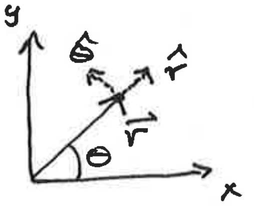
$\hat{r} = \cancel{*}\ \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$ $\qquad$ $\hat{r}\cdot\hat{r} = 1$ so $\hat{r}$ unit vector



$$\vec{r} = \cancel{\ \ } |r|\cos\theta\,\hat{i} + |r|\sin\theta\,\hat{j}$$

$$= x\,\hat{i} + y\,\hat{j} = |r|\,\hat{r} \qquad \hat{i} = \begin{pmatrix}1\\0\end{pmatrix} \quad \hat{j} = \begin{pmatrix}0\\1\end{pmatrix}$$

$\hat\theta$ defined as vector perpendicular to $\hat{r}$ so

$$\hat\theta = \pm\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right) \qquad \text{check } \hat\theta\cdot\hat{r} = 0$$

$$\hat\theta\cdot\hat\theta = 1 \quad (\hat\theta \text{ unit vector})$$

Conventionally take $\hat\theta = +(-\sin\theta, \cos\theta)$

so $\hat{r}$ & $\hat\theta$ are functions of θ. Lets compute:

$$\boxed{\begin{aligned} \frac{d\hat{r}}{d\theta} &= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \overset{①}{=} \hat\theta \\ \frac{d\hat\theta}{d\theta} &= -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} \overset{②}{=} -\hat{r} \end{aligned}}$$

$$\left\{\begin{aligned} \frac{d\hat{r}}{dr} &= 0 \\ \frac{d\hat\theta}{dr} &= 0 \end{aligned}\right.$$

after all $\hat{r}$, $\hat\theta$ are not functions of r

Notice that this is rather different to Cartesian unit vectors

$$\frac{d\hat{i}}{dx} = \frac{d\hat{j}}{dx} = \frac{d\hat{i}}{dy} = \frac{d\hat{j}}{dy} = 0$$

$$\left(\text{also } \frac{d\hat{i}}{d\theta} = \frac{d\hat{j}}{d\theta} = \frac{d\hat{i}}{dr} = \frac{d\hat{j}}{dr} = 0\right)$$

So if position $\vec{r}(t) = |r|\,\hat{r} \equiv r\,\hat{r}$ what is velocity?

$$\vec{v} = \frac{d(\vec{r})}{dt} = \frac{d}{dt}\left(\cancel{|}r\cancel{|}\,\hat{r}\right) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\,\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}$$

$$\overset{\text{using ①}}{=} \dot{r}\,\hat{r} + r\,\dot\theta\,\hat\theta$$

And acceleration

$$\vec{a} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}\left(\dot{r}\,\hat{r} + r\,\dot\theta\,\hat\theta\right) = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\dot\theta\,\hat\theta + r\frac{d\dot\theta}{dt}\hat\theta + r\dot\theta\frac{d\hat\theta}{dt}$$

$$= \ddot{r}\,\hat{r} + \dot{r}\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} + \dot{r}\dot\theta\,\hat\theta + r\ddot\theta\,\hat\theta + r\dot\theta\frac{d\hat\theta}{d\theta}\frac{d\theta}{dt}$$

$$\overset{\text{using ①+②}}{=} (\ddot{r} - r\dot\theta^2)\,\hat{r} + (2\dot{r}\dot\theta + r\ddot\theta)\,\hat\theta$$