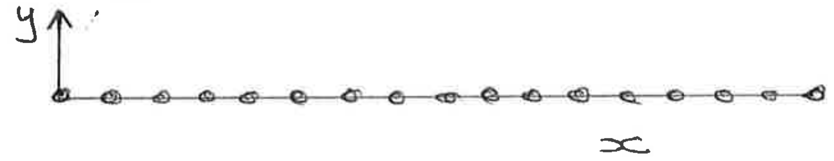
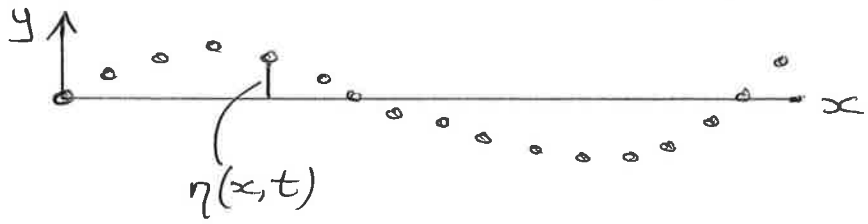
Motion of a string of particles

So far we have only considered single masses or particles. More complex situations arise if we want to model the motion of a 1D set of particles (a string), a 2D set (a membrane) or a 3D set (a fluid). Here we look at the 1D set.

A string is made up of a set of particles lying along the x direction.



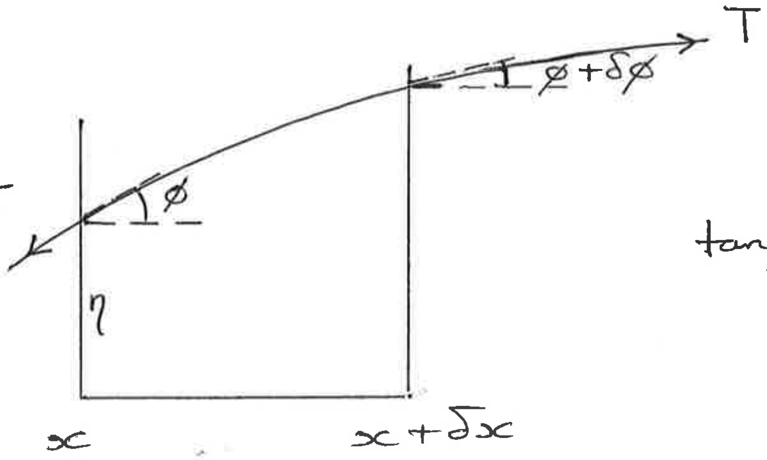
We allow the string to move in the transverse direction and denote the displacement by $\eta(x,t)$.



η depends on position x as well as time t, so is a function of two variables.

We obtain an equation for η using Newton's law.

Let T be the tension in the string and consider a small portion between x and $x + \delta x$.



$$\tan\phi = \frac{\partial \eta}{\partial x}$$

We assume small displacements so that $\frac{\partial \eta}{\partial x} \ll 1$.

Then $\tan\phi \approx \phi \approx \sin\phi \approx \frac{\partial \eta}{\partial x}$.

Apply Newton's law to element δx in the y direction, taking ρ to be the mass per unit length of the string:

$$\underbrace{-T\sin\phi + T\sin(\phi + \delta\phi)}_{\text{Force in } y \text{ direction}} = \underbrace{(\rho\,\delta x)}_{\text{Mass}} \underbrace{\frac{\partial^2 \eta}{\partial t^2}}_{\substack{\text{Acceleration in} \\ y \text{ direction}}}$$

$$-T\left(\frac{\partial \eta}{\partial x}\right)_x + T\left(\frac{\partial \eta}{\partial x}\right)_{x+\delta x} = \rho\,\delta x \frac{\partial^2 \eta}{\partial t^2}$$

$$\cancel{-T\left(\frac{\partial \eta}{\partial x}\right)_x} + T\left\{\cancel{\left(\frac{\partial \eta}{\partial x}\right)_x} + \delta x\left(\frac{\partial^2 \eta}{\partial x^2}\right)_x + \dots\right\} = \rho\,\delta x \frac{\partial^2 \eta}{\partial t^2}$$

$$\therefore \quad T\,\delta x \frac{\partial^2 \eta}{\partial x^2} + \underbrace{O((\delta x)^2)}_{\substack{\text{Terms involving} \\ (\delta x)^2 \text{ and higher} \\ \text{powers of } \delta x}} = \rho\,\delta x \frac{\partial^2 \eta}{\partial t^2}$$

Divide by δx

$$T\frac{\partial^2 \eta}{\partial x^2} + O(\delta x) = \rho \frac{\partial^2 \eta}{\partial t^2}$$

Let $\delta x \to 0$

$$\underline{\frac{\partial^2 \eta}{\partial t^2} = c^2 \frac{\partial^2 \eta}{\partial x^2}} \qquad \text{where} \quad c^2 = \frac{T}{\rho}$$

This is a partial differential equation for η, known as the 1D <u>wave equation</u>.

It is '1D' because it has one spatial dimension, x. In general it requires initial conditions that specify

$\eta(x,0)$, the displacement of the string at $t=0$

and

$\frac{\partial \eta}{\partial t}(x,0)$, the transverse speed of the string at $t=0$,

in order to obtain specific solutions.

D'Alembert's solution of the 1D wave equation

We transform $(x,t) \to (X,Y)$ where

$$X = x - ct, \quad Y = x + ct$$

Chain rule gives

$$\frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial \eta}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial \eta}{\partial X} + \frac{\partial \eta}{\partial Y}$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial X}\frac{\partial X}{\partial t} + \frac{\partial \eta}{\partial Y}\frac{\partial Y}{\partial t} = -c\frac{\partial \eta}{\partial X} + c\frac{\partial \eta}{\partial Y}$$

and then

$$\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}\right) = \frac{\partial}{\partial X}\left(\frac{\partial \eta}{\partial X} + \frac{\partial \eta}{\partial Y}\right) + \frac{\partial}{\partial Y}\left(\frac{\partial \eta}{\partial X} + \frac{\partial \eta}{\partial Y}\right)$$

$$= \frac{\partial^2 \eta}{\partial X^2} + 2\frac{\partial^2 \eta}{\partial X \partial Y} + \frac{\partial^2 \eta}{\partial Y^2}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial t}\right) = -c\frac{\partial}{\partial X}\left(-c\frac{\partial \eta}{\partial X} + c\frac{\partial \eta}{\partial Y}\right) + c\frac{\partial}{\partial Y}\left(-c\frac{\partial \eta}{\partial X} + c\frac{\partial \eta}{\partial Y}\right)$$

$$= c^2\left(\frac{\partial^2 \eta}{\partial X^2} - 2\frac{\partial^2 \eta}{\partial X \partial Y} + \frac{\partial^2 \eta}{\partial Y^2}\right)$$

Thus $\frac{\partial^2 \eta}{\partial t^2} - c^2\frac{\partial^2 \eta}{\partial x^2} = 0$ becomes $\frac{\partial^2 \eta}{\partial X \partial Y} = 0$

Integrate in Y $\Rightarrow$ $\frac{\partial \eta}{\partial X} = f(\cancel{X})$

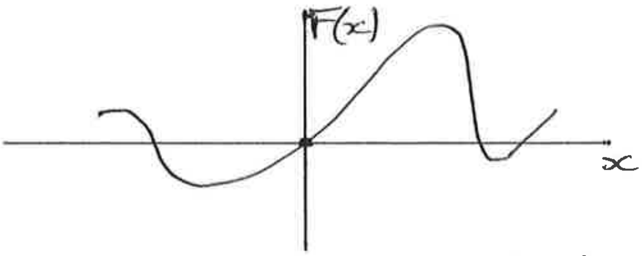
Integrate in X $\Rightarrow$ $\eta = \int f(X)\,dX + G(Y)$

$$= F(X) + G(Y)$$

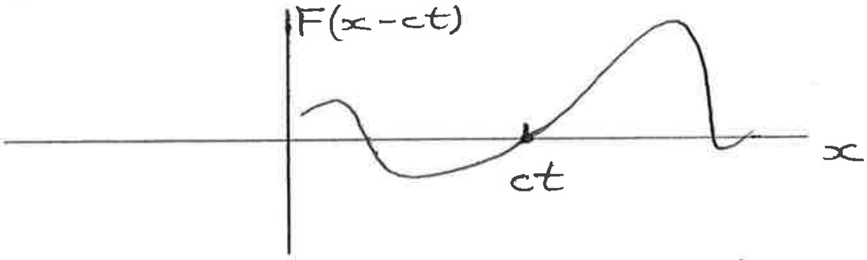
$$\therefore \quad \eta = F(x-ct) + G(x+ct) \qquad (1)$$

This is the general solution of the wave equation first found by d'Alembert, where F and G are arbitrary functions.

e.g. If F has the form



then its contribution to (1) is



i.e. it represents a wave travelling in the +ve x direction with speed c.

Similarly $G(x+ct)$ represents a wave travelling in the −ve x direction with speed c.

Thus the overall displacement η can be considered as composed of these two contributions.

Note that the particles of the string do <u>not</u> move in the x direction – they move transversely to create a wave travelling along the string.

Example Infinite string (Cauchy's solution)

Let initial state be

$$\eta = h(x), \qquad \frac{\partial \eta}{\partial t} = v(x) \qquad \text{at } t=0 \quad (2)$$

for $-\infty < x < \infty$.

We know the solution is

$$\eta = F(x-ct) + G(x+ct)$$

Set $t=0$ and use (2) $\Rightarrow$

$$h(x) = F(x) + G(x) \qquad (3)$$

Differentiate w.r.t t in d'Alembert solution $\Rightarrow$

$$\frac{\partial \eta}{\partial t} = -cF'(x-ct) + cG'(x+ct)$$

and now set $t=0$ and use (2) $\Rightarrow$

$$v(x) = -cF'(x) + cG'(x) \qquad (4)$$

Integrate (4) w.r.t x $\Rightarrow$

$$-F(x) + G(x) = \frac{1}{c}\int_{\xi=x_0}^{\xi=x} v(\xi)\,d\xi \qquad (4)'$$

where x_0 is constant of integration.

Combining (3) and (4)' gives $F(x)$ and $G(x)$:

$$(3)+(4)' \Rightarrow \qquad 2G(x) = h(x) + \frac{1}{c}\int_{\xi=x_0}^{\xi=x} v(\xi)\,d\xi$$

$$(3)-(4)' \Rightarrow \qquad 2F(x) = h(x) - \frac{1}{c}\int_{\xi=x_0}^{\xi=x} v(\xi)\,d\xi$$

$$\therefore \begin{cases} F(x-ct) = \frac{1}{2}h(x-ct) - \frac{1}{2c}\int_{\xi=x_0}^{\xi=x-ct} v(\xi)\,d\xi \\[2ex] G(x+ct) = \frac{1}{2}h(x+ct) + \frac{1}{2c}\int_{\xi=x_0}^{\xi=x+ct} v(\xi)\,d\xi \end{cases}$$

giving

$$\eta = \tfrac{1}{2}h(x-ct) + \tfrac{1}{2}h(x+ct) + \frac{1}{2c}\int_{\xi=x-ct}^{\xi=x_0} v(\xi)\,d\xi + \frac{1}{2c}\int_{\xi=x_0}^{\xi=x+ct} v(\xi)\,d\xi$$

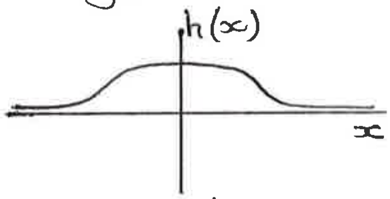
so

$$\eta = \tfrac{1}{2}h(x-ct) + \tfrac{1}{2}h(x+ct) + \frac{1}{2c}\int_{\xi=x-ct}^{\xi=x+ct} v(\xi)\,d\xi$$

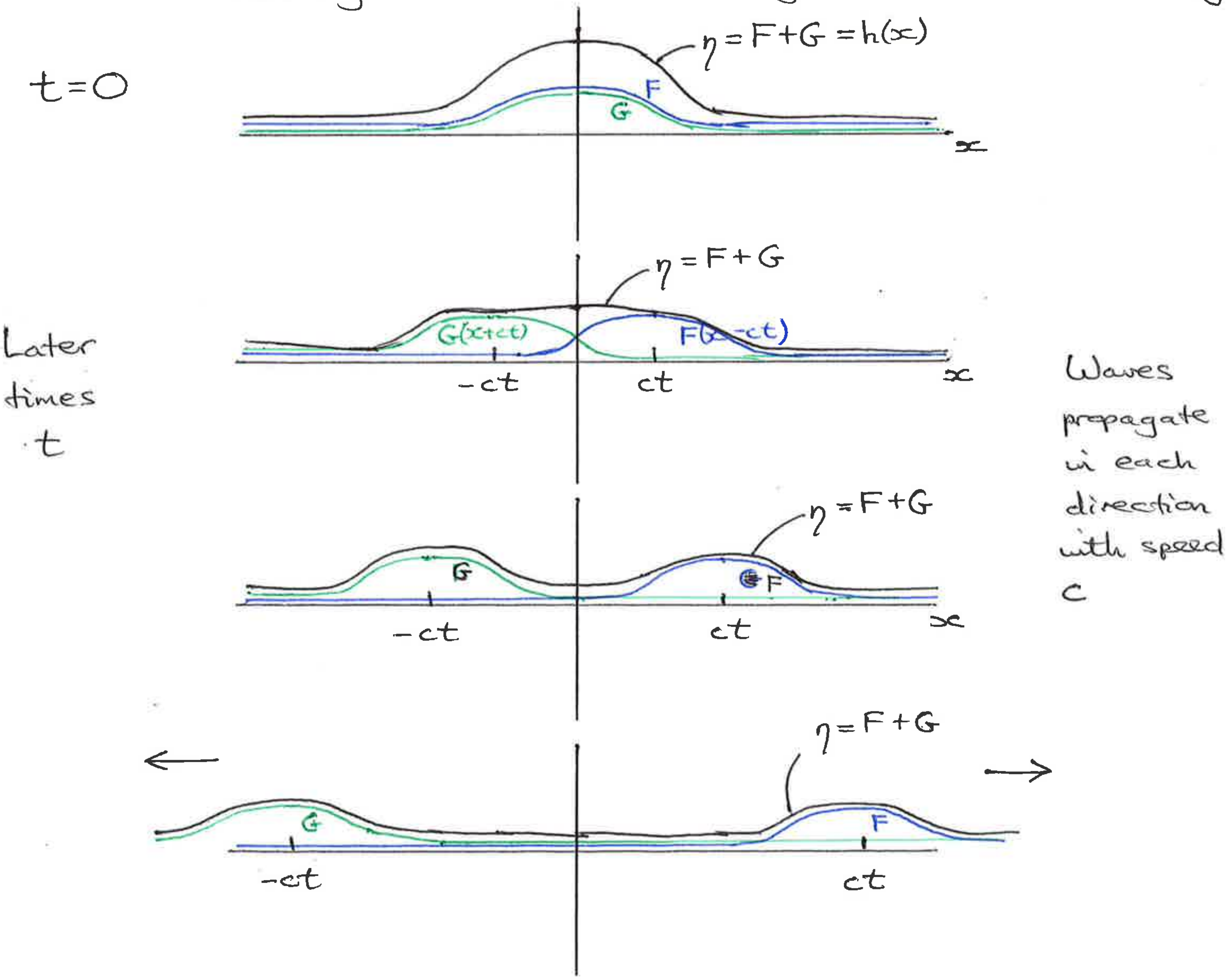
This is known as Cauchy's solution.

e.g. 1. Suppose $v(x) = 0$ (at rest initially)

Then $F(x) = G(x) = \tfrac{1}{2}h(x)$



We can construct η at successive times by shifting F and G to the right and left respectively.

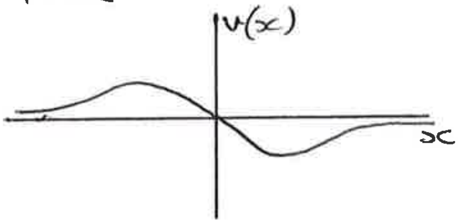


e.g. 2 Suppose $h(x) = 0$ (zero displacement initially)

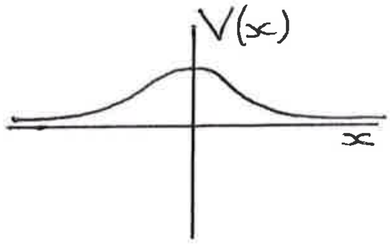
but now $v(x) \neq 0$.

Let $V(x) = \int_{-\infty}^{x} v(x)\,dx$

and take



so

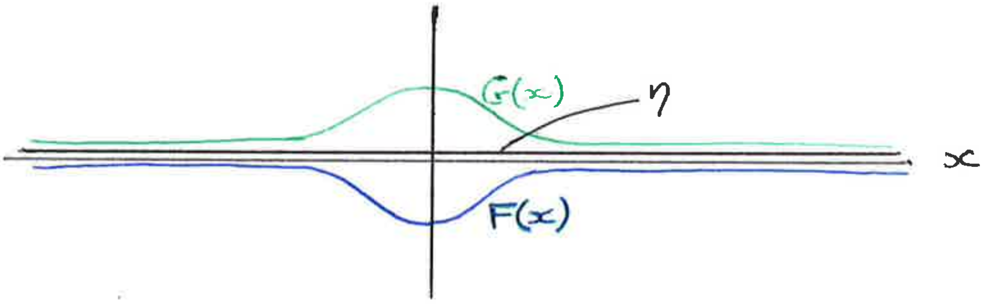


Then

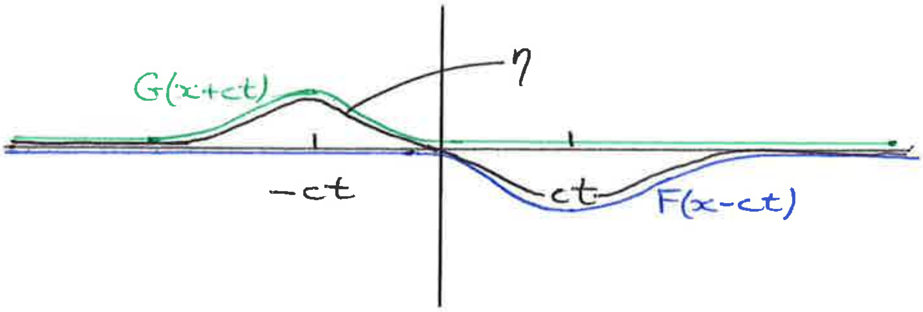
$$\frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\,d\xi = \frac{1}{2c}\left(V(x+ct) - V(x-ct)\right)$$

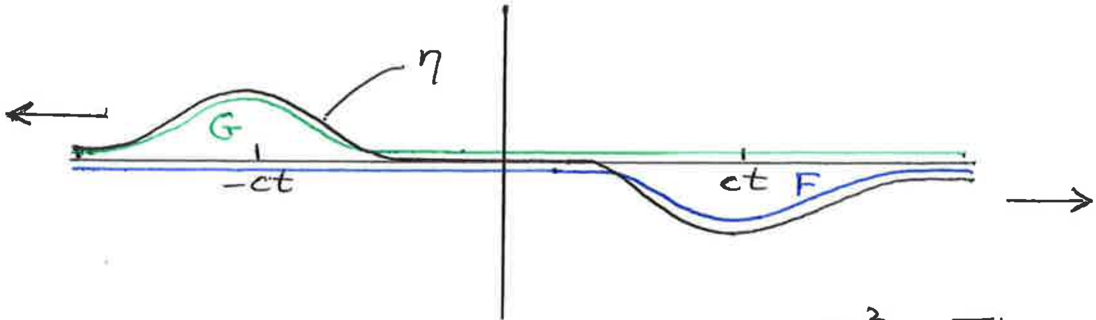
and $F(x) = -\frac{1}{2c}V(x)$, $G(x) = \frac{1}{2c}V(x)$

$t=0$



Later times t



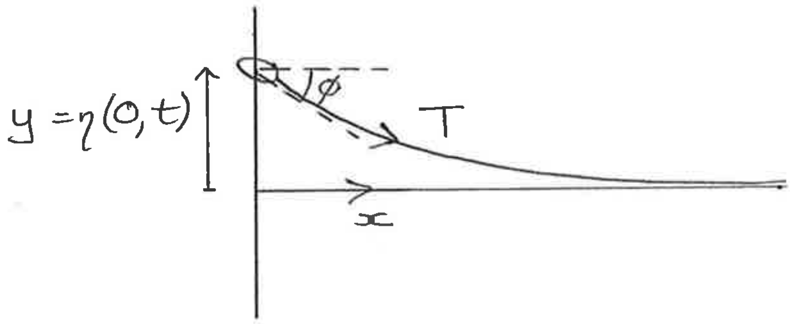


Analytical form: suppose $V(x) = e^{-x^2}$. Then

$$\left.\begin{aligned} F(x-ct) &= -\frac{1}{2c}e^{-(x-ct)^2} \\ G(x+ct) &= \frac{1}{2c}e^{-(x+ct)^2} \end{aligned}\right\} \text{ giving } \eta = \frac{1}{2c}\left\{e^{-(x+ct)^2} - e^{-(x-ct)^2}\right\}$$

Wave reflection: semi-infinite string

Let string be $x \geq 0$ and attached to a ring sliding on a vertical rod at $x = 0$.



Ring mass m
T tension

Resolve vertically at ring:

$$m \frac{\partial^2 \eta}{\partial t^2} = -T \sin\phi = -T \frac{\partial \eta}{\partial x}$$

$\therefore$ Boundary condition is

$$\frac{\partial \eta}{\partial x} = -\frac{m}{T} \frac{\partial^2 \eta}{\partial t^2} \quad \text{at} \quad x = 0, \ t > 0$$

If the ring is light ($m \simeq 0$) then

$$\frac{\partial \eta}{\partial x} = 0 \quad \text{at} \quad x = 0, \ t > 0$$

A wave ~~travelling~~ along the string will be reflected at $x = 0$. The solution

$$\eta = F(x - ct) + G(x + ct)$$

still applies. Suppose string is at rest at $t = 0$ with displacement $\eta = h(x)$, so

$$\eta = h(x), \quad \frac{\partial \eta}{\partial t} = 0 \quad \text{at} \quad t = 0 \text{ for } x > 0.$$

This gives

$$F(x) + G(x) = h(x), \quad x > 0 \qquad (1)$$

$$-cF'(x) + cG'(x) = 0, \quad x > 0 \qquad (2)$$

Also, assuming the ring is light, we have

$\frac{\partial \eta}{\partial x} = 0$ at $x = 0$ for $t > 0$ $\Rightarrow$

$$F'(-ct) + G'(ct) = 0, \quad t > 0 \qquad (3)$$

Integrate (3) $\Rightarrow$ $-F(-ct) + G(ct) = D$, $t > 0$

$\Rightarrow$ $-F(-x) + G(x) = D$, $x > 0$ $\qquad (3)'$

Integrate (2) $\Rightarrow$ $-F(x) + G(x) = E$, $x > 0$ $\qquad (2)'$

Also (1) $\Rightarrow$ $F(x) + G(x) = h(x)$, $x > 0$ $\qquad (1)'$

We use $(1)', (2)', (3)'$ to find $F(x)$ and $G(x)$.

$(1)'$ and $(2)'$ $\Rightarrow$

$$G(x) = \tfrac{1}{2}(h(x) + E), \quad x > 0$$
$$F(x) = \tfrac{1}{2}(h(x) - E), \quad x > 0$$

Then $(3)'$ $\Rightarrow$

$$F(-x) = G(x) - D = \tfrac{1}{2}(h(x) + E) - D, \quad x > 0$$

For continuity at $x = 0$, $D = E$, so

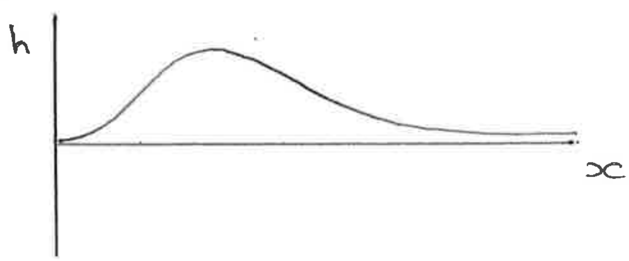
$$F(-x) = \tfrac{1}{2}(h(x) - E), \quad x > 0$$

We can take $E = 0$ wlog because it will cancel out when we add F and G to get η.

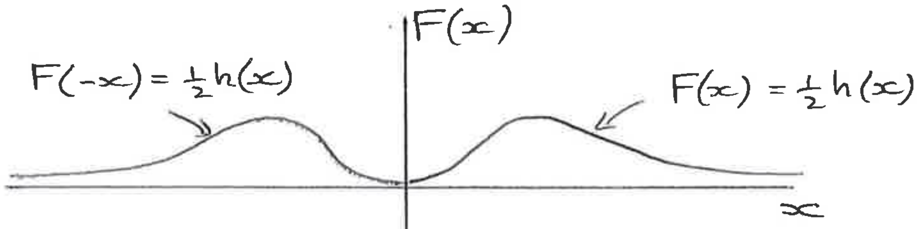
$\therefore$ Setting $E = 0$,

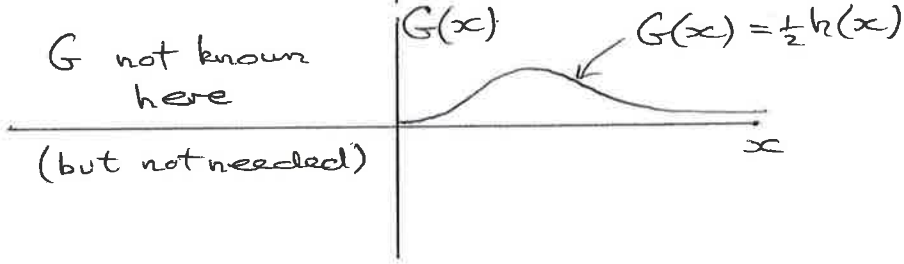
$$F(x) = G(x) = \tfrac{1}{2}h(x), \quad x > 0$$
$$F(-x) = \tfrac{1}{2}h(x), \quad x > 0$$

Suppose $h(x)$ has form:



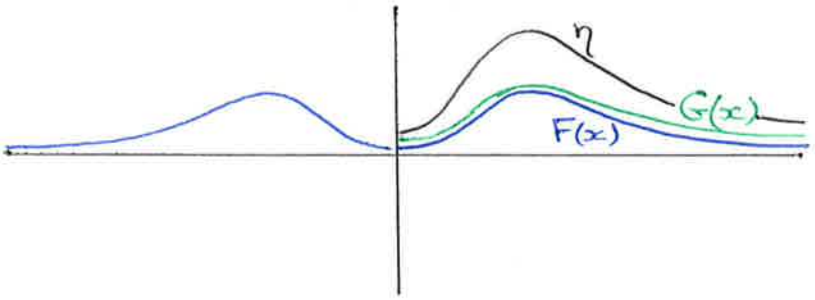
Then we can construct $F(x)$ and $G(x)$:



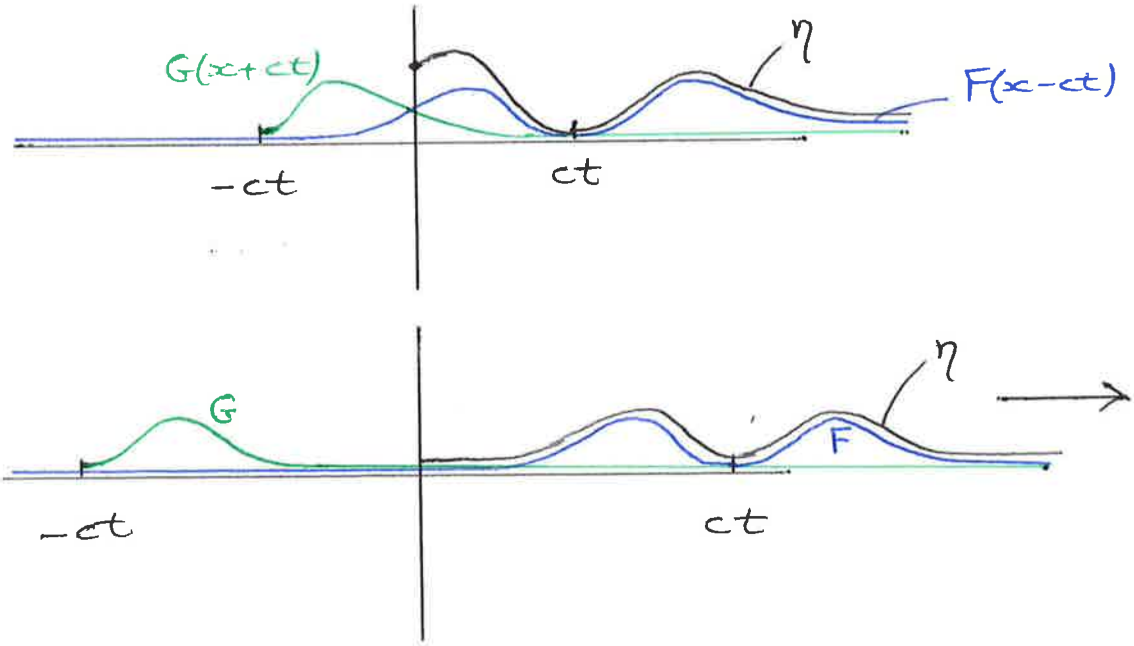


Now we add to get $\eta = F(x-ct) + G(x+ct)$

$t=0$



Later times



The inward travelling component is reflected, producing a 'double' profile travelling outwards.

Example $\quad h(x) = x^2 e^{-x}$

$$F(x) = G(x) = \tfrac{1}{2}x^2 e^{-x}, \quad x > 0 \qquad (1)$$

$$F(-x) = \tfrac{1}{2}x^2 e^{-x}, \quad x > 0 \qquad (2)$$

$\therefore$ For $x > ct$, (1) $\Rightarrow$

$$\eta = \tfrac{1}{2}(x-ct)^2 e^{-(x-ct)} + \tfrac{1}{2}(x+ct)^2 e^{-(x+ct)} \qquad (3)$$

For $0 \leq x \leq ct$, (2) and (1) $\Rightarrow$

$$\eta = \underbrace{\tfrac{1}{2}(ct-x)^2 e^{x-ct}}_{\substack{\text{From (2) with} \\ -x \to x-ct}} + \tfrac{1}{2}(x+ct)^2 e^{-(x+ct)} \qquad (4)$$

Check: We should have $\dfrac{\partial \eta}{\partial x} = 0$ at $x = 0$.

From (4),

$$\frac{\partial \eta}{\partial x} = \left\{-(ct-x) + \tfrac{1}{2}(ct-x)^2\right\} e^{x-ct} + \left\{(x+ct) - \tfrac{1}{2}(x+ct)^2\right\} e^{-(x+ct)}$$

$$= \left\{-ct + \tfrac{1}{2}(ct)^2 + ct - \tfrac{1}{2}(ct)^2\right\} e^{-ct} \quad \text{at } x = 0$$

$$= 0 \quad \text{as expected.}$$

Example A semi-infinite string is initially at rest with zero displacement and is set in motion by a displacement $\eta = te^{-t}$ at $x = 0$ for $t \geq 0$. Find the displacement as a function of x and t for $x \geq 0$ and $t > 0$.

By d'Alembert's solution, $\eta = F(x-ct) + G(x+ct)$

Initial conditions:

$\eta = 0$ at $t = 0 \Rightarrow F(x) + G(x) = 0 \quad , \quad x > 0$

$\dfrac{\partial \eta}{\partial t} = 0$ at $t = 0 \Rightarrow -cF'(x) + cG'(x) = 0 , \quad x > 0$

$\therefore -F(x) + G(x) = \text{Const} = 0 \quad$ (w.l.o.g)

$x > 0$

Boundary condition:

$\eta = te^{-t}$ at $x = 0 \Rightarrow F(-ct) + G(ct) = te^{-t}, \quad t > 0$

$\therefore$ We have

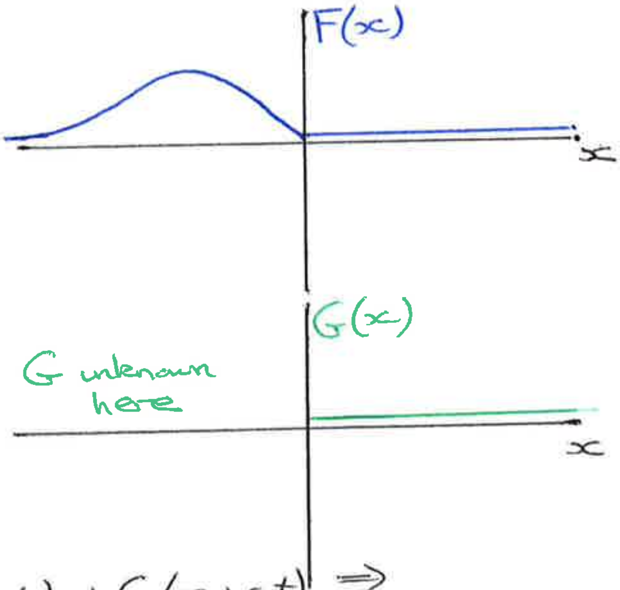
$$F(x) + G(x) = 0 \quad , \quad x > 0 \qquad (1)$$

$$-F(x) + G(x) = 0 \quad , \quad x > 0 \qquad (2)$$

$$F(-x) + G(x) = \frac{x}{c} e^{-\frac{x}{c}} , \quad x > 0 \qquad (3)$$

For $x > 0$, (1) and (2) $\Rightarrow$ $F(x) = G(x) = 0$ (4)

Then from (3), $F(-x) = \frac{x}{c} e^{-\frac{x}{c}}, \quad x > 0.$ (5)



$\eta = F(x - ct) + G(x + ct) \Rightarrow$

$$\begin{cases} \eta = 0 \quad \text{for} \quad x > ct & \text{(from (4))} \\ \eta = \dfrac{ct - x}{c} e^{-\frac{1}{c}(ct - x)} \quad \text{for} \quad 0 \le x \le ct & \text{(from (4) and (5))} \end{cases}$$