > 0$

Forced solution is now

$$x = \frac{a(\omega^2-\Omega^2)\cos\Omega t + 2ak\Omega\sin\Omega t}{(\omega^2-\Omega^2)^2 + 4k^2\Omega^2}$$

or

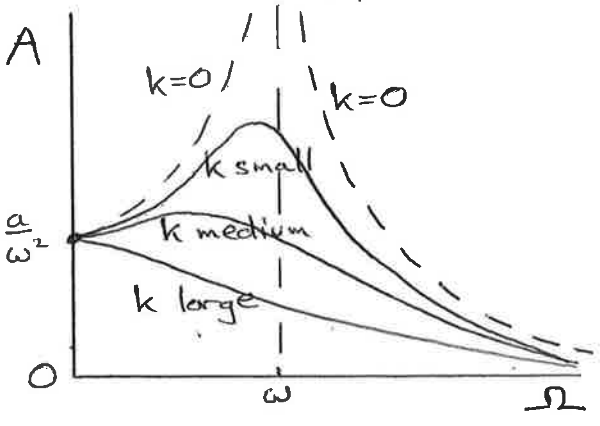
$$x = A(\cos\epsilon\cos\Omega t - \sin\epsilon\sin\Omega t)$$
$$= A\cos(\epsilon + \Omega t)$$

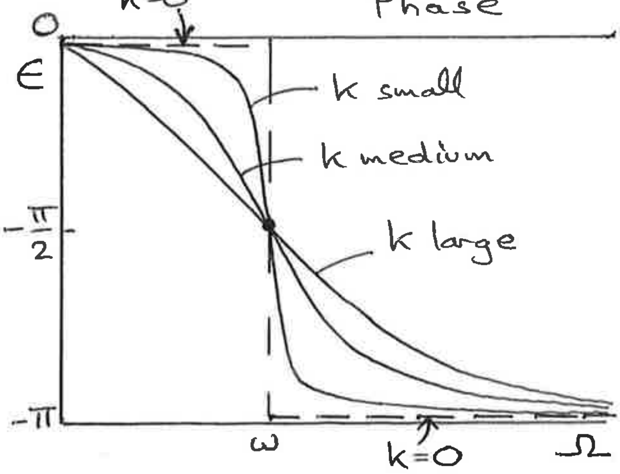
where

$$A\cos\epsilon = \frac{a(\omega^2-\Omega^2)}{(\omega^2-\Omega^2)^2+4k^2\Omega^2}, \quad A\sin\epsilon = \frac{-2k\Omega a}{(\omega^2-\Omega^2)^2+4k^2\Omega^2}$$

which gives

$$A = \underbrace{\frac{a}{((\omega^2-\Omega^2)^2+4k^2\Omega^2)^{1/2}}}_{\text{Amplitude}}, \quad \epsilon = \underbrace{\tan^{-1}\left(-\frac{2k\Omega}{\omega^2-\Omega^2}\right)}_{\text{Phase}}$$





Nonlinear oscillations

We consider conservative systems only, where

$$\ddot{x} = f(x)$$

and f depends only on x (not $\dot{x}$).

Then

$$\ddot{x}\dot{x} = f(x)\dot{x}$$

$$\therefore \quad \int \ddot{x}\dot{x}\,dt = \int f(x)\frac{dx}{dt}\,dt$$

$$= \int f(x)\,dx$$

$$\therefore \quad \tfrac{1}{2}\dot{x}^2 = \tfrac{1}{2}F(x) + \tfrac{1}{2}E \qquad (E \text{ const})$$

where $F(x) = 2\int^x f(x)\,dx$. This is effectively the energy equation. If $F+E>0$ then

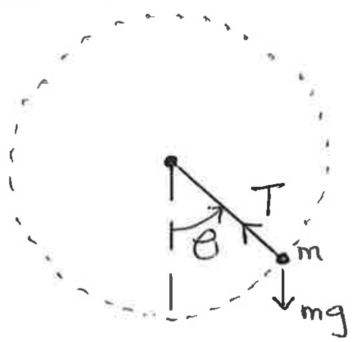
$$\dot{x} = \sqrt{F(x)+E}$$

and

$$\int \frac{dx}{\sqrt{F(x)+E}} = \int dt = t + t_0 \qquad (t_0 \text{ const})$$

In practice the integral in x may not have an analytical solution, so x cannot be obtained explicitly as a function of t. Instead we can use a <u>phase plane analysis</u> to find qualitative properties of the solution.

<u>Example</u> Pendulum with large swings



Length l
Tension T
Mass m

We have an energy eqtn

$$\tfrac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = E \quad , \text{ const}$$

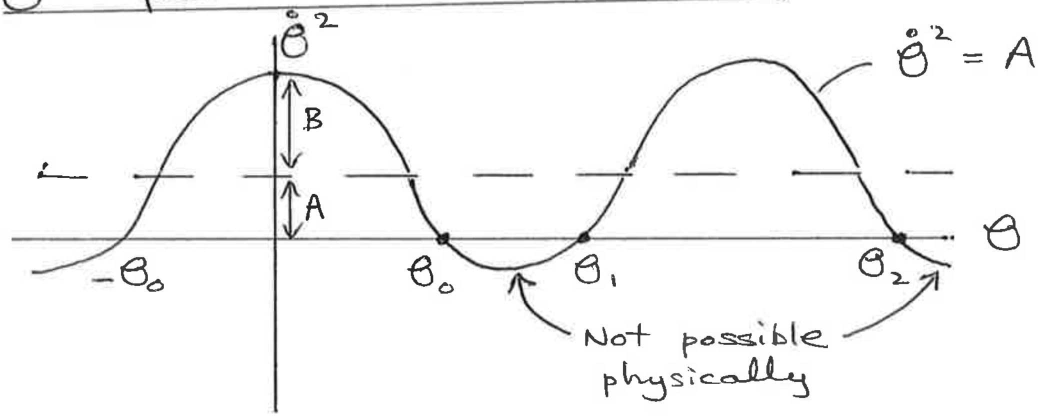
$$\therefore \quad \underline{\dot{\theta}^2 = A + B\cos\theta} \qquad (1)$$

where $A = \dfrac{2E}{ml^2}$, $B = \dfrac{2g}{l}$

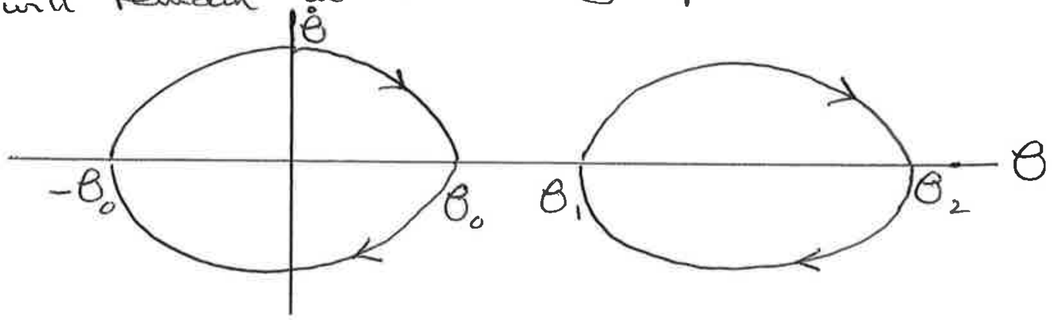
and $$t = \int \frac{d\theta}{(A+B\cos\theta)^{1/2}} + \text{const} \qquad (2)$$

We cannot integrate (2) so we consider the $\theta, \dot{\theta}^2$ relation (1) instead. There are three cases.

(i) $\dot{\theta}^2$ passes below the θ axis $\qquad (A < B)$

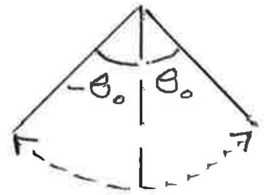


If initial conditions for $\theta, \dot{\theta}$ place the solution in the range $-\theta_0 < \theta < \theta_0$ (or $\theta_1 < \theta < \theta_2$ etc) it will remain in that range for all time.



The $\dot{\theta}, \theta$ graph is known as the phase plane. Arrows indicate the direction of motion with increasing time (if $\dot{\theta} > 0$, θ increases with t, for example).

A closed circuit in the phase plane represents an oscillatory motion. Here the pendulum swings between $-\theta_0$ and θ_0.



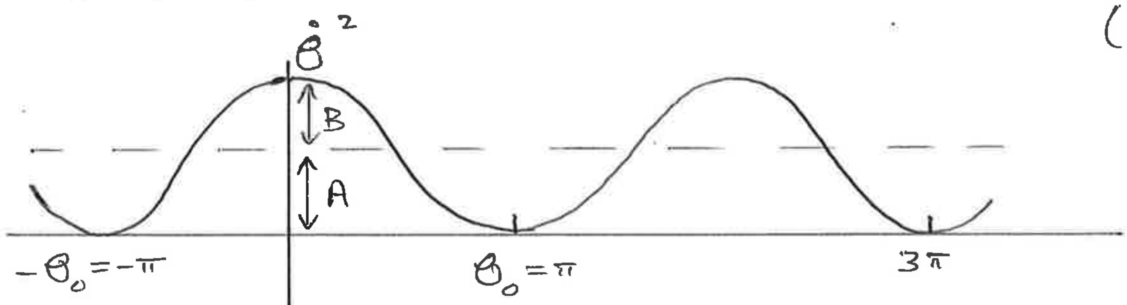
Period of oscillation is

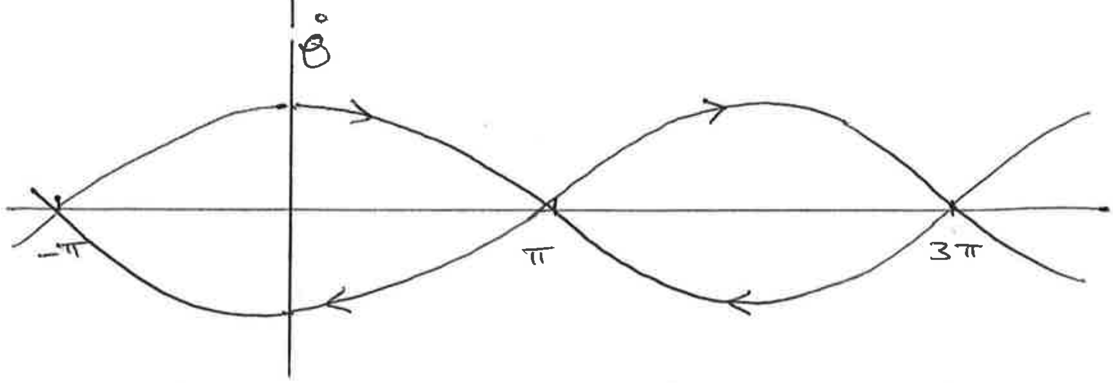
$$\int_{1\text{ cycle}} dt = \int \frac{d\theta}{\dot{\theta}} = 2\int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\dot{\theta}^2}}$$

Convergence of this integral to a finite answer requires $\dot{\theta} \sim (\theta - \theta_0)^n$ as $\theta \to \theta_0$ with $n < 1$. In our case above $n = \frac{1}{2}$, so integral converges and period is finite.

(ii) <u>$\dot{\theta}^2$ just touches the θ axis</u>

(B = A)





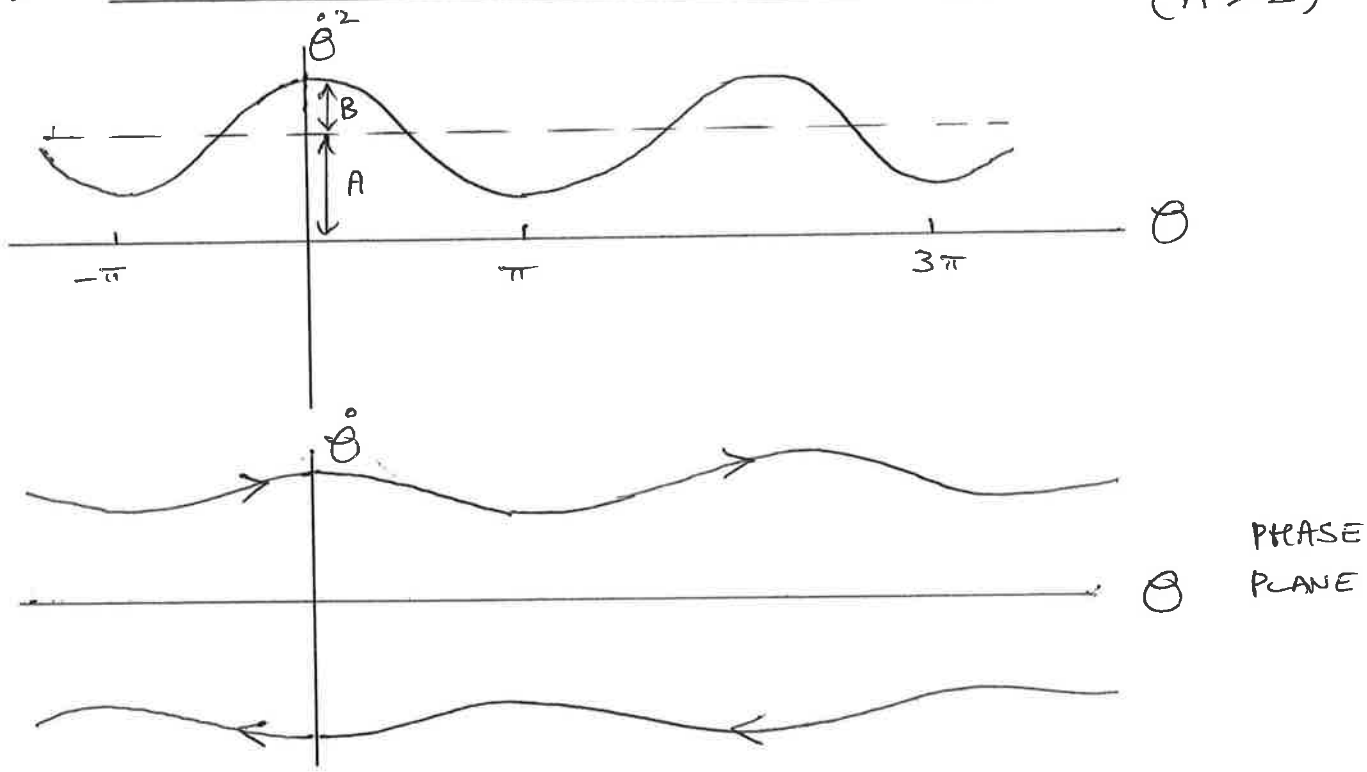
PHASE PLANE

Here $\dot{\theta} \sim (\theta - \theta_0)$ as $\theta \to \theta_0$ so $n = 1$.

Thus Period $\to \infty$ in this case.

The pendulum just reaches the top and takes an infinite time to get there

(iii) $\dot{\theta}^2$ entirely above the θ axis (A > B)



The pendulum completes whole circuits in either the clockwise or anticlockwise directions.

The size of A (which is proportional to the energy of the system) relative to B determines which kind of motion occurs. In general, the initial conditions will fix their values.

Equilibrium states

These are states where there is no motion, so

$$\dot{x} = \ddot{x} = 0$$

Since $\ddot{x} = f(x)$, this implies

$$f(x) = 0$$

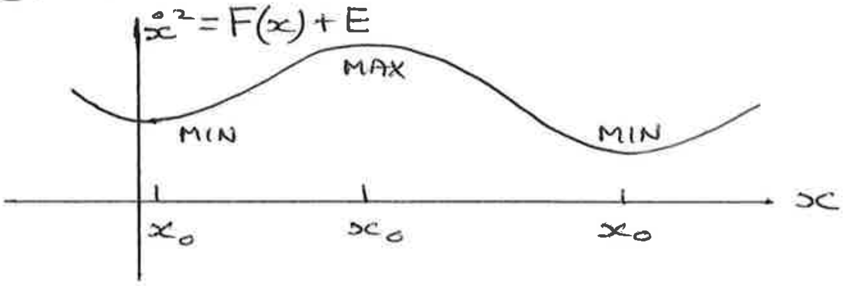
Thus equilibrium states, $x = x_0$, are given by the zeros of the function f, $f(x_0) = 0$.

Since $\quad F(x) = 2\int f(x)\,dx \quad$ it follows that

$$\frac{dF}{dx} = F' = 2f(x)$$

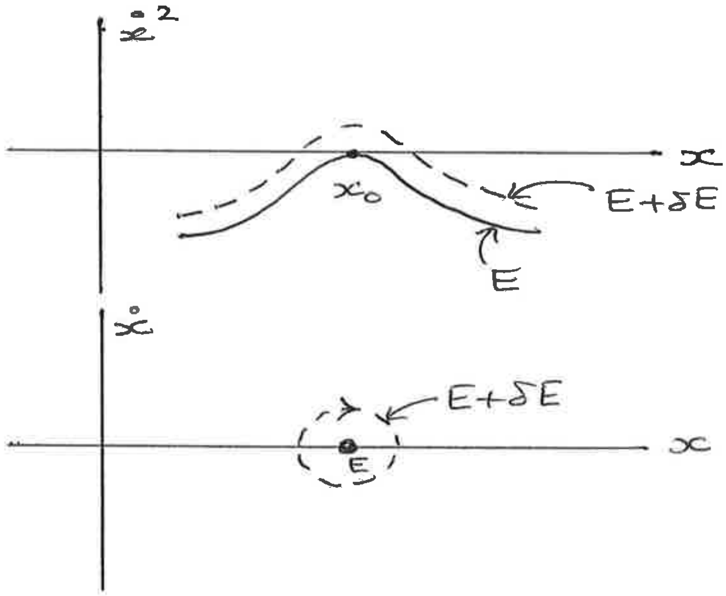
Thus zeros of f correspond to zeros of F'.

i.e. F has a max. or min.



Equilibrium points

<u>Case 1</u> $\quad F$ has a maximum at x_0

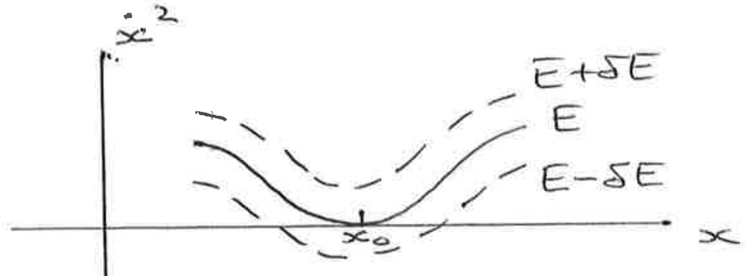


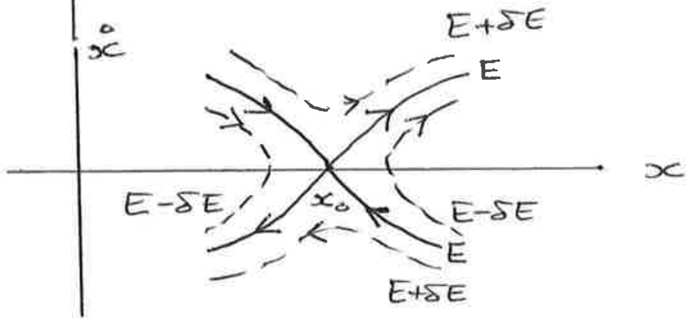
PHASE PLANE

If the system is disturbed by changing the energy by δE it remains 'close' to x_0, so x_0 is a position of <u>stable equilibrium</u>.

<u>Case 2</u> $\quad F$ has a minimum at x_0

If the system is disturbed by $\pm\delta E$ it no longer remains 'close' to x_0 as t increases, so this is a position of <u>unstable equilibrium</u>.





PHASE PLANE

We can analyse solutions near x_0 as follows.
Let x_0 be an equilibrium position such that $f(x_0)=0$.
Write $x = x_0 + y$ with $y \ll 1$.

Then
$$\ddot{x} = f(x)$$
gives
$$\ddot{y} - f(x_0+y) = 0$$
$$\ddot{y} - \{ f(x_0) + y f'(x_0) + \dots \} = 0$$
using Taylor expansion of f about x_0.
Since $f(x_0) = 0$ this gives to a first approximation
$$\ddot{y} - f'(x_0)\, y = 0$$

If $f'(x_0) = \frac{1}{2}F''(x_0) < 0$ (i.e. F is max)
$$y = A\cos\left(\sqrt{-f'(x_0)}\, t\right) + B\sin\left(\sqrt{-f'(x_0)}\, t\right)$$
so solution remains near x_0 and oscillates with frequency $\sqrt{-f'(x_0)}$ (i.e. period $2\pi/\sqrt{-f'(x_0)}$).

This is a <u>stable</u> state.

If $f'(x_0) = \frac{1}{2}F''(x_0) > 0$ (i.e. F is min)

$$y = Ae^{\sqrt{f'(x_0)}\,t} + Be^{-\sqrt{f'(x_0)}\,t}$$

so solution grows exponentially as t increases and does not remain close to x_0.

This is an <u>unstable</u> state.

<u>Example</u> Pendulum

$$\ddot{\theta} = -\omega^2 \sin\theta = f(\theta)$$

Equilibrium positions: $f(\theta) = 0$

$$\Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi.$$

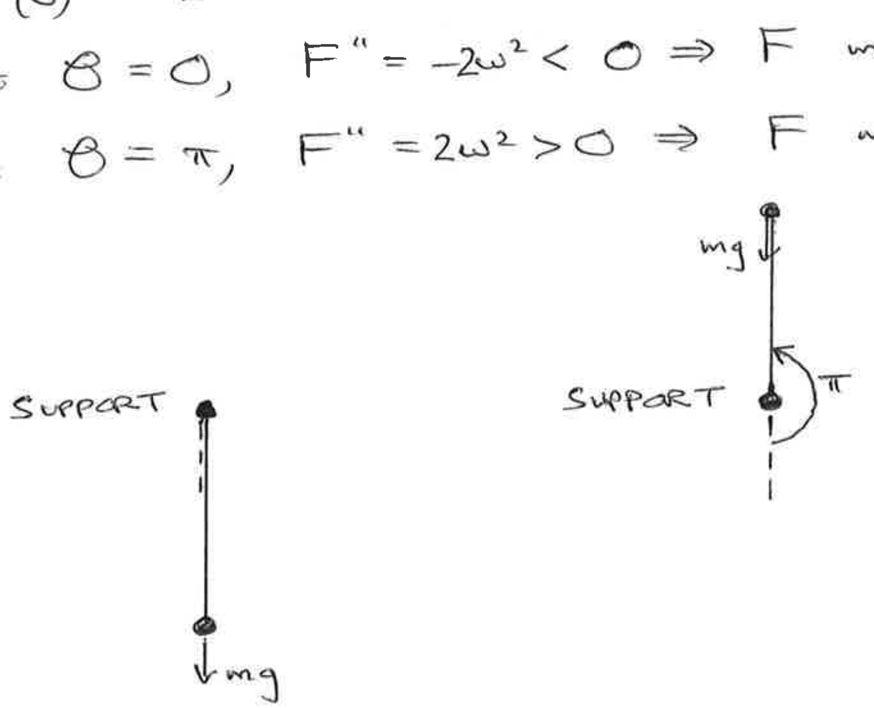
$$F(\theta) = 2\int f(\theta)\,d\theta = 2\omega^2\cos\theta$$

$$F'(\theta) = -2\omega^2\sin\theta$$

$$F''(\theta) = -2\omega^2\cos\theta$$

At $\theta = 0$, $F'' = -2\omega^2 < 0 \Rightarrow F$ max $\Rightarrow$ Stable

At $\theta = \pi$, $F'' = 2\omega^2 > 0 \Rightarrow F$ min $\Rightarrow$ Unstable



$\theta = 0$: stable state $\quad\quad$ $\theta = \pi$: unstable state