APPLIED MATHEMATICS SHEET 2: MOTION

1. A mass m is thrown vertically upwards with initial speed V. How high does it go before it begins to fall? How long does it take to reach the thrower again? Find and sketch the kinetic and potential energies of the mass as a function of time and show that their sum remains constant.

2. A mass m is rolled off a table at height h above the floor with horizontal speed V. Where does the mass land? What trajectory did it take and how long did it take to reach the floor?

3. A mass m is thrown with initial speed V at an angle α to the horizontal. Where does the mass land? What trajectory did it take? For what angle does the mass land farthest away from where it was thrown, assuming the same initial speed.

4. A simple pendulum consisting of a mass m attached to a light rod of length l is released at $t = 0$ with a small speed V from a position where the rod is at a small angle α to the downward vertical. Find the amplitude and period of the resulting motion.

5. Suppose that the inverse square law of gravitational attraction $F_r = -\mu/r^2$ per unit mass were replaced by an inverse cube law $F_r = -\mu/r^3$ where $\mu > 0$. Find the equation of orbits, distinguishing between the three cases that may arise.

6. A particle of mass m is subject to a force GMm/r^2 towards the centre of the Earth where M is the mass of the Earth inside the radius r. If the particle moves along a smooth straight tunnel through the Earth, show that it will perform simple harmonic motion of period $(3\pi/G\rho)^{1/2}$ where ρ is the uniform density of the Earth.

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7. A spring of natural length l and modulus kl is suspended vertically with a mass m attached at the lower end. If there is a frictional force rv where v is the speed of the mass, show that the downward displacement x of the mass from its equilibrium position is governed by

$$m\ddot{x} + r\dot{x} + kx = 0.$$

If $m = 1$, $r = 2$ and $k = 17$, and the mass is released from rest at $x = 2$ when $t = 0$, find x. What are the extreme values of x and when do they occur? Describe the motion.

8. Solve $\ddot{x} + 2k\dot{x} + k^2x = 0$ for $k > 0$ subject to $x = a$, $\dot{x} = V$ at $t = 0$, where $a > 0$ and $V > 0$. Describe the motion. When is x maximum?

9. Find the general solution of $\ddot{x} + 4\dot{x} + x = 0$. If $x = 1$ and $\dot{x} = V$ when $t = 0$, show that x reaches zero in a finite time provided V lies in a certain range. If $V = -9$, show that x is zero when $t = \frac{1}{2\sqrt{3}} \ln\left(\frac{7+\sqrt{3}}{7-\sqrt{3}}\right)$.

10. (a) A shot-putter releases the shot with speed V at an upward angle of 30° to the horizontal at a height h above the ground. If $V^2 = 9gh$, where g is the acceleration due to gravity, and air resistance is neglected, show that when the shot lands it has covered a horizontal distance of $3\sqrt{3}(3 + \sqrt{17})h/4$ from the point of release.

(b) A mass moves under the influence of an inverse-cube law of attraction and is governed by the equations

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^3}, \quad 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2} = 0$$

where r, θ are its plane polar coordinates and μ is a constant. Show that $r^2 d\theta/dt = h$, where h is a constant, and, by using the substitution $u = r^{-1}$, or otherwise, show that if $\mu < h^2$ then

$$\frac{1}{r} = C\cos\left((1 - \frac{\mu}{h^2})^{1/2}\theta + D\right)$$

where C and D are constants.