APPLIED MATHEMATICS SHEET 3: OSCILLATIONS AND WAVES

1. The displacement x of a system satisfies

$$3\ddot{x} + 8\dot{x} + 4x = 16 + 4y + 8\dot{y}$$

where $y = \sin 3t$. If $x = \dot{x} = 0$ at $t = 0$, find x and describe the motion that occurs at large times.

2. A mass m is suspended under gravity by a spring of natural length l and modulus λ. The downward displacement of the mass relative to its equilibrium position is denoted by x and there is a frictional force $3m\omega\dot{x}$ where $2\omega^2 = \lambda/ml$. If the spring support is moved vertically with downward displacement $\frac{1}{2}a\sin\omega t$, show that x satisfies

$$\ddot{x} + 3\omega\dot{x} + 2\omega^2 x = a\omega^2 \sin\omega t.$$

If the mass is released from rest at $x = 0$ when $t = 0$, find x as a function of t and show that for large times, x is approximately $\frac{a}{\sqrt{10}}\sin(\omega t - \epsilon)$ where $\epsilon = \tan^{-1} 3$.

3. The support of a pendulum is displaced a small horizontal distance $y = a\sin\omega t$ from a fixed vertical line, where $\omega = (g/l)^{1/2}$ is the natural frequency of the pendulum. At $t = 0$ the pendulum bob, of mass m, is at rest vertically below the support. Find the horizontal displacement x of the bob relative to the fixed vertical line for $t > 0$. Describe the motion as $t \to \infty$.

4. A spring of natural length l and modulus λ hangs vertically from a support with a mass m attached at the lower end. At $t = 0$ the system hangs in equilibrium under gravity. For $t > 0$ the support moves vertically upwards with speed $V\sin\Omega t$. Derive the equation for the upward displacement x of the mass from its equilibrium position. If $\Omega < \omega$ where $\omega = (\lambda/ml)^{1/2}$ is the natural frequency, and the mass is now released from rest at $x = V\Omega/(\omega^2 - \Omega^2)$ when $t = 0$, find x. Describe the motion of the mass relative to the support.

5. If $2\ddot{x} + 3x^2 - 2x = 0$, show that $\dot{x}^2 = A + x^2 - x^3$ where A is constant. Sketch the graph of $\dot{x}^2$ against x and discuss the possible motions for different values